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SEQUENTIAL PROPERTIES OF SUMS OVER STIRLING-PASCAL MATRIX

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Abstract. With the Stirling matrix S of the second kind and the Pascal matrix T, we study recurrence rules and sequences of certain sums over the matrix ST^k. We find a matrix E satisfying $ST = ES$ and interrelations of S and the Stirling matrix of the first kind.

1. Introduction

The Stirling number $s_{i,j}$ counts partitions of an i elements set into j subsets, and the Stirling matrix of the second kind $S = [s_{i,j}]$ $(i, j \geq 0)$ satisfies a recurrence rule $s_{i+1,j} = [s_{i,j-1}, s_{i,j}][1, j]^{tr}$. The sum of entries over i^{th}row of S is called the Bell number B_i that satisfy

$$s_{i,j} = \sum_{t=j-1}^{i-1} \binom{i-1}{t} s_{t,j-1} \text{ and } B_i = \sum_{t=0}^{i-1} \binom{i-1}{t} B_t \quad \text{(see [3], [7])}. \tag{1}$$

A Stirling-Pascal matrix T^kS with Pascal matrix T was studied in [2]. In fact $T^kS = S^{[k]} = [s^{[k]}_{i,j}]$ and its i^{th} row sum $B^{[k]}_i$ satisfy

$$s^{[k]}_{i+1,j} = [s^{[k]}_{i,j-1}, s^{[k]}_{i,j}][1, j+k]^{tr}, \;\; B^{[k+1]}_i = B^{[k]}_{i+1} - kB^{[k]}_i \text{ and}$$
$$[B^{[1]}_i, B^{[2]}_{i-1}, \cdots, B^{[i+1]}_0][[0]_{k-i}, k, k+1, \cdots, i+1]^{tr} = B^{[k]}_{i+1-k} - 1. \tag{2}$$

In this work we investigate the matrix ST^k $(k \geq 0)$ and its i^{th}row sum $B^{\langle k\rangle}_i$. Let $ST^k = S^{\langle k\rangle} = [s^{\langle k\rangle}_{i,j}]$. We prove recurrence rules of $s^{\langle k\rangle}_{i,j}$ and $B^{\langle k\rangle}_i$ in Theorem 4 and 7. We also find a matrix E satisfying $ST = ES$ that give relations of S and the Stirling matrix of the first kind.

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Throughout the work, $r_i(M)$ and $c_j(M)$ mean the i^{th}row and j^{th}column of a matrix M. Let $[0]_n$ denote either $[\underbrace{0,\cdots,0}_{n}]$ or $\left.\begin{bmatrix}0\\ \cdots\\ 0\end{bmatrix}\right\}_n$ depending on situations. $[[0]_n; r_i(M)]$ is a row matrix $[0]_n$ followed by $r_i(M)$, and $[r_i(M);[0]_n]$ is $r_i(M)$ followed by $[0]_n$. Let $\mathrm{di}[a,b,\cdots]$ be a diagonal matrix having diagonal $a,b,\cdots$, in particular, $\mathrm{di}[1,a,a^2,\cdots]=\mathrm{di}[a^i]_{i\ge 0}$.

2. $ST^k = S^{\langle k\rangle}$

Let $S=[s_{i,j}]$ $(i,j\ge 0)$ be the Stirling matrix, T be the Pascal matrix and $ST^k=S^{\langle k\rangle}=[s^{\langle k\rangle}_{i,j}]$ for $k\ge 0$. We begin to have the i^{th}row $r_i(T^k)$ and j^{th}column $c_j(T^k)$ of T^k, and the i^{th}row $r_i(S)$ of S.

Lemma 1. *$r_i(T^k)=r_i(T)di[k^i,\cdots,k,1]$, $c_j(T^k)=di[[0]_j,1,k,k^2,\cdots]$ $c_j(T)$, and $r_{i+1}(S)=[0;r_i(S)]+[r_i(S);0]di[0,1,\cdots,i+1]$ for $i,j\ge 0$.*

Proof. Since T^k is the arithmetic table of $(kx+1)^n$, $c_j(T^k)$ consists of the coefficients of x^j in the expansion of $(kx+1)^n$ for $n\ge 0$, so

$$c_j(T^k)=[[0]_j,\tbinom{j}{j},k\tbinom{j+1}{j},k^2\tbinom{j+2}{j},\cdots]^{tr}=\mathrm{di}[[0]_j,1,k,k^2,\cdots]c_j(T).$$

Moreover the Stirling recurrence of S yields

$$\begin{aligned}r_{i+1}(S)&=[0,s_{i,0},s_{i,1},\cdots,s_{i,i-1},s_{i,i}]+[s_{i,0},s_{i,1},2s_{i,2},\cdots,is_{i,i},0]\\&=[0;r_i(S)]+[r_i(S);0]\mathrm{di}[0,1,\cdots,i+1].\end{aligned}$$ □

Theorem 2. *Let $Y_j=\begin{bmatrix}1&&&\\ j&1&&\\ j^2&j&1&\\ j^3&j^2&j&1\\ \cdots&&&\end{bmatrix}$ and $c_j^*(S)$ be the nonzero part of $c_j(S)$ starting with $s_{j,j}$. Then $c_j^*(S)=Y_jc_{j-1}^*(S)$ and $c_j^*(S)=Y_j\cdots Y_2c_1^*(S)$.*

Proof. Clearly $c_j^*(S)=\begin{bmatrix}s_{j,j}\\ s_{j+1,j}\\ s_{j+2,j}\\ \cdots\end{bmatrix}=\begin{bmatrix}s_{j-1,j-1}\\ s_{j,j-1}\\ s_{j+1,j-1}\\ \cdots\end{bmatrix}+j\begin{bmatrix}0\\ s_{j,j}\\ s_{j+1,j}\\ \cdots\end{bmatrix}$, so we have

$$c_j^*(S)=\begin{bmatrix}s_{j-1,j-1}\\ s_{j,j-1}\\ s_{j+1,j-1}\\ s_{j+2,j-1}\\ \cdots\end{bmatrix}+j\begin{bmatrix}0\\ s_{j-1,j-1}\\ s_{j,j-1}\\ s_{j+1,j-1}\\ \cdots\end{bmatrix}+j^2\begin{bmatrix}[0]_2\\ s_{j-1,j-1}\\ s_{j,j-1}\\ s_{j+1,j-1}\\ \cdots\end{bmatrix}+j^3\begin{bmatrix}[0]_3\\ s_{j-1,j-1}\\ s_{j,j-1}\\ s_{j+1,j-1}\\ \cdots\end{bmatrix}+j^4\begin{bmatrix}[0]_4\\ s_{j,j}\\ s_{j+1,j}\\ s_{j+2,j}\\ \cdots\end{bmatrix}$$

$$=\cdots=\begin{bmatrix}1&0&0&0\\ j&1&0&0\\ j^2&j&1&0\\ j^3&j^2&j&1\\ \cdots\end{bmatrix}\begin{bmatrix}s_{j-1,j-1}\\ s_{j,j-1}\\ s_{j+1,j-1}\\ s_{j+2,j-1}\\ \cdots\end{bmatrix}=Y_j\begin{bmatrix}s_{j-1,j-1}\\ s_{j,j-1}\\ s_{j+1,j-1}\\ s_{j+2,j-1}\\ \cdots\end{bmatrix}=Y_jc_{j-1}^*(S).$$ □

Moreover $Y_jY_{j+1} = Y_{j+1}Y_j = [(j+1)^n - j^n]$, in fact $c_3^*(S) = Y_3Y_2c_1^*(S) =$
$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3^2-2^2 & 1 & 0 & 0 \\ 3^3-2^3 & 3^2-2^2 & 1 & 0 \\ 3^4-2^4 & \cdots & & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ \cdots \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 25 \\ 90 \\ \cdots \end{bmatrix}$, so $c_3(S) = [0,0,0,1,6,25,90,\cdots]^{tr}$.

THEOREM 3. *Let $T^2S = Z = [z_{i,j}]$ and $c_j^*(Z)$ be the j^{th}column from $z_{j,j}$. Then* $Y_j \cdots Y_3Y_2 = \begin{bmatrix} c_{j-2}^*(Z) & 0 & 0 \\ & c_{j-2}^*(Z) & 0 \\ & & c_{j-2}^*(Z) \\ & & \cdots \end{bmatrix}$ *and* $c_j^*(S) =$

$$\begin{bmatrix} z_{j-3,j-3} \\ \sum_{t=0}^{1} z_{j-3+t,j-3} \\ \cdots \\ \sum_{t=0}^{i-j} z_{j-3+t,j-3} \end{bmatrix} + \cdots + j^{i-j-1} \begin{bmatrix} [0]_{i-j-1} \\ z_{j-3,j-3} \\ \sum_{t=0}^{1} z_{j-3+t,j-3} \end{bmatrix} + j^{i-j} \begin{bmatrix} [0]_{i-j} \\ z_{j-3,j-3} \end{bmatrix}.$$

Proof. Observe $Z = T^2S = \begin{bmatrix} 1 \\ 2 & 1 \\ 4 & 5 & 1 \\ 8 & 19 & 9 & 1 \\ 16 & 65 & 55 & 14 \end{bmatrix}$, and $Y_2 = \begin{bmatrix} 1 \\ 2 & 1 \\ 4 & 2 & 1 \\ 8 & 4 & 2 & 1 \end{bmatrix}$, $Y_3Y_2 =$
$\begin{bmatrix} 1 \\ 5 & 1 \\ 19 & 5 & 1 \\ 65 & 19 & 5 & 1 \end{bmatrix}$, $Y_4Y_3Y_2 = \begin{bmatrix} 1 \\ 9 & 1 \\ 55 & 9 & 1 \\ 285 & 55 & 9 & 1 \end{bmatrix}$ are $\begin{bmatrix} c_{j-2}^*(Z) & 0 & 0 \\ & c_{j-2}^*(Z) & 0 \\ & & c_{j-2}^*(Z) \\ & & \cdots \end{bmatrix}$ for $j =$ $2,3,4$. Assume $Y_{j-1} \cdots Y_2 = \begin{bmatrix} c_{j-3}^*(Z) & 0 & 0 \\ & c_{j-3}^*(Z) & 0 \\ & & c_{j-3}^*(Z) \\ & & \cdots \end{bmatrix}$ for some j. Then

$$c_0(Y_jY_{j-1}\cdots Y_2) = Y_jc_0(Y_{j-1}\cdots Y_2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ j & 1 & 0 & 0 \\ j^2 & j & 1 & 0 \\ j^3 & j^2 & j & 1 \\ \cdots \end{bmatrix} c_{j-3}^*(Z)$$

$$= \begin{bmatrix} z_{j-3,j-3} \\ z_{j-2,j-3} + jz_{j-3,j-3} \\ z_{j-1,j-3} + jz_{j-2,j-3} + j^2z_{j-3,j-3} \\ \cdots \end{bmatrix} = \begin{bmatrix} z_{j-2,j-2} \\ z_{j-1,j-2} \\ z_{j,j-2} \\ \cdots \end{bmatrix} = c_{j-2}^*(Z),$$

because $z_{j-2,j-3} + jz_{j-2,j-2} = z_{j-1,j-2}$ and $z_{j-1,j-3} + jz_{j-2,j-3} + j^2z_{j-2,j-2} = z_{j-1,j-3} + j(z_{j-2,j-3} + jz_{j-2,j-2}) = z_{j,j-2}$ etc. by the recurrence (2).

Moreover $c_1(Y_jY_{j-1}\cdots Y_2) = Y_jc_1(Y_{j-1}\cdots Y_2) = Y_j[0; c_{j-3}^*(Z)] = [0; c_{j-2}^*(Z)]$, and $c_2(Y_jY_{j-1}\cdots Y_2) = [0,0; c_{j-2}^*(Z)]$, and so on. Thus

$$Y_j \cdots Y_3Y_2 = \begin{bmatrix} c_{j-2}^*(Z) & 0 & 0 \\ & c_{j-2}^*(Z) & 0 \\ & & c_{j-2}^*(Z) \\ & & \cdots \end{bmatrix}.$$

Furthermore $c_j^*(S) = Y_j \cdots Y_2c_1^*(S)$ with $c_1^*(S) = [1,\cdots,1]^{tr}$ implies

$$
\begin{aligned}
c_j^*(S) &= \begin{bmatrix} z_{j-2,j-2} \\ z_{j-2,j-2} + z_{j-1,j-2} \\ z_{j-2,j-2} + z_{j-1,j-2} + z_{j,j-2} \\ z_{j-2,j-2} + z_{j-1,j-2} + z_{j,j-2} + z_{j+1,j-2} \\ \cdots \end{bmatrix} \\
&= \begin{bmatrix} z_{j-3,j-3} \\ \sum_{t=0}^{1} z_{j-3+t,j-3} \\ \cdots \\ \sum_{t=0}^{i-j} z_{j-3+t,j-3} \end{bmatrix} + j \begin{bmatrix} 0 \\ z_{j-2,j-2} \\ \sum_{t=0}^{1} z_{j-2+t,j-2} \\ \cdots \\ \sum_{t=0}^{i-j-1} z_{j-2+t,j-2} \end{bmatrix}.
\end{aligned}
$$

Continuing this process, we therefore have

$$
\begin{aligned}
c_j^*(S) &= \begin{bmatrix} z_{j-3,j-3} \\ \sum_{t=0}^{1} z_{j-3+t,j-3} \\ \cdots \\ \sum_{t=0}^{i-j} z_{j-3+t,j-3} \end{bmatrix} + j \begin{bmatrix} 0 \\ z_{j-3,j-3} \\ \sum_{t=0}^{1} z_{j-3+t,j-3} \\ \cdots \\ \sum_{t=0}^{i-j-1} z_{j-3+t,j-3} \end{bmatrix} + j^2 \begin{bmatrix} 0 \\ 0 \\ z_{j-2,j-2} \\ \sum_{t=0}^{1} z_{j-2+t,j-2} \\ \cdots \\ \sum_{t=0}^{i-j-2} z_{j-2+t,j-2} \end{bmatrix} \\
&= \begin{bmatrix} z_{j-3,j-3} \\ \sum_{t=0}^{1} z_{j-3+t,j-3} \\ \cdots \\ \sum_{t=0}^{i-j} z_{j-3+t,j-3} \end{bmatrix} + j \begin{bmatrix} 0 \\ z_{j-3,j-3} \\ \sum_{t=0}^{1} z_{j-3+t,j-3} \\ \cdots \\ \sum_{t=0}^{i-j-1} z_{j-3+t,j-3} \end{bmatrix} + \cdots + j^{i-j} \begin{bmatrix} [0]_{i-j} \\ z_{j-3,j-3} \end{bmatrix}. \qquad \square
\end{aligned}
$$

The next theorem shows a recurrence of $S^{\langle k \rangle} = ST^k = [s_{i,j}^{\langle k \rangle}]$ that can be compared to (2) of $T^k S = [s_{i,j}^{[k]}]$.

THEOREM 4. *$s_{i+1,j}^{\langle k \rangle} = [s_{i,j-1}^{\langle k \rangle}, s_{i,j}^{\langle k \rangle}, s_{i,j+1}^{\langle k \rangle}][1, j+k, (j+1)k]^{tr}$ for all i, j.*

Proof. Over $S^{\langle 1 \rangle}$, observe $[5, 10, 6][1, 2, 2]^{tr} = 37$, $[37, 31, 10][1, 3, 3]^{tr} = 16$ and $[31, 10, 1][1, 4, 4]^{tr} = 75$. Since $s_{i,j}^{\langle k \rangle} = r_i(S) c_j(T^k)$,

$$
\begin{aligned}
& s_{i+1,j}^{\langle k \rangle} = r_{i+1}(S) c_j(T^k) \\
&= \big([0; r_i(S)] + [r_i(S); 0]\mathrm{di}[0, 1, \cdots, i+1]\big) \left[[0]_j, \tbinom{j}{j}, k\tbinom{j+1}{j}, \cdots, k^{i-j+1}\tbinom{i+1}{j}\right]^{tr} \\
&= [0; r_i(S)]\left[[0]_j, \tbinom{j}{j}, k\tbinom{j+1}{j}, k^2\tbinom{j+2}{j}, \cdots, k^{i-j+1}\tbinom{i+1}{j}\right]^{tr} \\
&\quad + [r_i(S); 0]\mathrm{di}[0, 1, \cdots, i+1]\left[[0]_j, \tbinom{j}{j}, k\tbinom{j+1}{j}, k^2\tbinom{j+2}{j}, \cdots, k^{i-j+1}\tbinom{i+1}{j}\right]^{tr} \\
&= [0; r_i(S)]\left[[0]_j, \tbinom{j}{j}, k\tbinom{j+1}{j}, k^2\tbinom{j+2}{j}, \cdots, k^{i-j+1}\tbinom{i+1}{j}\right]^{tr} \\
&\quad + [r_i(S); 0]\left[[0]_j, j\tbinom{j}{j}, (j+1)k\tbinom{j+1}{j}, (j+2)k^2\tbinom{j+2}{j}, \cdots, (i+1)k^{i-j+1}\tbinom{i+1}{j}\right]^{tr} \\
&= r_i(S)\Big[[0]_{j-1}, \tbinom{j}{j}, k\tbinom{j+1}{j} + j\tbinom{j}{j}, k^2\tbinom{j+2}{j} + (j+1)k\tbinom{j+1}{j}, \\
&\qquad k^3\tbinom{j+3}{j} + (j+2)k^2\tbinom{j+2}{j}, \cdots, k^{i-j+1}\tbinom{i+1}{j} + ik^{i-j}\tbinom{i}{j}\Big]^{tr} \qquad (3)
\end{aligned}
$$

by Lemma 1. On the other hand, Lemma 1 also implies

$$[s^{\langle k\rangle}_{i,j-1}, s^{\langle k\rangle}_{i,j}, s^{\langle k\rangle}_{i,j+1}]\begin{bmatrix}1\\ j+k\\ (j+1)k\end{bmatrix} = r_i(S)\big[c_{j-1}(T^k)|c_j(T^k)|c_{j+1}(T^k)\big]\begin{bmatrix}1\\ j+k\\ (j+1)k\end{bmatrix}$$

$$= r_i(S)\begin{bmatrix}[0]_{j-1}\\ \binom{j-1}{j-1}\\ k\binom{j}{j-1}+(j+k)\binom{j}{j}\\ k^2\binom{j+1}{j-1}+(j+k)k\binom{j+1}{j}+(j+1)k\binom{j+1}{j+1}\\ \cdots\\ k^{i-j+1}\binom{i}{j-1}+(j+k)k^{i-j}\binom{i}{j}+(j+1)k\cdot k^{i-j-1}\binom{i}{j+1}\end{bmatrix}.$$

We note some identities of binomial coefficients that $k\binom{j}{j-1}+(j+k)\binom{j}{j} = k(\binom{j}{j-1}+\binom{j}{j})+j\binom{j}{j} = k\binom{j+1}{j}+j\binom{j}{j}$, $k^2\binom{j+1}{j-1}+(j+k)k\binom{j+1}{j}+(j+1)k\binom{j+1}{j+1} = k^2\binom{j+2}{j}+k(j+1)\binom{j+1}{j}$, and $k^{i-j+1}\binom{i}{j-1}+(j+k)k^{i-j}\binom{i}{j}+(j+1)k\cdot k^{i-j-1}\binom{i}{j+1} = k^{i-j+1}\binom{i+1}{j}+k^{i-j}i\binom{i}{j}$. These identities together with (3) show

$$[s^{\langle k\rangle}_{i,j-1}, s^{\langle k\rangle}_{i,j}, s^{\langle k\rangle}_{i,j+1}]\begin{bmatrix}1\\ j+k\\ (j+1)k\end{bmatrix} = r_i(S)\begin{bmatrix}[0]_{j-1}\\ \binom{j}{j}\\ k\binom{j+1}{j}+j\binom{j}{j}\\ k^2\binom{j+2}{j}+(j+1)k\binom{j+1}{j}\\ \cdots\\ k^{i-j+1}\binom{i+1}{j}+ik^{i-j}\binom{i}{j}\end{bmatrix} = s^{\langle k\rangle}_{i+1,j}.$$

□

THEOREM 5. $[s^{\langle k\rangle}_{i+1,1},\cdots,s^{\langle k\rangle}_{i+1,i+1}] = r_i(S^{\langle k\rangle})\begin{bmatrix}1 & & & \\ k+1 & 1 & & \\ 2k & k+2 & & \\ & 3k & \ddots 1 & \\ & & ik & k+i\,1\end{bmatrix}.$

Proof. Since $s^{\langle k\rangle}_{3,1} = s^{\langle k\rangle}_{2,0}+(k+1)s^{\langle k\rangle}_{2,1}+2ks^{\langle k\rangle}_{2,2}$ and $s^{\langle k\rangle}_{3,2} = s^{\langle k\rangle}_{2,1}+(k+2)s^{\langle k\rangle}_{2,2}$, we have $[s^{\langle k\rangle}_{3,1}, s^{\langle k\rangle}_{3,2}, s^{\langle k\rangle}_{3,3}] = [s^{\langle k\rangle}_{2,0}, s^{\langle k\rangle}_{2,1}, s^{\langle k\rangle}_{2,2}]\begin{bmatrix}1 & 0 & 0\\ k+1 & 1 & 0\\ 2k & k+2 & 1\end{bmatrix}$. And $s^{\langle k\rangle}_{4,1} = s^{\langle k\rangle}_{3,0}+(k+1)s^{\langle k\rangle}_{3,1}+2ks^{\langle k\rangle}_{3,2}$, $s^{\langle k\rangle}_{4,2} = s^{\langle k\rangle}_{3,1}+(k+2)s^{\langle k\rangle}_{3,2}+3ks^{\langle k\rangle}_{3,3}$ and $s^{\langle k\rangle}_{4,3} = s^{\langle k\rangle}_{3,2}+(k+3)s^{\langle k\rangle}_{3,3}$ imply

$$[s^{\langle k\rangle}_{4,1}, s^{\langle k\rangle}_{4,2}, s^{\langle k\rangle}_{4,3}, s^{\langle k\rangle}_{4,4}] = [s^{\langle k\rangle}_{3,0}, s^{\langle k\rangle}_{3,1}, s^{\langle k\rangle}_{3,2}, s^{\langle k\rangle}_{3,3}]\begin{bmatrix}1 & & & \\ k+1 & 1 & & \\ 2k & k+2 & 1 & \\ 0 & 3k & k+3 & 1\end{bmatrix}.$$

Now generally, since $s^{\langle k\rangle}_{i+1,1} = s^{\langle k\rangle}_{i,0}+(k+1)s^{\langle k\rangle}_{i,1}+2ks^{\langle k\rangle}_{i,2}$, $s^{\langle k\rangle}_{i+1,2} = s^{\langle k\rangle}_{i,1}+(k+2)s^{\langle k\rangle}_{i,2}+3ks^{\langle k\rangle}_{i,3}$, and $s^{\langle k\rangle}_{i+1,t} = s^{\langle k\rangle}_{i,t-1}+(k+t)s^{\langle k\rangle}_{i,t}+(t+1)ks^{\langle k\rangle}_{i,t+1}$ for all $t < i$, we have

$$[s^{\langle k\rangle}_{i+1,1},\cdots,s^{\langle k\rangle}_{i+1,i+1}] = [s^{\langle k\rangle}_{i,0},\cdots,s^{\langle k\rangle}_{i,i}]\begin{bmatrix}1 & & & & \\ k+1 & 1 & & & \\ 2k & k+2 & 1 & & \\ 0 & 3k & k+3 & 1 & \\ 0 & 0 & 4k & \cdots & \\ 0 & 0 & 0 & ik & k+i\,1\end{bmatrix}.$$ □

We may refer [5] for the tri-diagonal matrix above. We note more identities for next use.

THEOREM 6. $s_{2,0}^{\langle k\rangle} = k(k+1)$, $s_{2,1}^{\langle k\rangle} = 2k+1$, $s_{3,0}^{\langle k\rangle} = k(k^2+3k+1)$, $s_{3,1}^{\langle k\rangle} = 3k^2+6k+1$, $s_{3,2}^{\langle k\rangle} = 3k+3$ *and* $s_{i+1,i}^{\langle k\rangle} = (i+1)(k+\frac{i}{2})$.

Proof. With $s_{i+1,i}^{\langle k\rangle} = s_{i,i-1}^{\langle k\rangle} + (i+k)$, we shall only show
$s_{i+1,i}^{\langle k\rangle} = s_{i-1,i-2}^{\langle k\rangle} + (i-1+k) + (i+k) = s_{i-1,i-2}^{\langle k\rangle} + 2k + 2i - 1$
$= s_{i-2,i-3}^{\langle k\rangle} + (i-2+k) + 2k + 2i - 1 = s_{i-2,i-3}^{\langle k\rangle} + 3k + 3i - (1+2) = \cdots$
$= s_{i-(i-1),i-i}^{\langle k\rangle} + ik + i^2 - (1+2+\cdots+(i-1)) = (i+1)(k+\frac{i}{2})$. $\square$

3. i^{th}row sum of ST^k

Let $B_i^{\langle k\rangle}$ be the i^{th}row sum of $ST^k = S^{\langle k\rangle}$ and $B^{\langle k\rangle} = \{B_i^{\langle k\rangle} | i \geq 0\}$. Let $\widetilde{B^{\langle *\rangle}} = [B^{\langle 0\rangle}|B^{\langle 1\rangle}|B^{\langle 2\rangle}|\cdots]$ be a lower triangular matrix placing $B^{\langle k\rangle}$ in each k^{th}column. Observe $B^{\langle 0\rangle} = \{1,1,2,5,15,52,\cdots\}$, $B^{\langle 1\rangle} = \{1,2,6,22,94,\cdots\}$ and $B^{\langle 2\rangle} = \{1,3,12,57,\ 309,\cdots\}$, so we have

$$\widetilde{B^{\langle *\rangle}} = [B^{[0]}|B^{[1]}|B^{[2]}|\cdots] = \begin{bmatrix} B_0^{\langle 0\rangle} & & & \\ B_1^{\langle 0\rangle} & B_0^{\langle 1\rangle} & & \\ B_2^{\langle 0\rangle} & B_1^{\langle 1\rangle} & B_0^{\langle 2\rangle} & \\ B_3^{\langle 0\rangle} & B_2^{\langle 1\rangle} & B_1^{\langle 2\rangle} & B_0^{\langle 3\rangle} \\ & \ddots & & \end{bmatrix} = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 2 & 1 & & & \\ 5 & 6 & 3 & 1 & & \\ 15 & 22 & 12 & 4 & 1 & \\ 52 & 94 & 57 & 20 & 5 & 1 \end{bmatrix}$$

For the matrix $\widetilde{B^{\langle *\rangle}}$, we may refer [1] and OEIS A189233. The next theorem provides relationships of $B_i^{\langle k\rangle}$ with either $B_{i+1}^{\langle k\rangle}$ or $B_i^{\langle k+1\rangle}$.

THEOREM 7. $B_{i+1}^{\langle k\rangle} = B_i^{\langle k\rangle} + r_i(S^{\langle k\rangle})[0,1,\cdots,i]^{tr} + r_i(S^{\langle k\rangle})k[1,2,\cdots,(i+1)]^{tr}$ *and* $B_i^{\langle k+1\rangle} = B_i^{\langle k\rangle} + r_i(S^{\langle k\rangle})[0,1,\cdots,2^i-1]^{tr} = r_i(S^{\langle k\rangle})[1,2,\cdots,2^i]^{tr}$. *And* $[B_0^{\langle k+1\rangle} - B_0^{\langle k\rangle},\cdots,B_i^{\langle k+1\rangle} - B_i^{\langle k\rangle}]^{tr} = S^{\langle k\rangle}[0,1,2^2-1,\cdots,2^i-1]^{tr}$.

Proof. The recurrence of $S^{\langle k\rangle} = ST^k$ in Theorem 4 shows
$B_{i+1}^{\langle k\rangle} - B_i^{\langle k\rangle} = (s_{i+1,0}^{\langle k\rangle} + s_{i+1,1}^{\langle k\rangle} + \cdots + s_{i+1,i+1}^{\langle k\rangle}) - (s_{i,0}^{\langle k\rangle} + s_{i,1}^{\langle k\rangle} + \cdots + s_{i,i}^{\langle k\rangle})$
$= s_{i,0}^{\langle k\rangle}k + s_{i,1}^{\langle k\rangle}(1+2k) + s_{i,2}^{\langle k\rangle}(2+3k) + \cdots + s_{i,i}^{\langle k\rangle}(i+(i+1)k)$
$= r_i(S^{\langle k\rangle})[0,1,\cdots,i]^{tr} + r_i(S^{\langle k\rangle})k[1,2,\cdots,(i+1)]^{tr}$.
And $S^{\langle k+1\rangle} = S^{\langle k\rangle}T$ implies $s_{i,j}^{\langle k+1\rangle} = r_i(S^{\langle k\rangle})c_j(T)$, so we have
$B_i^{\langle k+1\rangle} = r_i(S^{\langle k\rangle})(c_0(T) + c_1(T) + \cdots + c_i(T))$
$= r_i(S^{\langle k\rangle})\ T[1,1,\cdots,1]^{tr} = r_i(S^{\langle k\rangle})[1,2,2^2,\cdots,2^i]^{tr}$

$$= r_i(S^{\langle k\rangle})[1,\cdots,1]^{tr} + r_i(S^{\langle k\rangle})[0,1,2^2-1,\cdots,2^i-1]^{tr}$$
$$= B_i^{\langle k\rangle} + r_i(S^{\langle k\rangle})[0,1,2^2-1,\cdots,2^i-1]^{tr}.$$

Thus $B_0^{\langle k+1\rangle} - B_0^{\langle k\rangle} = r_0(S^{\langle k\rangle})[0]$, $B_1^{\langle k+1\rangle} - B_1^{\langle k\rangle} = r_1(S^{\langle k\rangle})[0,1]^{tr}$, $B_2^{\langle k+1\rangle} - B_2^{\langle k\rangle} = r_2(S^{\langle k\rangle})[0,1,3]^{tr}$ and $B_i^{\langle k+1\rangle} - B_i^{\langle k\rangle} = r_i(S^{\langle k\rangle})[0,1,\cdots,2^i-1]^{tr}$ yield $[B_0^{\langle k+1\rangle} - B_0^{\langle k\rangle},\cdots,B_i^{\langle k+1\rangle} - B_i^{\langle k\rangle}] = S^{\langle k\rangle}[0,1,\cdots,2^i-1]^{tr}$. □

THEOREM 8. *Let* $\rho_1 = [1,1]$, $\rho_2 = [2,2,1]$, $\rho_3 = [5,6,3,1]$ *and* $\rho_4 = [15,20,12,4,1]$. *Then* $\rho_1[B_0^{\langle k\rangle}, B_1^{\langle k\rangle}]^{tr} = B_1^{\langle k+1\rangle}$, $\rho_2[B_0^{\langle k\rangle}, B_1^{\langle k\rangle}, B_2^{\langle k\rangle}]^{tr} = B_2^{\langle k+1\rangle}$, $\rho_3[B_0^{\langle k\rangle},\cdots,B_3^{\langle k\rangle}]^{tr} = B_3^{\langle k+1\rangle}$ *and* $\rho_4[B_0^{\langle k\rangle},\cdots,B_4^{\langle k\rangle}]^{tr} = B_4^{\langle k+1\rangle}$.

Proof. We observe that the first few $B_i^{\langle k\rangle}$ satisfy the identities:

$$\begin{cases}\rho_2[1,2,6]^{tr} = 12 = B_2^{\langle 2\rangle}\\ \rho_2[1,3,12]^{tr} = 20 = B_2^{\langle 3\rangle}\end{cases} \qquad \begin{cases}\rho_3[1,2,6,22]^{tr} = 57 = B_3^{\langle 2\rangle}\\ \rho_3[1,3,12,57]^{tr} = 116 = B_3^{\langle 3\rangle}\end{cases}$$

Due to Theorem 5 and Theorem 7, we have

$$B_2^{\langle k+1\rangle} - B_2^{\langle k\rangle} = [s_{2,1}^{\langle k\rangle}, s_{2,2}^{\langle k\rangle}]\begin{bmatrix}1\\3\end{bmatrix} = [s_{1,0}^{\langle k\rangle}, s_{1,1}^{\langle k\rangle}]\begin{bmatrix}1 & 0\\ k+1 & 1\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix}$$
$$= [s_{1,0}^{\langle k\rangle}, s_{1,1}^{\langle k\rangle}]\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix} + [s_{1,0}^{\langle k\rangle}, s_{1,1}^{\langle k\rangle}]\begin{bmatrix}0 & 0\\ k & 1\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix}$$
$$= B_1^{\langle k\rangle} + (k+3) = B_1^{\langle k\rangle} + (k+1) + 2 = [2,2][B_0^{\langle k\rangle}, B_1^{\langle k\rangle}]^{tr}, \tag{4}$$

for $k+1 = s_{1,0} + s_{1,1} = B_1^{\langle k\rangle}$. Theorem 5 and 7 imply

$$B_3^{\langle k+1\rangle} - B_3^{\langle k\rangle} = [s_{3,1}^{\langle k\rangle}, s_{3,2}^{\langle k\rangle}, s_{3,3}^{\langle k\rangle}][1,3,7]^{tr}$$
$$= [s_{2,0}^{\langle k\rangle}, s_{2,1}^{\langle k\rangle}, s_{2,2}^{\langle k\rangle}]\begin{bmatrix}1 & 0 & 0\\ k+1 & 1 & 0\\ 2k & k+2 & 1\end{bmatrix}\begin{bmatrix}1\\3\\7\end{bmatrix} = s_{2,0}^{\langle k\rangle} + [s_{2,1}^{\langle k\rangle}, s_{2,2}^{\langle k\rangle}]\begin{bmatrix}k+4\\5k+13\end{bmatrix}$$
$$= s_{2,0}^{\langle k\rangle} + [s_{2,1}^{\langle k\rangle}, s_{2,2}^{\langle k\rangle}]\begin{bmatrix}1\\3\end{bmatrix} + [s_{2,1}^{\langle k\rangle}, s_{2,2}^{\langle k\rangle}]\begin{bmatrix}k+3\\5k+10\end{bmatrix}$$
$$= [2,2][B_0^{\langle k\rangle}, B_1^{\langle k\rangle}]^{tr} + s_{2,0}^{\langle k\rangle} + [s_{2,1}^{\langle k\rangle}, s_{2,2}^{\langle k\rangle}]\begin{bmatrix}k+3\\5k+10\end{bmatrix} \tag{5}$$

by (4). Since $2s_{2,0}^{\langle k\rangle} = k(2k+1) + k = ks_{2,1}^{\langle k\rangle} + k$ (Theorem 6), we have

$$s_{2,0}^{\langle k\rangle} + [s_{2,1}^{\langle k\rangle}, s_{2,2}^{\langle k\rangle}]\begin{bmatrix}k+3\\5k+10\end{bmatrix} = s_{2,0}^{\langle k\rangle} + (k+3)s_{2,1}^{\langle k\rangle} + (5k+10)s_{2,2}^{\langle k\rangle}$$
$$= s_{2,0}^{\langle k\rangle} + ks_{2,1}^{\langle k\rangle} + k + 3s_{2,1}^{\langle k\rangle} + 4k + 10 = s_{2,0}^{\langle k\rangle} + 2s_{2,0}^{\langle k\rangle} + 3s_{2,1}^{\langle k\rangle} + 4k + 10$$
$$= 3s_{0,0}^{\langle k\rangle} + 4(s_{1,0}^{\langle k\rangle} + s_{1,1}^{\langle k\rangle}) + 3(s_{2,0}^{\langle k\rangle} + s_{2,1}^{\langle k\rangle} + s_{2,2}^{\langle k\rangle}) = [3,4,3][B_0^{\langle k\rangle}, B_1^{\langle k\rangle}, B_2^{\langle k\rangle}]^{tr}.$$

Therefore from (5), we have

$$B_3^{\langle k+1\rangle} - B_3^{\langle k\rangle} = [5,6,3][B_0^{\langle k\rangle}, B_1^{\langle k\rangle}, B_2^{\langle k\rangle}]^{tr}. \tag{6}$$

Moreover $\rho_4[1,1,2,5,15]^{tr} = 94 = B_4^{\langle 1\rangle}$ and

$$B_4^{\langle k+1\rangle} - B_4^{\langle k\rangle} = [s_{4,1}^{\langle k\rangle}, s_{4,2}^{\langle k\rangle}, s_{4,3}^{\langle k\rangle}, s_{4,4}^{\langle k\rangle}][1,3,7,15]^{tr}$$
$$= s_{3,0}^{\langle k\rangle} + [s_{3,1}^{\langle k\rangle}, s_{3,2}^{\langle k\rangle}, s_{3,3}^{\langle k\rangle}]\begin{bmatrix}k+4\\5k+13\\16k+36\end{bmatrix}$$

$$= s_{3,0}^{\langle k\rangle} + 3[s_{3,1}^{\langle k\rangle}, s_{3,2}^{\langle k\rangle}, s_{3,3}^{\langle k\rangle}]\begin{bmatrix}1\\3\\7\end{bmatrix} + [s_{3,1}^{\langle k\rangle}, s_{3,2}^{\langle k\rangle}, s_{3,3}^{\langle k\rangle}]\begin{bmatrix}k+1\\5k+4\\16k+15\end{bmatrix}$$

$$= 3[5,6,3][B_0^{\langle k\rangle}, B_1^{\langle k\rangle}, B_2^{\langle k\rangle}]^{tr} + [s_{3,0}^{\langle k\rangle}, s_{3,1}^{\langle k\rangle}, s_{3,2}^{\langle k\rangle}, s_{3,3}^{\langle k\rangle}]\begin{bmatrix}1\\k+1\\5k+4\\16k+15\end{bmatrix},$$

for $[s_{3,1}^{\langle k\rangle}, s_{3,2}^{\langle k\rangle}, s_{3,3}^{\langle k\rangle}][1,3,7]^{tr} = B_3^{\langle k+1\rangle} - B_3^{\langle k\rangle} = [5,6,3][B_0^{\langle k\rangle}, B_1^{\langle k\rangle}, B_2^{\langle k\rangle}]^{tr}$ by (6). But $[s_{3,0}^{\langle k\rangle}, s_{3,1}^{\langle k\rangle}, s_{3,2}^{\langle k\rangle}, s_{3,3}^{\langle k\rangle}]\begin{bmatrix}1\\k+1\\5k+4\\16k+15\end{bmatrix} = (k+4)(4k+7)(k+1) = [2,3,4][B_1^{\langle k\rangle}, B_2^{\langle k\rangle}, B_3^{\langle k\rangle}]^{tr}$ from the table $S^{\langle k\rangle}$ imply

$$B_4^{\langle k+1\rangle} - B_4^{\langle k\rangle} = 3[5,6,3][B_0^{\langle k\rangle}, B_1^{\langle k\rangle}, B_2^{\langle k\rangle}]^{tr} + [2,3,4][B_1^{\langle k\rangle}, B_2^{\langle k\rangle}, B_3^{\langle k\rangle}]^{tr}$$
$$= [15,20,12,4][B_0^{\langle k\rangle}, B_1^{\langle k\rangle}, B_2^{\langle k\rangle}, B_3^{\langle k\rangle}]^{tr}.$$

It shows $\rho_4[B_0^{\langle k\rangle}, \cdots, B_4^{\langle k\rangle}]^{tr} = B_4^{\langle k+1\rangle}$ with $\rho_4 = [15,20,12,4,1]$. □

With $\rho_5 = [52,75,50,20,5,1]$ and $\rho_6 = [203,312,225,100,30,6,1]$, see $\begin{cases}\rho_5[1,1,2,5,15,52]^{tr} = 454\\ \rho_5[1,2,6,22,94,454]^{tr} = 1866,\end{cases}$ $\begin{cases}\rho_6[1,1,2,5,15,52,203]^{tr} = 2430\\ \rho_6[1,2,6,22,94,454,2430]^{tr} = 12351\end{cases}$ and $\rho_i[B_0^{\langle k\rangle}, \cdots, B_i^{\langle k\rangle}]^{tr} = B_i^{\langle k+1\rangle}$ $(i = 5,6)$. Let $E = \begin{bmatrix}1 \\ 1 & 1 \\ 2 & 2 & 1 \\ 5 & 6 & 3 & 1 \\ 15 & 20 & 12 & 4 & 1 \\ 52 & 75 & 50 & 20 & 5 & 1 \\ 203 & \cdots\end{bmatrix}$ be a matrix having ρ_i as i^{th}row. E is the exponential matrix $exp(T)$ of T scaled by $exp(1)$ (see [5], OEIS A056857).

Theorem 9. *Let $E = [e_{i,j}]$ $(i,j \geq 0)$ be the matrix above. Then $je_{i,j} = ie_{i-1,j-1}$ and $e_{i,0} = \sum_{t=0}^{i-1} e_{i-1,t} = B_i$. And $r_i(E) = [B_i, \cdots, B_1, B_0]$ $di[\binom{i}{0}, \binom{i}{1}, \cdots, \binom{i}{i}]$ and $c_j^*(E) = di[\binom{j}{j}, \binom{j+1}{j}, \binom{j+2}{j}, \cdots][B_0, B_1, B_2, \cdots]^{tr}$.*

Proof. Since $e_{i,j} = \binom{i}{j}B_{i-j}$ (OEIS A056860), $e_{i,j} = \frac{i}{j}\binom{i-1}{j-1}B_{i-j} = \frac{i}{j}e_{i-1,j-1}$ and $\sum_{t=0}^{i-1} e_{i-1,t} = \sum_{t=0}^{i-1}\binom{i-1}{t}B_{i-1-t} = B_i = e_{i,0}$. Thus $r_i(E) = [e_{i,0}, \cdots, e_{i,i}] = [B_i, \cdots, B_0]\begin{bmatrix}\binom{i}{0} \\ & \binom{i}{1} \\ & & \ddots \\ & & & \binom{i}{i}\end{bmatrix}$ and $c_j^*(E) = \begin{bmatrix}\binom{j}{j} \\ & \binom{j+1}{j} \\ & & \binom{j+2}{j} \\ & & & \ddots\end{bmatrix}\begin{bmatrix}B_0\\B_1\\B_2\\\cdots\end{bmatrix}$. □

Theorem 10. *The matrix E satisfies $ES = ST$.*

Proof. Clearly $ES = ST$ with small size matrices. Since $ST = S^{\langle 1\rangle} = [s^{\langle 1\rangle}_{i,j}]$ satisfies $s^{\langle 1\rangle}_{i+1,j} = [s^{\langle 1\rangle}_{i,j-1}, s^{\langle 1\rangle}_{i,j}, s^{\langle 1\rangle}_{i,j+1}][1, j+1, j+1]^{tr}$, it is enough to prove ES holds the same type of recurrence rule. That is, let $ES = [x_{i,j}]$ and we will show $x_{i+1,j} - x_{i,j-1} = (j+1)(x_{i,j} + x_{i,j+1})$ for $i, j \geq 0$.

Since $x_{i,j} = r_i(E)c_j(S) = [e_{i,0}, \cdots, e_{i,i}][[0]_j, s_{j,j}, \cdots, s_{i,j}]^{tr} = [e_{i,j}, \cdots, e_{i,i}]$ $[s_{j,j}, \cdots, s_{i,j}]^{tr}$, the identity $e_{i,j} = \frac{i}{j}e_{i-1,j-1}$ in Theorem 9 shows

$$\begin{aligned} x_{i+1,j} &= [e_{i+1,j}, e_{i+1,j+1}, \cdots, e_{i+1,i}, e_{i+1,i+1}][s_{j,j}, s_{j+1,j}, \cdots, s_{i,j}, s_{i+1,j}]^{tr} \\ &= [\tfrac{i+1}{j}e_{i,j-1}, \tfrac{i+1}{j+1}e_{i,j}, \cdots, \tfrac{i+1}{i}e_{i,i-1}, e_{i,i}][s_{j,j}, s_{j+1,j}, \cdots, s_{i,j}, s_{i+1,j}]^{tr} \\ &= [e_{i,j-1}, e_{i,j}, \cdots, e_{i,i-1}, e_{i,i}][\tfrac{i+1}{j}s_{j,j}, \tfrac{i+1}{j+1}s_{j+1,j}, \cdots, \tfrac{i+1}{i}s_{i,j}, s_{i+1,j}]^{tr}. \end{aligned}$$

Similarly

$$\begin{aligned} x_{i+1,j} - x_{i,j-1} &= [e_{i,j-1}, e_{i,j}, \cdots, e_{i,i-1}, e_{i,i}] \begin{bmatrix} \frac{i+1}{j}s_{j,j} - s_{j-1,j-1} \\ \frac{i+1}{j+1}s_{j+1,j} - s_{j,j-1} \\ \cdots \\ \frac{i+1}{i}s_{i,j} - s_{i-1,j-1} \\ s_{i+1,j} - s_{i,j-1} \end{bmatrix} \\ &= \frac{i-j+1}{j}e_{i,j-1} + [e_{i,j}, \cdots, e_{i,i-1}, e_{i,i}] \begin{bmatrix} \frac{i+1}{j+1}s_{j+1,j} - s_{j,j-1} \\ \frac{i+1}{j+2}s_{j+2,j} - s_{j+1,j-1} \\ \cdots \\ \frac{i+1}{i}s_{i,j} - s_{i-1,j-1} \\ s_{i+1,j} - s_{i,j-1} \end{bmatrix} \\ &= \binom{i}{j}B_{i-j+1} + [e_{i,j}, \cdots, e_{i,i-1}, e_{i,i}] \begin{bmatrix} \frac{i-j}{j+1}s_{j+1,j} + js_{j,j} \\ \frac{i-j-1}{j+2}s_{j+2,j} + js_{j+1,j} \\ \cdots \\ \frac{1}{i}s_{i,j} + js_{i-1,j} \\ js_{i,j} \end{bmatrix} \\ &= \binom{i}{j}B_{i-j+1} + [e_{i,j}, \cdots, e_{i,i}] \begin{bmatrix} \frac{i-j}{j+1}s_{j+1,j} \\ \frac{i-j-1}{j+2}s_{j+2,j} \\ \cdots \\ \frac{1}{i}s_{i,j} \\ 0 \end{bmatrix} + [e_{i,j}, \cdots, e_{i,i}]j \begin{bmatrix} s_{j,j} \\ s_{j+1,j} \\ \cdots \\ s_{i-1,j} \\ s_{i,j} \end{bmatrix} \qquad (7) \end{aligned}$$

because $\frac{i-j+1}{j}e_{i,j-1} = \frac{i-j+1}{j}\binom{i}{j-1}B_{i-j+1} = \binom{i}{j}B_{i-j+1}$, $\frac{i+1}{j+1}s_{j+1,j} - s_{j,j-1} = \frac{i-j}{j+1}s_{j+1,j} + s_{j+1,j} - s_{j,j-1} = \frac{i-j}{j+1}s_{j+1,j} + js_{j,j}$, $\frac{i+1}{j+2}s_{j+2,j} - s_{j+1,j-1} = \frac{i-j-1}{j+2}s_{j+2,j} + s_{j+2,j} - s_{j+1,j-1} = \frac{i-j-1}{j+2}s_{j+2,j} + js_{j+1,j}$, $\frac{i+1}{i}s_{i,j} - s_{i-1,j-1} = \frac{1}{i}s_{i,j} + js_{i-1,j}$ and $s_{i+1,j} - s_{i,j-1} = js_{i,j}$.

On the other hand, $x_{i,j} + x_{i,j+1} = r_i(E)(c_j(S) + c_{j+1}(S))$ shows

$$\begin{aligned} (j+1)(x_{i,j} + x_{i,j+1}) &= (j+1)[e_{i,j}, e_{i,j+1}, \cdots, e_{i,i}] \begin{bmatrix} s_{j,j} \\ s_{j+1,j} + s_{j+1,j+1} \\ \cdots \\ s_{i-1,j} + s_{i-1,j+1} \\ s_{i,j} + s_{i,j+1} \end{bmatrix} \\ &= [e_{i,j}, \cdots, e_{i,i}]j \begin{bmatrix} s_{j,j} \\ s_{j+1,j} \\ \cdots \\ s_{i-1,j} \\ s_{i,j} \end{bmatrix} + [e_{i,j}, \cdots, e_{i,i}] \begin{bmatrix} s_{j+1,j+1} \\ s_{j+2,j+1} \\ \cdots \\ s_{i,j+1} \\ s_{i+1,j+1} \end{bmatrix}. \qquad (8) \end{aligned}$$

We let Γ_1 and Γ_2 be

$$\Gamma_1 = \binom{i}{j} B_{i-j+1} + [e_{i,j}, \cdots, e_{i,i}] \begin{bmatrix} \frac{i-j}{j+1} s_{j+1,j} \\ \frac{i-j-1}{j+2} s_{j+2,j} \\ \cdots \\ \frac{1}{i} s_{i,j} \\ 0 \end{bmatrix}, \Gamma_2 = [e_{i,j}, \cdots, e_{i,i}] \begin{bmatrix} s_{j+1,j+1} \\ s_{j+2,j+1} \\ \cdots \\ s_{i,j+1} \\ s_{i+1,j+1} \end{bmatrix}.$$

Then comparing (7) and (8), it suffices to show $\Gamma_1 = \Gamma_2$.

Now for Γ_1, the identity $B_i = \sum_{t=0}^{i-1} \binom{i-1}{t} B_t$ in (1) yields

$$\binom{i}{j} B_{i-j+1} = \binom{i}{j}\left(\binom{i-j}{0} B_0 + \binom{i-j}{1} B_1 + \cdots + \binom{i-j}{i-j-1} B_{i-j-1} + B_{i-j}\right)$$

$$= [B_{i-j}, \cdots, B_1, B_0] \begin{bmatrix} \binom{i}{j} \\ \binom{i}{j}\binom{i-j}{i-j-1} \\ \cdots \\ \binom{i}{j}\binom{i-j}{1} \\ \binom{i}{j}\binom{i-j}{0} \end{bmatrix} = [B_{i-j}, \cdots, B_1, B_0] \begin{bmatrix} \binom{i}{j} \\ \binom{i}{j}\binom{i-j}{1} \\ \cdots \\ \binom{i}{j}\binom{i-j}{i-j-1} \\ \binom{i}{j}\binom{i-j}{0} \end{bmatrix}$$

and

$$[e_{i,j},, \cdots, e_{i,i-1}, e_{i,i}] \begin{bmatrix} \frac{i-j}{j+1} s_{j+1,j} \\ \frac{i-j-1}{j+2} s_{j+2,j} \\ \cdots \\ \frac{1}{i} s_{i,j} \\ 0 \end{bmatrix} = [B_{i-j}, \cdots, B_1, B_0] \begin{bmatrix} \frac{i-j}{j+1}\binom{i}{j} s_{j+1,j} \\ \frac{i-j-1}{j+2}\binom{i}{j+1} s_{j+2,j} \\ \cdots \\ \frac{1}{i}\binom{i}{i-1} s_{i,j} \\ 0 \end{bmatrix}$$

$$= [B_{i-j}, \cdots, B_1, B_0] \begin{bmatrix} \binom{i}{j+1} s_{j+1,j} \\ \binom{i}{j+2} s_{j+2,j} \\ \cdots \\ \binom{i}{i} s_{i,j} \\ 0 \end{bmatrix}, \text{ hence we have}$$

$$\Gamma_1 = [B_{i-j}, \cdots, B_0] \begin{bmatrix} \binom{i}{j} \\ \binom{i}{j}\binom{i-j}{1} \\ \cdots \\ \binom{i}{j}\binom{i-j}{i-j-1} \\ \binom{i}{j}\binom{i-j}{i-j} \end{bmatrix} + [B_{i-j}, \cdots, B_0] \begin{bmatrix} \binom{i}{j+1} s_{j+1,j} \\ \binom{i}{j+2} s_{j+2,j} \\ \cdots \\ \binom{i}{i} s_{i,j} \\ 0 \end{bmatrix}.$$

Now for Γ_2, due to (1), we have $s_{j+2,j+1} = \binom{j+1}{j} s_{j,j} + \binom{j+1}{j+1} s_{j+1,j}$, $\cdots$, $s_{i,j+1} = \binom{i-1}{j} s_{j,j} + \binom{i-1}{j+1} s_{j+1,j} + \cdots + \binom{i-1}{i-2} s_{i-2,j} + \binom{i-1}{i-1} s_{i-1,j}$, and $s_{i+1,j+1} = \binom{i}{j} s_{j,j} + \binom{i}{j+1} s_{j+1,j} + \cdots + \binom{i}{i-1} s_{i-1,j} + \binom{i}{i} s_{i,j}$.
So using a binomial identity $\binom{a}{b}\binom{b}{c} = \binom{a}{c}\binom{a-c}{b-c}$ for $a \geq b \geq c$, we have

$$\Gamma_2 = [e_{i,j}, \cdots, e_{i,i}] \begin{bmatrix} s_{j+1,j+1} \\ s_{j+2,j+1} \\ \cdots \\ s_{i,j+1} \\ s_{i+1,j+1} \end{bmatrix} = [B_{i-j}, \cdots, B_1, B_0] \begin{bmatrix} \binom{i}{j} s_{j+1,j+1} \\ \binom{i}{j+1} s_{j+2,j+1} \\ \cdots \\ \binom{i}{i-1} s_{i,j+1} \\ s_{i+1,j+1} \end{bmatrix}$$

$$= [B_{i-j}, \cdots, B_1, B_0]$$

$$\begin{bmatrix} \binom{i}{j}s_{j+1,j+1} \\ \binom{i}{j+1}\left(\binom{j+1}{j}s_{j,j}+\binom{j+1}{j+1}s_{j+1,j}\right) \\ \cdots \\ \binom{i}{i-1}\left(\binom{i-1}{j}s_{j,j}+\binom{i-1}{j+1}s_{j+1,j}+\cdots+\binom{i-1}{i-2}s_{i-2,j}+\binom{i-1}{i-1}s_{i-1,j}\right) \\ \binom{i}{i}\left(\binom{i}{j}s_{j,j}+\binom{i}{j+1}s_{j+1,j}+\binom{i}{j+2}s_{j+2,j}+\cdots+\binom{i}{i-1}s_{i-1,j}+\binom{i}{i}s_{i,j}\right) \end{bmatrix}$$

$$= [B_{i-j},\cdots,B_1,B_0]\begin{bmatrix} 0 \\ \binom{i}{j+1}\binom{i-j-1}{0}s_{j+1,j} \\ \binom{i}{j+1}\binom{i-j-1}{1}s_{j+1,j}+\binom{i}{j+2}\binom{i-j-2}{0}s_{j+2,j} \\ \cdots \\ \binom{i}{j+1}s_{j+1,j}+\binom{i}{j+2}s_{j+2,j}+\cdots+\binom{i}{i}s_{i,j} \end{bmatrix}$$

$$+ [B_{i-j},\cdots,B_1,B_0]\begin{bmatrix} \binom{i}{j} \\ \binom{i}{j}\binom{i-j}{1}s_{j,j} \\ \cdots \\ \binom{i}{j}\binom{i-j}{i-j-1}s_{j,j} \\ \binom{i}{j}s_{j,j} \end{bmatrix}$$

But since $s_{j,j}=1$, $\Gamma_2-\Gamma_1$ is equal to

$$\begin{bmatrix} B_{i-j} \\ \cdots \\ B_1 \\ B_0 \end{bmatrix}^{tr}\begin{bmatrix} 0-\binom{i}{j+1}s_{j+1,j} \\ \binom{i}{j+1}\binom{i-j-1}{0}s_{j+1,j}-\binom{i}{j+2}s_{j+2,j} \\ \cdots \\ \binom{i}{j+1}\binom{i-j-1}{i-j-2}s_{j+1,j}+\binom{i}{j+2}\binom{i-j-2}{i-j-3}s_{j+2,j}+\cdots-\binom{i}{i}s_{i,j} \\ \binom{i}{j+1}s_{j+1,j}+\binom{i}{j+2}s_{j+2,j}+\cdots+\binom{i}{i-1}s_{i-1,j}+\binom{i}{i}s_{i,j} \end{bmatrix}$$

$$=\binom{i}{j+1}s_{j+1,j}\theta_{j+1,j}+\binom{i}{j+2}s_{j+2,j}\theta_{j+2,j}+\cdots+\binom{i}{i-1}s_{i-1,j}\theta_{i-1,i}+\binom{i}{i}s_{i,j}\theta_{i,i}$$

where $\theta_{j+1,j}=-B_{i-j}+\binom{i-j-1}{0}B_{i-j-1}+\cdots+\binom{i-j-1}{i-j-2}B_1+B_0$, $\theta_{j+2,j}=-B_{i-j-1}+\binom{i-j-2}{0}B_{i-j-2}+\cdots+\binom{i-j-2}{i-j-3}B_1+B_0,\cdots$, and $\theta_{i,i}=-B_1+B_0$.

But $\theta_{j+1,j}=\theta_{j+2,j}=\cdots=\theta_{i-1,i}=\theta_{i,i}=0$, for $B_k=\sum_{t=0}^{k-1}\binom{k-1}{t}B_t$ in (1). So $\Gamma_2-\Gamma_1=0$ and $x_{i+1,j}=[x_{i,j-1},x_{i,j},x_{i,j+1}][1,j+1,j+1]^{tr}$. □

Corollary 11. *For any* $k\geq 0$, $ES^{\langle k\rangle}=S^{\langle k+1\rangle}$.

Theorem 12. $r_i(E)[B_0^{\langle k\rangle},B_1^{\langle k\rangle},\cdots,B_i^{\langle k\rangle}]^{tr}=B_i^{\langle k+1\rangle}$.

Proof. By Theorem 8, we assume $r_i(E)[B_0^{\langle k-1\rangle},\cdots,B_i^{\langle k-1\rangle}]^{tr}=B_i^{\langle k\rangle}$ for some k. Then $r_i(E)S^{\langle k-1\rangle}=r_i(S^{\langle k\rangle})$ in Corollary 11 shows

$$\begin{aligned} & r_i(E)[B_0^{\langle k\rangle},B_1^{\langle k\rangle},\cdots,B_i^{\langle k\rangle}]^{tr}-B_i^{\langle k\rangle} \\ &= r_i(E)[B_0^{\langle k\rangle}-B_0^{\langle k-1\rangle},B_1^{\langle k\rangle}-B_1^{\langle k-1\rangle},\cdots,B_i^{\langle k\rangle}-B_i^{\langle k-1\rangle}]^{tr} \\ &= r_i(S^{\langle k\rangle})[0,1,2^2-1,\cdots,2^i-1]^{tr}=B_i^{\langle k+1\rangle}-B_i^{\langle k\rangle}, \end{aligned}$$

by Theorem 7. So $r_i(E)[B_0^{\langle k\rangle},B_1^{\langle k\rangle},\cdots,B_i^{\langle k\rangle}]^{tr}=B_i^{\langle k+1\rangle}$. □

Note that the inverse of the Stirling matrix S (of the second kind) equals the signed Stirling matrix s of the first kind [4], i.e., if $s = [a_{i,j}]$ then $S^{-1} = [(-1)^{i+j} a_{i,j}]$. While only a few relations are known about S and s ([6]), Theorem 10 gives a way for finding relationships of $S^{\langle 1 \rangle} = [s^{\langle 1 \rangle}_{i,j}]$ and the first kind Stirling matrix $s = [a_{i,j}]$ in terms of B_i.

Theorem 13. *$\sum_{k=j}^{i} (-1)^{k+j} s^{\langle 1 \rangle}_{i,k} a_{k,j} = \binom{i}{j} B_{i-j}$ for any $i \geq j \geq 0$.*

Proof. From $E = STS^{-1} = S^{\langle 1 \rangle} S^{-1}$ in Theorem 10, we have

$$\binom{i}{j} B_{i-1} = e_{i,j} = r_i(S^{\langle 1 \rangle}) c_j(S^{-1})$$
$$= [s^{\langle 1 \rangle}_{i,0}, \cdots, s^{\langle 1 \rangle}_{i,i}][(-1)^j a_{0,j}, \cdots, a_{j,j}, (-1) a_{j+1,j}, \cdots, (-1)^{i+j} a_{i,j}]^{tr}$$
$$= [s^{\langle 1 \rangle}_{i,j}, s^{\langle 1 \rangle}_{i,1}, \cdots, s^{\langle 1 \rangle}_{i,i}][a_{j,j}, (-1) a_{j+1,j}, \cdots, (-1)^{i+j} a_{i,j}]^{tr}$$
$$= s^{\langle 1 \rangle}_{i,j} a_{j,j} - s^{\langle 1 \rangle}_{i,1} a_{j+1,j} + \cdots + (-1)^{i+j} s^{\langle 1 \rangle}_{i,i} a_{i,j} = \sum_{k=j}^{i} (-1)^{k+j} s^{\langle 1 \rangle}_{i,k} a_{k,j}. \quad \square$$

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