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A NOTE ON g-SEMISIMPLICITY OF A FINITE-DIMENSIONAL MODULE OVER THE RATIONAL CHEREDNIK ALGEBRA OF TYPE A

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Abstract. The purpose of this paper is to show that a certain finite dimensional representation of the rational Cherednik algebra of type A has a basis consisting of simultaneous eigenvectors for the actions of a certain family of commuting elements, which are introduced in the author's previous paper. To this end, we introduce a combinatorial object, which is called a restricted arrangement of colored beads, and consider an action of the affine symmetric group on the set of the arrangements.

1. Introduction

Throughout this paper, let n be a fixed integer greater than or equal to 3, and let r a positive integer with $\gcd(r, n) = 1$, unless otherwise stated.

In the representation theory of the symmetric group, Young's seminormal construction describes each irreducible representation of the symmetric group in terms of a basis consisting of simultaneous eigenvectors for the Jucys-Murphy elements. In the construction, each simultaneous eigenspace for the Jucys-Murphy elements is one-dimensional or zero.

In [4], the author analyzes a certain representation of the rational Cherednik algebra (rational double affine Hecke algebra) of type $\mathfrak{gl}_n$ by similar methods. To be precise, the representation is described with a basis consisting of simultaneous eigenvectors for the Dunkl-Cherednik elements, introduce by Cherednik([2]), which are analogs to the Jucys-Murphy elements.

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In this article, we develop an analogous theory for $\mathfrak{sl}_n$. We study a finite dimensional representation of the rational Cherednik algebra of type $\mathfrak{sl}_n$ in terms of the actions of the modified Dunkl-Cherednik elements which are introduced in [5]. The purpose of this paper is to show that the finite dimensional representation of the rational Cherednik algebra of type $\mathfrak{sl}_n$ has a basis consisting of simultaneous eigenvectors for the actions of the modified Dunkl-Cherednik elements; moreover, each simultaneous eigenspace is one-dimensional or zero.

The outline of this paper is as follows. In Section 2, we introduce a combinatorial object, which is called a restricted arrangement of colored beads, and consider an action of the affine symmetric group on the set of the arrangements. In Section 3, we review the definition of the rational Cherednik algebra of type A and several facts about its representations. In Section 4, we prove the main theorem by investigating the relationship between the action on restricted arrangements and properties of representations of the rational Cherednik algebra of type A.

2. Restriced arrangements of colored beads

Let n be a fixed positive integer. Recall that an (extended) affine permutation is a bijection $w\colon \mathbb{Z} \to \mathbb{Z}$ subject to the condition:

$$w(m+n) = w(m) + n \text{ for all } m \in \mathbb{Z}.$$

For example, the simple transpositions s_i $(1 \le i \le n-1)$ and the translation π are affine permutations, where $s_i, \pi\colon \mathbb{Z} \to \mathbb{Z}$ are defined as

$$s_i(m) = \begin{cases} m+1 & \text{if } m \equiv i \pmod{n}, \\ m-1 & \text{if } m \equiv i+1 \pmod{n}, \\ m & \text{otherwise}, \end{cases}$$

and $\pi(m) = m+1$ for all $m \in \mathbb{Z}$. We denote by $\widetilde{\mathfrak{S}}_n$ the group of the affine permutations.

Let $(a_m)_{m=-\infty}^{\infty}$ be a bi-infinite sequence of integers. We say that (a_m) is n-**periodic** if $a_{m+n} = a_m - 1$ for every $m \in \mathbb{Z}$. By periodicity, an n-periodic bi-infinite sequence $(a_m)_{m=\infty}^{\infty}$ is completely determined by the n terms $a_1, a_2, \dots, a_n$. For convenience, we also denote by $[a_1, a_2, \dots, a_n]$ the n-periodic bi-infinite sequence (a_m). Note that the group $\widetilde{\mathfrak{S}}_n$ naturally acts on the set of the n-periodic bi-infinite sequences of integers as follows:

$$w \cdot (a_m) = (a_{w^{-1}(m)}) \quad (w \in \widetilde{\mathfrak{S}}_n).$$

Definition 2.1. Let $n(\geq 3)$ and r be positive integers with $\gcd(r, n) = 1$, and let $(a_m)_{m=-\infty}^{\infty}$ an $(n-1)$-periodic bi-infinite sequence of integers. A transposition $s_i \in \widetilde{\mathfrak{S}}_{n-1}(1 \leq i \leq n-2)$ is said to be an (r, n)**-admissible shift of type I** for (a_m) if $a_i \neq a_{i+1}$. Also, we say that $\pi \in \widetilde{\mathfrak{S}}_{n-1}$ is a (r, n)**-admissible shift of type II** for (a_m) if $a_{n-1} \neq r-1$.

Example 2.2. *In case of* $r = 2$ *and* $n = 5$*, the transposition* s_3 *is a* $(2, 5)$*-admissible shift of type I for* $[0, 1, 1, 0]$*, while* s_2 *is not admissible for* $[0, 1, 1, 0]$*. Also, in this case, the translation* π *is a* $(2, 5)$*-admissible shift of type II for* $[0, 1, 1, 0]$.

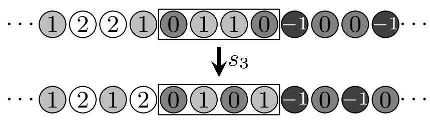

Figure 1. $(2, 5)$-admissible shift of type I for $[0, 1, 1, 0]$

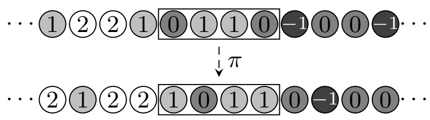

Figure 2. $(2, 5)$-admissible shift of tpye II for $[0, 1, 1, 0]$

Definition 2.3. An $(n-1)$-periodic bi-infinite sequence $[a_1, \ldots, a_{n-1}]$ is called an (r, n)**-restricted arrangement of $\mathbb{Z}$-colored beads** if either $[a_1, \ldots, a_{n-1}] = [0, \ldots, 0]$ or there exists a finite sequence of affine permutations $w_1, w_2, \ldots, w_t \in \{s_1, \ldots, s_{n-2}, \pi\}$ such that

$$[a_1, \ldots, a_{n-1}] = w_t \cdots w_2 w_1 \cdot [0, \ldots, 0]$$

and w_j is an (r, n)-admissible shift of either type I or type II for $w_{j-1} \cdots w_1 \cdot [0, \ldots, 0]$ for each $2 \leq j \leq t$.

Example 2.4. *In case of* $r = 2$ *and* $n = 5$*, the sequence* $[0, 1, 1, 0]$ *is a* $(2, 5)$*-restricted arrangement of* $\mathbb{Z}$*-colored beads because*

$$[0, 1, 1, 0] = s_1 s_2 \pi\pi \cdot [0, 0, 0, 0].$$

$$[0, 0, 0, 0] \xrightarrow{\pi} [1, 0, 0, 0] \xrightarrow{\pi} [1, 1, 0, 0] \xrightarrow{s_2} [1, 0, 1, 0] \xrightarrow{s_1} [0, 1, 1, 0]$$

Example 2.5. *The $(n-1)$-periodic bi-infinite sequence $[r,\ldots,r]$ is not an (r,n)-restricted arrangement. In fact, if $[r,\ldots,r]$ is a restricted arrangement, then there exists an (r,n)-admissible shift $w \in \{s_1,\ldots,s_{n-2},\pi\}$ for $w^{-1}\cdot[r,\cdots,r]$. However, for each $1 \le i \le n-2$, the transposition s_i is not admissible for $s_i^{-1}\cdot[r,\cdots,r]=[r,\cdots,r]$, and π is not admissible for $\pi^{-1}\cdot[r,\cdots,r,r]=[r,\cdots,r,r-1]$.*

We denote by $\mathcal{RA}_{(r,n)}$ the set of all (r,n)-restricted arrangements of $\mathbb{Z}$-colored beads.

Theorem 2.6. *The set $\mathcal{RA}_{(r,n)}$ of all (r,n)-restricted arrangement of $\mathbb{Z}$-colored beads consists of exactly r^{n-1} elements. Moreover,*

$$\mathcal{RA}_{(r,n)} = \{[a_1,\cdots,a_{n-1}] \mid a_1,\ldots,a_{n-1} \in \{0,1,\ldots,r-1\}\}.$$

Proof. Consider the map $\ell\colon \mathcal{RA}_{(r,n)} \longrightarrow \mathbb{Z}$ defined by

$$\ell([a_1,\ldots,a_{n-1}]) = \sum_{m=1}^{n-1} a_m.$$

By straightforward computation, we have

$\ell([0,\ldots,0]) = 0$,

$\ell(s_i\cdot[a_1,\ldots,a_{n-1}]) = \ell([a_1,\ldots,a_{n-1}])$ for each $1 \le i \le n-2$, and

$\ell(\pi\cdot[a_1,\ldots,a_{n-1}]) = \ell([a_1,\ldots,a_{n-1}])+1$.

From the definition of $\mathcal{RA}_{(r,n)}$, we directly see that for every restricted arrangement $[a_1,\ldots,a_{n-1}] \in \mathcal{RA}_{(r,n)}$, each a_i is nonnegative; hence, we have $\ell([a_1,\ldots,a_{n-1}]) \ge 0$.

We first show that if $[a_1,\ldots,a_{n-1}] \in \mathcal{RA}_{(r,n)}$, then $0 \le a_i \le r-1$ for all $1 \le i \le n-1$. Assume to the contrary that there exists an (r,n)-restricted arrangement $[a_1,\ldots,a_{n-1}]$ such that $a_i \ge r$ for some $1 \le i \le n-1$. We may assume that $[a_1,\cdots,a_{n-1}]$ is a restricted arrangment with $a_i \ge r$ for some $1 \le i \le n-1$ of minimal $\ell_0 = \ell([a_1,\cdots,a_{n-1}])$. Then we can directly see $\ell_0 \ge r > 0$; in particular, $[a_1,\ldots,a_{n-1}] \ne [0,\ldots,0]$. Since $[a_1,\ldots,a_{n-1}] \in \mathcal{RA}_{(r,n)}$, there exists a sequence $w_1,w_2,\ldots,w_t \in \{s_1,\ldots,s_{n-2},\pi\}$ such that $[a_1,\ldots,a_{n-1}] = w_t\cdots w_2w_1\cdot[0,\ldots,0]$ and w_j is an (r,n)-admissible shift of either type I or type II for $w_{j-1}\cdots w_1\cdot[0,\ldots,0]$ for each $2 \le j \le t$. Note that $\ell([a_1,\ldots,a_{n-1}]) > 0$ and $\ell([0,\ldots,0]) = 0$, from which it follows that there exists a maximal index $k (1 \le k \le t)$ such that w_k is an admissible shift of type II. For $w_k\cdots w_1\cdot[0,\ldots,0] = [b_1,\ldots,b_{n-1}]$, the tuple $(b_1,\ldots,b_{n-1})$ is a rearrangement of $(a_1,\ldots,a_{n-1})$ since $w_{k+1},\ldots,w_t \in \{s_1,\ldots,s_{n-2}\}$. Note that w_k is an (r,n)-admissible shift of type II for

$w_k^{-1} \cdot [b_1, \ldots, b_{n-1}] = [b_2, \ldots, b_{n-1}, b_1 - 1] \in \mathcal{RA}_{(r,n)}$, or equivalently $b_1 \neq r$. Thus, $(b_2, \ldots, b_{n-1}, b_1 - 1)$ still has at least one component greater than or equal to r. However, we have $\ell(w_k^{-1} \cdot [b_1, \ldots, b_{n-1}]) = \ell([b_2, \ldots, b_{n-1}, b_1 - 1]) = \ell_0 - 1$, which contradicts the minimality of ℓ_0.

We now assume that $[c_1, \ldots, c_{n-1}]$ $(c_1, \ldots, c_{n-1} \in \{0, 1, \ldots, r-1\})$ is an $(n-1)$-periodic sequence of minimal $\ell_0 = \ell([c_1, \ldots, c_{n-1}])$ which does not belong to $\mathcal{RA}_{(r,n)}$. Since $[0, \ldots, 0] \in \mathcal{RA}_{(r,n)}$, we may assume $c_1 \neq 0$ after applying the actions of $s_1, \ldots, s_{n-2}$ if necessary. Note that $\ell(\pi^{-1} \cdot [c_1, \ldots, c_{n-1}]) = \ell([c_2, \ldots, c_{n-1}, c_1 - 1]) = \ell_0 - 1 < \ell_0$ and $0 \leq c_1 - 1 < r - 1$, we have $[c_2, \ldots, c_{n-1}, c_1 - 1] \in \mathcal{RA}_{(r,n)}$ by the minimality of ℓ_0. Thus, $[c_1, \ldots, c_{n-1}] = \pi \cdot [c_2, \ldots, c_{n-1}, c_1 - 1] \in \mathcal{RA}_{(r,n)}$, which is absurd. $\square$

EXAMPLE 2.7. *In case of $r = 2$ and $n = 5$, there are exactly $2^4 (= 16)$ $(2,5)$-restricted arrangements of $\mathbb{Z}$-colored beads:*

$$\begin{gathered}
[0,0,0,0], [1,0,0,0], [0,1,0,0], [0,0,1,0], \\
[0,0,0,1], [1,1,0,0], [1,0,1,0], [1,0,0,1], \\
[0,1,0,1], [0,0,1,1], [0,1,1,0], [1,1,1,0], \\
[1,1,0,1], [1,0,1,1], [0,1,1,1], [1,1,1,1].
\end{gathered}$$

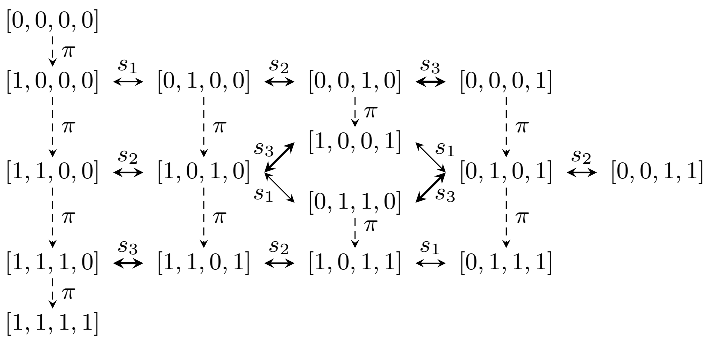

FIGURE 3. $(2,5)$-restricted arrangements of $\mathbb{Z}$-colored beads

3. Rational Cherednik algebra

In this section, we briefly review the definition of the rational Cherednik algebra $\mathcal{H}_n(c)$ of type $\mathfrak{sl}_n$, and give several facts about its representations in [1], [3], and [5].

Definition 3.1. Let $c \in \mathbb{C} \setminus \{0\}$. The **rational cherednik algebra** (abbreviated **RCA**) of type $\mathfrak{gl}_n$ associated with c, denoted by $\overline{\mathcal{H}}_n(c)$, is the unital associative $\mathbb{C}$-algebra defined by the following presentation.

$$\begin{array}{rlll}
\text{generators:} & \mathsf{x}_1, \dots, \mathsf{x}_n,\ s_1, \dots, s_{n-1},\ \mathsf{y}_1, \dots, \mathsf{y}_n & & \\
\text{relations:} & s_i^2 = 1 & & (1 \le i \le n-1), \\
& s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & & (1 \le i < n-1), \\
& s_i s_j = s_j s_i & & (|i-j| > 1), \\
& \mathsf{x}_i \mathsf{x}_j = \mathsf{x}_j \mathsf{x}_i, & \mathsf{y}_i \mathsf{y}_j = \mathsf{y}_j \mathsf{y}_i & (1 \le i, j \le n), \\
& s_i \mathsf{x}_i = \mathsf{x}_{i+1} s_i, & s_i \mathsf{y}_i = \mathsf{y}_{i+1} s_i & (1 \le i \le n-1), \\
& s_i \mathsf{x}_j = \mathsf{x}_j s_i, & s_i \mathsf{y}_j = \mathsf{y}_j s_i & (j \ne i, i+1), \\
& \mathsf{y}_i \mathsf{x}_j - \mathsf{x}_j \mathsf{y}_i = \begin{cases} c s_{ij} & \text{if } i \ne j, \\ 1 - c \sum_{k \ne i} s_{i,k} & \text{if } i = j, \end{cases} & &
\end{array}$$

where $s_{i,i+1} = s_{i+1,i} = s_i$ and $s_{i,j} = \begin{cases} s_i s_{i+1} \dots s_{j-1} \dots s_{i+1} s_i & \text{if } i < j, \\ s_i s_{i-1} \dots s_{j+1} \dots s_{i-1} s_i & \text{if } i > j \end{cases}$ for $i, j \in \{1, \dots, n-1\}$ with $i \ne j$.

The subalgebra $\mathcal{H}_n(c)$ of $\overline{\mathcal{H}}_n(c)$ generated by $x_2, \dots, x_n$, $s_1, \dots, s_{n-1}$, and $y_2, \dots, y_n$ is called the RCA of type $\mathfrak{sl}_n$ associated with c, where

$$x_i = \mathsf{x}_i - \mathsf{x}_1, \quad y_i = \mathsf{y}_i - \tfrac{1}{n}(\mathsf{y}_1 + \cdots + \mathsf{y}_n)$$

for $i \in \{2, \dots, n\}$.

Remark 3.2. It is known that the subalgebra $\mathfrak{X}$ (resp. $\mathfrak{Y}$) generated by $x_2, \dots, x_n$ (resp. $y_2, \dots, y_n$) is isomorphic to the polynomial algebra $\mathbb{C}[T_2, \dots, T_n]$ via the isomorphism given by $x_i \mapsto T_i$ (resp. $y_i \mapsto T_i$), and that the subalgebra generated by $s_1, \dots, s_{n-1}$ is isomorphic to the group algebra $\mathbb{C}\mathfrak{S}_n$. Also, it is well known as the PBW theorem for rational Cherednik algebras that the elements

$$x_2^{\alpha_2} \cdots x_n^{\alpha_n} w y_2^{\beta_2} \cdots y_n^{\beta_n} \quad (\alpha_i \ge 0, \beta_j \ge 0, \text{ and } w \in \mathfrak{S}_n)$$

form a basis of the vector space $\mathcal{H}_n(c)$ over $\mathbb{C}$, where the subalgebra generated by $s_1, \dots, s_{n-1}$ is identified with $\mathbb{C}\mathfrak{S}_n$. See [3] for details.

Let $\mathfrak{B}$ be the subalgebra generated by $\mathfrak{Y}$ and $\mathbb{C}\mathfrak{S}_n$, and consider a one-dimensional $\mathfrak{B}$-module $\mathbb{C}$ on which the action of $\mathfrak{B}$ is defined by:

$$y_i \cdot 1 = 0 \ (i \in \{2, \dots, n\}), \qquad w \cdot 1 = 1 (w \in \mathfrak{S}_n).$$

By induction, we can construct an infinite dimensional module over $\mathcal{H}_n(c)$:

$$V_n(c) = \mathrm{Ind}_{\mathfrak{B}}^{\mathcal{H}_n(c)} \mathbb{C} = \mathcal{H}_n(c) \otimes_{\mathfrak{B}} \mathbb{C},$$

which is isomorphic to $\mathfrak{X} \otimes \mathbb{C}$ as a vector space over $\mathbb{C}$ due to the PBW theorem. It is the fact that the module $V_n(c)$ has a unique maximal proper submodule $M_n(c)$, from which it follows that the quotient $L_n(c) = V_n(c)/M_n(c)$ is a unique simple quotient module. See [3] for details.

It is known that if $c = \frac{r}{n}$ with $\gcd(r,n) = 1$, then $L_n(\frac{r}{n})$ is a finite dimensional $\mathcal{H}_n(\frac{r}{n})$-module of dimension r^{n-1}. See [1] for more details. In the next section, we will prove that the finite dimensional $\mathcal{H}_n(\frac{r}{n})$-module $L_n(\frac{r}{n})$ has a basis consisting of simultaneous eigenvectors for the actions of the modified Dunkl-Cherednik elements $g_2, \dots, g_n$ which are defined below.

Definition 3.3.

1. For each $i \in \{2, 3, \dots, n\}$, we define the **modified Dunkl-Cherednik element** g_i to be the element

$$g_i = -\frac{1}{c} x_i y_i + \sum_{j=1}^{i-1} s_{ji} \in \mathcal{H}_n(c).$$

2. For each $i \in \{2, 3, \cdots, n-1\}$, we defined the **intertwining element** φ_i to be the element

$$\varphi_i = 1 + s_i(g_i - g_{i+1}) \in \mathcal{H}_n(c).$$

3. The element $r = x_2 s_2 s_3 \cdots s_{n-1} \in \mathcal{H}_n(c)$ is called the **raising element**.
4. The element $l = -\frac{1}{c} s_{n-1} \cdots s_3 s_2 (s_1 y_2 - y_2 g_2)$ is called the **lowering element**.

Proposition 3.4 ([5]). *Let g_i, φ_i, r, l be as defined above. Then the following relations hold.*

1. $g_i g_j = g_j g_i$ *for all* $i, j \in \{2, 3, \cdots, n\}$
2. $g_j \varphi_i = \varphi_i g_{s_i(j)}$ *for* $i \in \{2, 3, \dots, n-1\}$ *and* $j \in \{2, 3, \cdots, n\}$.
3. $\varphi_i^2 = (1 + g_i - g_{i+1})(1 - g_i + g_{i+1})$ *for all* $i \in \{2, 3, \dots, n-1\}$.
4. $g_2 r = r(g_n - \frac{1}{c})$ *and* $g_i r = r g_{i-1}$ *for* $i \in \{3, \cdots, n\}$.
5. $rl = (1 + g_i)(1 - g_1)$ *and* $lr = (1 + g_{n-1} - \frac{1}{c})(1 - g_{n-1} + \frac{1}{c})$.

Let $\mathcal{G}$ denote the commutative subalgebra of $\mathcal{H}_n(c)$ generated by the elements $g_2, g_3, \dots, g_n$. Given a module M over $\mathcal{H}_n(c)$, we consider simultaneous eigenvectors in M for the actions of all elements of $\mathcal{G}$.

Definition 3.5. Let M be an $\mathcal{H}_n(c)$-module. An $(n-1)$-tuple $\mathbf{a} = (a_2, a_3, \cdots, a_n) \in \mathbb{C}^{n-1}$ is called a **weight** of M if there exists a nonzero vector $v \in M$ such that $g_i \cdot v = a_i v$ for all $i \in \{2, 3, \dots, n\}$, and such a vector v is called a **weight vector** of weight $\mathbf{a}$. Also, for a weight $\mathbf{a} \in \mathbb{C}^{n-1}$ of M, the subspace

$$M_{\mathbf{a}} = \{v \in M \mid g_i \cdot v = a_i v \text{ for all } i\}$$

is called the **weight space** with respect to the weight $\mathbf{a}$.

The following result can be obtained directly from Proposition 3.4.

Proposition 3.6 ([5] Theorem 4.3). *Let M be an $\mathcal{H}_n(c)$-module.*

1. *If a vector $v \in M$ is a weight vector of weight $(a_2, \cdots, a_n)$ with $a_i - a_{i+1} \neq \pm 1$ for some $2 \leq i \leq n-1$, then $\varphi_i \cdot v$ is a weight vector of weight $(a_2, \cdots, a_{i+1}, a_i, \cdots, a_n)$.*
2. *If a vector $v \in M$ is a weight vector of weight $(a_2, \cdots, a_n)$ with $a_n - \frac{1}{c} \neq \pm 1$, then $r \cdot v$ is a weight vector of weight $(a_n - \frac{1}{c}, a_2, \cdots, a_{n-1})$.*

Definition 3.7. An $\mathcal{H}_n(c)$-module M is said to be g**-semisimple** if M has a basis consisting of weight vectors; in other words, M can be decomposed into a direct sum of weight spaces:

$$M = \bigoplus_{\mathbf{a}} M_{\mathbf{a}},$$

where the direct sum is taken over all weights $\mathbf{a}$ of M.

4. Main theorem: g-semisimplicity

Let $\mathbf{a} = [a_1, \ldots, a_{n-1}]$ be an (r, n)-restricted arrangement of $\mathbb{Z}$-colored beads. Consider the bijection $f_{\mathbf{a}} \colon \{1, 2, \cdots, n-1\} \to \{1, 2, \cdots, n-1\}$ satisfying the following conditions:

1. if $a_i < a_j$, then $f_{\mathbf{a}}(i) < f_{\mathbf{a}}(j)$, and
2. if $a_i = a_j$ and $i < j$, then $f_{\mathbf{a}}(i) < f_{\mathbf{a}}(j)$.

We define the map $\mathrm{wt} \colon \mathcal{RA}_{(r,n)} \to \mathbb{C}^{n-1}$ by

$$\mathrm{wt}(\mathbf{a}) = \left(f_{\mathbf{a}}(1) - a_1 \cdot \frac{n}{r}, f_{\mathbf{a}}(2) - a_2 \cdot \frac{n}{r}, \cdots, f_{\mathbf{a}}(n-1) - a_{n-1} \cdot \frac{n}{r}\right).$$

Example 4.1. *In the case of $r = 2$ and $n = 5$, let $\mathbf{a} = [0, 1, 1, 0]$. Then we have $f_{\mathbf{a}}(1) = 1$, $f_{\mathbf{a}}(2) = 3$, $f_{\mathbf{a}}(3) = 4$, and $f_{\mathbf{a}}(4) = 2$. Also, we directly see* $\mathrm{wt}(\mathbf{a}) = (0, \frac{1}{2}, \frac{3}{2}, 1)$.

Lemma 4.2. *The map $\mathrm{wt} \colon \mathcal{RA}_{(r,n)} \to \mathbb{C}^{n-1}$, as defined above, is injective.*

Proof. In order to show that the map wt is injective, assume that $\mathrm{wt}(\mathbf{a}) = \mathrm{wt}(\mathbf{b})$ for $\mathbf{a} = [a_1, \ldots, a_{n-1}], \mathbf{b} = [b_1, \ldots, b_{n-1}] \in \mathcal{RA}_{(r,n)}$. Then for each $1 \leq i \leq n-1$, we have $f_{\mathbf{a}}(i) - a_i \cdot \frac{n}{r} = f_{\mathbf{b}}(i) - b_i \cdot \frac{n}{r}$, or equivalently, $f_{\mathbf{a}}(i) - f_{\mathbf{b}}(i) = (a_i - b_i) \cdot \frac{n}{r}$. Hence, $(a_i - b_i) \cdot \frac{n}{r}$ is an integer. Since $-r+1 \leq a_i - b_i \leq r-1$ and $\gcd(n,r) = 1$, we have $a_i = b_i$ for each $1 \leq i \leq n-1$. □

LEMMA 4.3. *Let $\mathbf{a} = [a_1, \ldots, a_{n-1}] \in \mathcal{RA}_{(r,n)}$, and let M an $\mathcal{H}_n(\frac{r}{n})$-module.*

1. *Assume that s_i is an (r,n)-admissible shift of type I for $\mathbf{a}$, and that a nonzero vector $v \in M$ is a weight vector of weight $\mathrm{wt}(\mathbf{a})$. Then $\varphi_{i+1} \cdot v$ is a weight vector of weight $\mathrm{wt}(s_i \cdot \mathbf{a})$.*
2. *Assume that π is an (r,n)-admissible shift of type II for $\mathbf{a}$, and that a nonzero vector $v \in M$ is a weight vector of weight $\mathrm{wt}(\mathbf{a})$. Then $r \cdot v$ is a weight vector of weight $\mathrm{wt}(\pi \cdot \mathbf{a})$.*

Proof. We first assume that s_i is an (r,n)-admissible shift of type I for $\mathbf{a} = [a_1, \cdots, a_{n-1}]$, i.e., $a_i \neq a_{i+1}$. Since $0 \leq a_i, a_{i+1} \leq r-1$ and $\gcd(r,n) = 1$, $(a_i - a_{i+1}) \cdot \frac{n}{r}$ is not an integer, from which it follows that $(f_{\mathbf{a}}(i) - a_i \cdot \frac{n}{r}) - (f_{\mathbf{a}}(i+1) - a_{i+1} \cdot \frac{n}{r}) \neq \pm 1$. Hence, by Proposition 3.6, for a weight vector $v \in M$ of weight $\mathrm{wt}(\mathbf{a})$, $\varphi_{i+1} \cdot v$ is a weight vector of weight $\mathrm{wt}(s_i \cdot \mathbf{a})$.

We now assume that π is an (r,n)-admissible shift of type II for $\mathbf{a}$, i.e., $a_{n-1} \neq r-1$. Then $(a_{n-1}+1) \cdot \frac{n}{r}$ is not an integer because $\gcd(r,n) = 1$. Thus, we obtain $\left(f_{\mathbf{a}}(n-1) - a_{n-1} \cdot \frac{n}{r}\right) - \frac{n}{r} \neq \pm 1$. It follows that $r \cdot v$ is a weight vector of weight

$$\left(f_{\mathbf{a}}(n-1) - (a_{n-1}+1) \cdot \frac{n}{r}, f_{\mathbf{a}}(1) - a_1 \cdot \frac{n}{r}, \ldots f_{\mathbf{a}}(n-2) - a_{n-2} \cdot \frac{n}{r}\right),$$

which is equal to $\mathrm{wt}(\pi \cdot \mathbf{a})$ by Proposition 3.6. □

THEOREM 4.4. *The finite dimensional representation $L_n(\frac{r}{n})$ of the algebra $\mathcal{H}_n(\frac{r}{n})$ is g-semisimple. Moreover, each weight space of $L_n(\frac{r}{n})$ is one-dimensional.*

Proof. By construction, $1 \otimes 1 \in V_n(\frac{r}{n})$ is a weight vector of weight $(1, 2, \ldots, n-1)$. Since there is no a proper submodule of $V_n(\frac{r}{n})$ containing $1 \otimes 1$, the maximal proper submodule $M_n(\frac{r}{n})$ does not contain $1 \otimes 1$. Hence $\mathrm{wt}([0, 0, \ldots, 0]) = (1, 2, \cdots, n-1)$ is a weight of the quotient $L_n(\frac{r}{n})$. By applying Lemma 4.3 inductively, we directly see that $\mathrm{wt}(\mathbf{a})$ for every $\mathbf{a} \in \mathcal{RA}_{(r,n)}$ is a weight of $L_n(\frac{r}{n})$. Thus, there are r^{n-1} weight vectors in $L_n(\frac{r}{n})$ of distinct weights because the map wt is injective. Since the weight vecotrs of r^{n-1} distinct weights are linearly independent and $\dim L_n(\frac{r}{n}) = r^{n-1}$, the representation $L_n(\frac{r}{n})$ has a basis consisting of weight vectors, and each weight space is one-dimensional. $\square$

Example 4.5. *Let $r = 2$ and $n = 5$. The finite dimensional representation $L_5(\frac{2}{5})$ has exactly 16 weights. Figure 4 describes all weights of $L_5(\frac{2}{5})$ and local actions of the intertwining elements φ_i and the raising element r among the corresponding weight spaces.*

$$\begin{tikzcd}
(1,2,3,4) \arrow[d, dashed, "r"] & & & & \\
(\frac{3}{2},1,2,3) \arrow[r, leftrightarrow, "\varphi_2"] \arrow[dd, dashed, "r"] & (1,\frac{3}{2},2,3) \arrow[r, leftrightarrow, "\varphi_3"] \arrow[dd, dashed, "r"] & (1,2,\frac{3}{2},3) \arrow[r, leftrightarrow, "\varphi_4"] \arrow[d, dashed, "r"] & (1,2,3,\frac{3}{2}) \arrow[dd, dashed, "r"] & \\
& & (\frac{1}{2},1,2,\frac{3}{2}) \arrow[dr, leftrightarrow, "\varphi_2"] & & \\
(\frac{1}{2},\frac{3}{2},1,2) \arrow[r, leftrightarrow, "\varphi_3"] \arrow[dd, dashed, "r"] & (\frac{1}{2},1,\frac{3}{2},2) \arrow[ur, leftrightarrow, "\varphi_4"] \arrow[dr, leftrightarrow, "\varphi_2"'] \arrow[dd, dashed, "r"] & & (1,\frac{1}{2},2,\frac{3}{2}) \arrow[r, leftrightarrow, "\varphi_3"] \arrow[dd, dashed, "r"] & (1,2,\frac{1}{2},\frac{3}{2}) \\
& & (1,\frac{1}{2},\frac{3}{2},2) \arrow[ur, leftrightarrow, "\varphi_4"'] \arrow[d, dashed, "r"] & & \\
(-\frac{1}{2},\frac{1}{2},\frac{3}{2},1) \arrow[r, leftrightarrow, "\varphi_4"] \arrow[d, dashed, "r"] & (-\frac{1}{2},\frac{1}{2},1,\frac{3}{2}) \arrow[r, leftrightarrow, "\varphi_3"] & (-\frac{1}{2},1,\frac{1}{2},\frac{3}{2}) \arrow[r, leftrightarrow, "\varphi_2"] & (1,-\frac{1}{2},\frac{1}{2},\frac{3}{2}) & \\
(-\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2}) & & & &
\end{tikzcd}$$

Figure 4. Decomposition of $L_5(\frac{2}{5})$ into a direct sum of its weight spaces

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