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CONVERGENCE OF MODIFIED VISCOSITY INEXACT MANN ITERATION FOR A FAMILY OF NONLINEAR MAPPINGS FOR VARIATIONAL INEQUALITY IN $CAT(0)$ SPACES

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Abstract. The purpose of this paper, we prove convergence theorems of the modified viscosity inexact Mann iteration process for a family of asymptotically quasi-nonexpansive type mappings in $CAT(0)$ spaces. We also show that the limit of the modified viscosity inexact Mann iteration $\{x_n\}$ solves the solution of some variational inequality.

1. Introduction

Let C be a nonempty subset of a metric space (X, d) and Y be a nonempty subset of C.

(1) The mapping $T : C \to C$ is said to be nonexpansive respect to Y if for each $x \in C$ and $y \in Y$,

$$d(Tx, Ty) \leq d(x, y).$$

If $Y = C$, T is called nonexpansive and if $Y = F(T) = \{x \in C : Tx = x\}$, T is called quasi-nonexpansive.

(2) The mapping T is said to be asymptotically nonexpansive respect to Y if there exists a sequence $\{k_n\}$ of positive real numbers such that

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$k_n \to 1$ and for all $x \in C$ and $y \in Y$,

$$d(T^n x, T^n y) \leq k_n d(x, y).$$

If $Y = C$, T is called asymptotically nonexpansive and if $Y = F(T)$, T is called asymptotically quasi-nonexpansive.

(3) The mapping T is said to be asymptotically nonexpansive type respect to Y if

$$\limsup_{n\to\infty} \sup_{y\in Y} (d(T^n x, T^n y) - d(x, y)) \leq 0,$$

for all $x \in C$. If $Y = C$, T is called asymptotically nonexpansive type and if $Y = F(T)$, T is called asymptotically quasi-nonexpansive type.

It is clear that nonexpansive mappings(quasi-nonexpansive mappings) and asymptotically nonexpansive mappings(asymptotically quasi-nonexpansive mappings) are asymptotically nonexpansive type mappings (resp. asymptotically quasi-nonexpansive type mappings).

(4) The sequence $\{T_n\}$ of self mappings on C is called a family of asymptotically nonexpansive mappings respect to Y if for each T_i, there exists a sequence $\{k_{n,i}\}$ of positive real numbers such that $k_{n,i} \to 1$, as $n \to \infty$, and for all $x \in C$ and $y \in Y$,

$$d(T_i^n x, T_i^n y) \leq k_{n,i} d(x, y).$$

If $Y = C$, the sequence $\{T_n\}$ is called a family of asymptotically nonexpansive mappings and if $Y = \bigcap_{n=1}^{\infty} F(T_n)$, the sequence $\{T_n\}$ is called a family of asymptotically quasi-nonexpansive mappings.

(5) The sequence $\{T_n\}$ of self mappings on C is called a family of asymptotically nonexpansive type mappings respect to Y if each T_i satisfies

$$\limsup_{n\to\infty} \sup_{y\in Y} (d(T_i^n x, T_i^n y) - d(x, y)) \leq 0,$$

for all $x \in C$. If $Y = C$, the sequence $\{T_n\}$ is called a family of asymptotically nonexpansive type mappings and if $Y = \bigcap_{n=1}^{\infty} F(T_n)$, the sequence $\{T_n\}$ is called a family of asymptotically quasi-nonexpansive type mappings.

(6) A mapping $f : C \to C$ is called contractive respect to Y with coefficient $k \in (0, 1)$ if for each $x \in C$ and $y \in Y$,

$$d(f(x), f(y)) \leq k d(x, y).$$

If $Y = C$, f is called a contraction with coefficient $k \in (0, 1)$. f has a unique fixed point when C is a nonempty, closed, and subset of a complete metric space was guaranteed by Banach's contraction principle [2].

The existence theorems of fixed points and convergence theorems for various mappings in $CAT(0)$ spaces have been investigated by many authors [1, 8, 10, 12, 17], [19]-[24], [27], [29]-[34].

Let us to introduce the $CAT(0)$ spaces.

Let (X, d) be a metric space. A *geodesic path* joining $p_1 \in X$ to $p_2 \in X$ (or, a *geodesic* from p_1 to p_2) is a mapping g from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $g(0) = p_1, g(l) = p_2$, and

$$d(g(t), g(t')) = |t - t'|, \ \ \forall\, t, t' \in [0, l].$$

In particular, g is an isometry and $d(p_1, p_2) = l$. The image α of g is said to be a *geodesic segment* (or, *metric segment*) joining p_1 and p_2. When it is unique, this geodesic segment is denoted by $[p_1, p_2]$. The space (X, d) is called a *geodesic space* if every two points of X are joined by a geodesic segment, and X is called *uniquely geodesic segment* if there is exactly one geodesic segment joining p_1 and p_2 for each $p_1, p_2 \in X$. A subset $Y \subseteq X$ is called *convex* if Y includes every geodesic segment joining any two of its points.

A *geodesic triangle* $\triangle(p_1, p_2, p_3)$ is a geodesic metric space (X, d) consists of three *vertices* of $\triangle$(the points $p_1, p_2, p_3 \in X$) and the *edges* of $\triangle$(a geodesic segment between each pair of vertices). A *comparison triangle* for the geodesic triangle $\triangle(p_1, p_2, p_3)$ in (X, d) is a triangle $\bar{\triangle}(p_1, p_2, p_3) = \triangle(\bar{p_1}, \bar{p_2}, \bar{p_3})$ in $\mathbb{R}^2$ such that

$$d_{\mathbb{R}^2}(\bar{p_i}, \bar{p_j}) = d(p_i, p_j), \ \ i, j \in \{1, 2, 3\}.$$

A comparison triangle for the geodesic triangle always exists(see, [4], [30]).

A geodesic metric space is called a $CAT(0)$ *space*(this term is due to Gromov [15] and it is an acronym for Cartan, Aleksandrov and Toponogov) if all geodesic triangles of appropriate size satisfy the following $CAT(0)$ comparison axiom.

> Let $\triangle$ be a geodesic triangle in (X, d) and let $\bar{\triangle} \subset \mathbb{R}^2$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ *inequality* if for all vertices $p_1, p_2 \in \triangle$ and all comparison points $\bar{p_1}, \bar{p_2} \in \bar{\triangle}$,
>
> $$d(p_1, p_2) \leq d_{\mathbb{R}^2}(\bar{p_1}, \bar{p_2}).$$

Let p, p_1, p_2 are points in $CAT(0)$ space, if p_0 is the midpoint of the segment $[p_1, p_2]$, which we will denote by $\frac{p_1 \oplus p_2}{2}$, then the $CAT(0)$ inequality implies

$$d^2\left(p, \frac{p_1 \oplus p_2}{2}\right) = d^2(p, p_0) \leq \frac{1}{2} d^2(p, p_1) + \frac{1}{2} d^2(p, p_2) - \frac{1}{4} d^2(p_1, p_2).$$

This inequality is called the *(CN) inequality* ([6]).

Remark 1.1. A geodesic metric space (X, d) is a $CAT(0)$ space if and only if satisfies the (CN) inequality (*cf.* [4, p.163]).

The above (CN) inequality has been extended as

$$d^2(p, \alpha p_1 \oplus (1-\alpha)p_2) \le \alpha d^2(p, p_1) + (1-\alpha)d^2(p, p_2) \\ - \alpha(1-\alpha)d^2(p_1, p_2), \quad \forall\, p, p_1, p_2 \in X \tag{CN*}$$

for all $0 \le \alpha \le 1$ ([12]).

In the recent years, $CAT(0)$ spaces have attracted many researchers as they treated a very important role in different directions of geometry and mathematics (see [4], [5], [7], [14], [23]). Some examples of $CAT(0)$ spaces are pre-Hilbert spaces (see [4]), $\mathbb{R}$-trees (see [23]), Euclidean buildings (see [5]), the complex Hilbert ball with a hyperbolic metric (see [14]), Hadamard manifolds and many others. Complete $CAT(0)$ spaces are often called *Hadamard spaces* (see [23]).

It is well known that a normed linear space satisfies the (CN) inequality if and only if it satisfies the parallelogram identity, that is, it is a pre-Hilbert space ([4]). Hence it is not so unusual to have an inner product-like notion in Hadamard spaces. In [3], they introduced the concept of quasilinearization as follows:

Let us usually denote a pair $(x, y) \in X^2 = X \times X$ by $\overrightarrow{xy}$ and call it a *vector*. Then *quasilinearization* is defined as a mapping $\langle \cdot, \cdot \rangle : X^2 \times X^2 \to \mathbb{R}$ by

$$\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle = \frac{1}{2}(d^2(x, v) + d^2(y, u) - d^2(x, u) - d^2(y, v)), \quad \forall\, x, y, u, v \in X.$$

It is easily seen that

$$\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle = \langle \overrightarrow{uv}, \overrightarrow{xy} \rangle, \quad \langle \overrightarrow{xy}, \overrightarrow{uv} \rangle = -\langle \overrightarrow{yx}, \overrightarrow{uv} \rangle$$

and

$$\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle = \langle \overrightarrow{xw}, \overrightarrow{uv} \rangle + \langle \overrightarrow{wy}, \overrightarrow{uv} \rangle$$

for all $x, y, u, v, w \in X$. We say that X satisfies the *Cauchy-Schwarz inequality* if

$$\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle \le d(x, y)d(u, v), \quad \forall\, x, y, u, v \in X. \tag{1.1}$$

Remark 1.2. A geodesically connected metric space is a $CAT(0)$ space if and only if it satisfies the Cauchy-Schwarz inequality([3, Corollary 3]).

In [10], they introduced the concept of duality mapping in $CAT(0)$ spaces, by using the concept of quasilinearization, and studied its relation with subdifferential. Also they proved a characterization of metric projection in $CAT(0)$ spaces as follows.

Theorem 1.3. ([10, Theorem 2.4]) *Let C be a nonempty convex subset of a complete $CAT(0)$ space X. Then*

$$p = P_C x \quad \Leftrightarrow \quad \langle \overrightarrow{yp}, \overrightarrow{px} \rangle \geq 0, \ \ \forall\, y \in C$$

for all $x \in X$ and $p \in C$.

Let C be a nonempty closed subset of a $CAT(0)$ space X and let $T : C \to C$ be an asymptotically nonexpansive type mapping. The Krasnoselski-Mann iteration starting from $x_1 \in C$ is defined by

$$x_{n+1} = \alpha_n T^n(x_n) \oplus (1-\alpha_n)x_n, \quad n \geq 1, \tag{1.2}$$

where $\{\alpha_n\}$is a sequence in $[0,1]$. In 2011, Zhang and Cui [35] consider the convergence of the above iteration (1.2) for continuous mappings of asymptotically nonexpansive mappings.

In 2016, Ranjbar and Khatibzadeh [29] extended the results of Zhang and Cui [35] to a family of asymptotically quasi-nonexpansive mappings in the setting of complete $CAT(0)$ spaces. They consider the sequence given by the modified inexact Mann iteration

$$x_{n+1} = \alpha_n P y_n \oplus (1-\alpha_n) T_n^n P y_n, \quad d(y_n, x_n) \leq e_n, \quad x_0 \in C, \tag{1.3}$$

where $\{T_n\}$ is a family of asymptotically nonexpansive type self-mappings on a closed and convex subset C of a complete $CAT(0)$ space X, $\{\alpha_n\} \subset [0,1]$, $\{e_n\} \subset \mathbb{R}$ and P is the nearest point projection on C. They prove $\triangle$-convergence of the sequence given by (1.3) to be a common fixed point of the sequence $\{T_n\}$ under appropriate assumptions on $\{\alpha_n\}$ and $\{e_n\}$ in complete $CAT(0)$ spaces.

In 2015, using the concept of quasilinearization, Wangkeeree *et al.* [34] proved the strong convergence theorems of the following Moudafi's viscosity iterations for an asymptotically nonexpansive mapping T: for given a contraction mapping f defined on C and $0 < \alpha_n < 1$, let $x_n \in C$ be the unique fixed point of the contraction $x \mapsto \alpha_n f(x) \oplus (1-\alpha_n)T^n x$, that is,

$$x_n = \alpha_n f(x_n) \oplus (1-\alpha_n) T^n x_n, \ \ \forall\, n \geq 1 \tag{1.4}$$

and $x_1 \in C$ is arbitrary chosen and

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n) T^n x_n, \ \ \forall\, n \geq 1. \tag{1.5}$$

They proved the iterative schemes $\{x_n\}$ defined by (1.4) and (1.5) strongly convergent to the same point $\bar{x} \in F(T)$ with $\bar{x} = P_{F(T)} f(\bar{x})$ which is the unique solution of the variational inequality

$$\langle \overrightarrow{\bar{x} f(\bar{x})}, x\bar{x} \rangle \geq 0, \ \ x \in F(T),$$

where $F(T) = \{x : Tx = x\}$.

The purpose of this paper is to prove convergence theorems of the modified viscosity inexact Mann iteration process

$$\begin{aligned} x_{n+1} &= \alpha_n f(Py_n) \oplus (1-\alpha_n) T_n^n (Py_n), \\ d(y_n, x_n) &\le e_n, \\ x_0 &\in C \end{aligned} \tag{1.6}$$

for a family of asymptotically quasi-nonexpansive type mappings $\{T_n\}$ in $CAT(0)$ spaces, where f is given contraction mapping and P is the nearest point projection on C. We also show that the limit of the modified viscosity inexact Mann iteration $\{x_n\}$ generated by (1.6) solves the solution of some variational inequality.

2. Preliminaries

Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers. Let C be a nonempty subset of a metric space (X, d). $\mathcal{F} := \bigcap_{n=1}^{\infty} F(T_n)$, where $F(T_n) = \{x : T_n x = x\}$ denotes the set of fixed points of T_n.

We write $(1-t)p_1 \oplus tp_2$ for the unique point p in the geodesic segment joining from p_1 to p_2 such that

$$d(p, p_1) = td(p_1, p_2) \quad \text{and} \quad d(p, p_2) = (1-t)d(p_1, p_2).$$

We also denote by $[p_1, p_2]$ the geodesic segment joining from p_1 to p_2, that is, $[p_1, p_2] = \{(1-t)p_1 \oplus tp_2 : t \in [0,1]\}$. A subset C of a $CAT(0)$ space is convex if $[p_1, p_2] \subset C$ for all $p_1, p_2 \in C$.

Now, we give the concept of $\triangle$-convergence and its some basic properties.

Kirk and Panyanak [24] insisted the concept of $\triangle$-convergence in $CAT(0)$ spaces that was introduced by Lim [25] in 1976 is very similar to the weak convergence in a Banach space setting.

Let $\{x_n\}$ be a bounded sequence in $CAT(0)$ spaces X. For $p \in X$, we set

$$r(p, \{x_n\}) = \limsup_{n\to\infty} d(p, x_n).$$

The asymptotic radius $A_r(\{x_n\})$ of $\{x_n\}$ is given by

$$A_r(\{x_n\}) = \inf\{r(p, \{x_n\}) : p \in X\}$$

and the asymptotic center $A_c(\{x_n\})$ of $\{x_n\}$ is the set

$$A_c(\{x_n\}) = \{p \in X : r(p, \{x_n\}) = A_r(\{x_n\})\}.$$

It is well known that asymptotic center $A_c(\{x_n\})$ consists of exactly one point (see, e.g., [11, Proposition 7, p.767]) in a complete $CAT(0)$ space.

Definition 2.1. ([24]) A sequence $\{x_n\}$ in a complete $CAT(0)$ space X is said to $\triangle$-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$, that is, $A_c(\{u_n\}) = \{x\}$. In this case one can write

$$x_n \xrightarrow{\triangle} x \quad \text{or} \quad \triangle - \lim_{n\to\infty} x_n = x$$

and call x the $\triangle$-limit of $\{x_n\}$.

The concept of $\triangle$-convergence has been studied by many authors and extend the notion of weak convergence of Hilbert space to $CAT(0)$ spaces.

Lemma 2.2. ([12]) *Let X be a $CAT(0)$ space, $p_1, p_2, z \in X$ and $t \in [0,1]$. Then*

$$d^2(tp_1 \oplus (1-t)p_2, z) \le td^2(p_1, z) + (1-t)d^2(p_2, z) - t(1-t)d^2(p_1, p_2).$$

Lemma 2.3. ([28]) *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{\lambda_n\}$ are nonnegative sequences such that*

$$a_{n+1} \le (1+\lambda_n)a_n + b_n, \quad n \ge 1$$

with $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$. Then $\lim \alpha_n$ exists. Moreover, if $\liminf_{n\to\infty} \alpha_n = 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4. ([24]) *Every bounded sequence in a complete $CAT(0)$ space always has a $\triangle$-convergent subsequence.*

Remark 2.5. In a $CAT(0)$ space, strong convergence in the metric implies $\triangle$-convergence (see, [17, 19]).

Lemma 2.6. ([17, Theorem 2.6]) *Let X be a complete $CAT(0)$ space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ $\triangle$-converges to x if and only if*

$$\limsup_{n\to\infty} \langle \overrightarrow{xx_n}, \overrightarrow{xy} \rangle \le 0, \ \ \forall\, y \in X.$$

The following two useful lemmas can be found in [34].

Lemma 2.7. ([34]) *Let X be a complete $CAT(0)$ space. Then the following inequality holds*

$$d^2(p, r) \le d^2(q, r) + 2\langle \overrightarrow{pq}, \overrightarrow{pr} \rangle, \ \ \forall\, p, q, r \in X.$$

Lemma 2.8. ([34]) *Let X be a $CAT(0)$ space. For any $l \in (0,1)$ and $x, y \in X$, let*

$$x_l = lx \oplus (1-l)y.$$

Then, for all $u, v \in X$,

(i) $\langle \overrightarrow{x_l u}, \overrightarrow{x_l v} \rangle \le l \langle \overrightarrow{xu}, \overrightarrow{x_l v} \rangle + (1-l) \langle \overrightarrow{yu}, \overrightarrow{x_l v} \rangle$,

(ii) $\langle \overrightarrow{x_l u}, \overrightarrow{xv} \rangle \le l \langle \overrightarrow{xu}, \overrightarrow{xv} \rangle + (1-l) \langle \overrightarrow{yu}, \overrightarrow{xv} \rangle$ *and*
$\langle \overrightarrow{x_l u}, \overrightarrow{yv} \rangle \le l \langle \overrightarrow{xu}, \overrightarrow{yv} \rangle + (1-l) \langle \overrightarrow{yu}, \overrightarrow{yv} \rangle$.

3. Main results

In this section, we prove the convergence of the modified viscosity inexact Mann iteration $\{x_n\}$ generated by (1.6) such that the family $\{T_n\}$ of asymptotically (quasi-)nonexpansive type self-mappings on subset C in a $CAT(0)$ space (X, d) satisfies the following condition:

For subsequence $\{T_{n_j}\}$ of $\{T_n\}$ and $\{x_{n_j}\} \subset C$ such that

$$x_{n_j} \xrightarrow{\triangle} x \quad \text{and} \quad d(x_{n_j}, T_{n_j}^{n_j} x_{n_j}) \to 0.$$

$$\text{Then } x \in \mathcal{F} = \bigcap_{n=1}^{\infty} F(T_n). \tag{3.1}$$

Theorem 3.1. *Suppose that C is a closed and convex subset of a complete $CAT(0)$ space (X, d) and $\{T_n\}$ is a family of asymptotically nonexpansive type self-mappings on C such that $\mathcal{F} := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{\alpha_n\} \subset [0, 1]$, $\{e_n\} \subset [0, \infty)$ and $\{y_n\} \subset X$ be sequences such that the modified viscosity inexact Mann iteration $\{x_n\}$ is generated by (1.6). Suppose $\sum_{n=1}^{\infty} e_n < \infty$ and $\{\alpha_n\} \subset [a, b]$ with $a, b \in (0, 1)$. Then we have the following statements.*

(i) *Let $\{T_n\}$ be a sequence of asymptotically nonexpansive type mappings such that the condition (3.1) is satisfied. Set*

$$c_{ni} = \max\{0, \sup_{x,y \in C}(d(T_i^n x, T_i^n y) - d(x, y))\}.$$

If $\sum_{n=1}^{\infty} c_{nn} < \infty$, $\sum_{n=1}^{\infty} \alpha_n < \infty$, $d(x_n, x_{n+1}) = o(\alpha_n)$ and $\lim_{n\to\infty} \frac{c_{nn}}{\alpha_n} = \lim_{n\to\infty} \frac{e_n}{\alpha_n} = 0$, then $\{x_n\}$ is convergent to $q \in \mathcal{F}$;

(ii) *Let $e_n \equiv 0$ and $\{T_n\}$ be a sequence of asymptotically quasi-nonexpansive type mappings such that the condition (3.1) is satisfied. Set*

$$c_{ni} = \max\{0, \sup\{(d(T_i^n x, T_i^n p) - d(x, p)) : x \in C, p \in \mathcal{F}\}.$$

If $\sum_{n=1}^{\infty} c_{nn} < \infty$, $\sum_{n=1}^{\infty} \alpha_n < \infty$, $d(x_n, x_{n+1}) = o(\alpha_n)$ and $\lim_{n\to\infty} \frac{c_{nn}}{\alpha_n} = \lim_{n\to\infty} \frac{e_n}{\alpha_n} = 0$, then $\{x_n\}$ is convergent to $q \in \mathcal{F}$. Moreover,

$$x^* = P_{\mathcal{F}} f(x^*),$$

which is equivalent to the following variational inequality

$$\langle \overrightarrow{x^* f(x^*)}, \overrightarrow{xx^*} \rangle \ge 0, \quad \forall\, x \in \mathcal{F} = \bigcap_{n=1}^{\infty} F(T_n). \tag{3.2}$$

Proof. Let $\{T_n\}$ be a sequence of asymptotically quasi-nonexpansive type mappings. Suppose $q \in \mathcal{F} \subset C$. Then

$$\begin{aligned} d(x_{n+1}, q) &= d(\alpha_n f(Py_n) \oplus (1-\alpha_n) T_n^n(Py_n), q) \\ &\le \alpha_n d(f(Py_n), f(q)) + \alpha_n d(f(q), q) + (1-\alpha_n) d(T_n^n(Py_n), q) \\ &\le \alpha \alpha_n d(Py_n, q) + \alpha_n d(f(q), q) + (1-\alpha_n)(c_{nn} + d(Py_n, q)) \\ &\le d(y_n, q) + \alpha_n d(f(q), q) + (1-\alpha_n) c_{nn} \\ &\le d(x_n, q) + e_n + \alpha_n d(f(q), q) + c_{nn}, \end{aligned}$$

so, by the assumption and Lemma 2.3, $\lim_{n\to\infty} d(x_n, q)$ exists for all $q \in \mathcal{F}$ and $\{x_n\}$, $\{y_n\}$ and $\{Py_n\}$ are bounded. Also are $\{f(Py_n)\}$ and $\{T_n(Py_n)\}$.

I. We claim that $\lim_{n\to\infty} d(x_n, T_n^n x_n) = 0$.

We have

$$\begin{aligned} d(x_{n+1}, T_n^n(Py_n)) &= d(\alpha_n f(Py_n) \oplus (1-\alpha_n) T_n^n(Py_n), T_n^n(Py_n)) \\ &\le \alpha_n d(f(Py_n), T_n^n(Py_n)). \end{aligned} \tag{3.3}$$

Since

$$\begin{aligned} d(f(Py_n), T_n^n(Py_n)) &\le d(f(Py_n), x_{n+1}) + d(x_{n+1}, T_n^n(Py_n)) \\ &= d(f(Py_n), \alpha_n f(Py_n) \oplus (1-\alpha_n) T_n^n(Py_n)) \\ &\quad + d(x_{n+1}, T_n^n(Py_n)) \\ &\le (1-\alpha_n) d(f(Py_n), T_n^n(Py_n)) + d(x_{n+1}, T_n^n(Py_n)), \end{aligned}$$

we obtain

$$\alpha_n d(f(Py_n), T_n^n(Py_n)) \le d(x_{n+1}, T_n^n(Py_n)). \tag{3.4}$$

Since $\sum_{n=1}^{\infty} \alpha_n < \infty$, from (3.3) and (3.4), we have

$$\begin{aligned} \lim_{n\to\infty} \alpha_n d(f(Py_n), T_n^n(Py_n)) &= \lim_{n\to\infty} d(x_{n+1}, T_n^n(Py_n)) \\ &= 0. \end{aligned} \tag{3.5}$$

From (3.5), we get

$$\begin{aligned} d(x_{n+1}, x_n) &= d(\alpha_n f(Py_n) \oplus (1-\alpha_n) T_n^n(Py_n), x_n) \\ &\le \alpha_n d(f(Py_n), x_n) + (1-\alpha_n) d(T_n^n(Py_n), x_n) \\ &\le \alpha_n [d(f(Py_n), T_n^n(Py_n)) + d(T_n^n(Py_n), x_n)] \\ &\quad + (1-\alpha_n) d(T_n^n(Py_n), x_n) \\ &= \alpha_n d(f(Py_n), T_n^n(Py_n)) + d(x_n, T_n^n(Py_n)) \\ &\to 0. \end{aligned} \tag{3.6}$$

Since

$$\begin{aligned} d(x_n, T_n^n x_n) &\le d(x_n, x_{n+1}) + d(x_{n+1}, T_n^n(Py_n)) + d(T_n^n(Py_n), T_n^n x_n) \\ &= d(x_n, x_{n+1}) + d(x_{n+1}, T_n^n(Py_n)) + c_{nn} + d(Py_n, x_n) \\ &\le d(x_n, x_{n+1}) + d(x_{n+1}, T_n^n(Py_n)) + c_{nn} + d(y_n, x_n) \\ &= d(x_n, x_{n+1}) + d(x_{n+1}, T_n^n(Py_n)) + c_{nn} + e_n, \end{aligned}$$

from (3.5) and (3.6), we obtain

$$\lim_{n\to\infty} d(x_n, T_n^n x_n) = 0. \tag{3.7}$$

Since $\{x_n\}$ is bounded, by Lemma 2.4, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which $\triangle$-converges to x^*. Therefore, from (3.7), the condition (3.1) guaranties that $x^* \in \mathcal{F}$.

II. Next, we will show that $\{x_n\}$ contains a subsequence converging strongly to x^* such that $x^* = P_{\mathcal{F}} f(x^*)$, which is equivalent to the following variational inequality

$$\langle \overrightarrow{x^* f(x^*)}, \overrightarrow{x x^*} \rangle \ge 0, \ \ \forall\, x \in \mathcal{F} = \bigcap_{n=1}^{\infty} F(T_n).$$

II-1. It follows from Lemma 2.8 (i) that

$$\begin{aligned} d^2(x_{n_j}, x^*) &= \langle \overrightarrow{x_{n_j} x^*}, \overrightarrow{x_{n_j} x^*} \rangle \\ &\le \alpha_{n_j} \langle \overrightarrow{f(Py_{n_j}) x^*}, \overrightarrow{x_{n_j} x^*} \rangle + (1-\alpha_{n_j}) \langle \overrightarrow{T_{n_j}^{n_j}(Py_{n_j}) x^*}, \overrightarrow{x_{n_j} x^*} \rangle \\ &\le \alpha_{n_j} \langle \overrightarrow{f(Py_{n_j}) x^*}, \overrightarrow{x_{n_j} x^*} \rangle + (1-\alpha_{n_j}) d(T_{n_j}^{n_j}(Py_{n_j}), x^*) d(x_{n_j}, x^*) \\ &\le \alpha_{n_j} \langle \overrightarrow{f(Py_{n_j}) x^*}, \overrightarrow{x_{n_j} x^*} \rangle + (1-\alpha_{n_j}) c_{n_j n_j} d(x_{n_j}, x^*) \\ &\quad + (1-\alpha_{n_j}) d(Py_{n_j}, x^*) d(x_{n_j}, x^*) \\ &\le \alpha_{n_j} \langle \overrightarrow{f(Py_{n_j}) x^*}, \overrightarrow{x_{n_j} x^*} \rangle + (1-\alpha_{n_j}) c_{n_j n_j} d(x_{n_j}, x^*) \\ &\quad + (1-\alpha_{n_j})(d(x_{n_j}, x^*) + e_{n_j}) d(x_{n_j}, x^*), \end{aligned}$$

thus

$$\alpha_{n_j} d^2(x_{n_j}, x^*) \le \alpha_{n_j} \langle \overrightarrow{f(Py_{n_j}) x^*}, \overrightarrow{x_{n_j} x^*} \rangle + (1-\alpha_{n_j})(c_{n_j n_j} + e_{n_j}) d(x_{n_j}, x^*),$$

$$\begin{aligned}
d^2(x_{n_j}, x^*) &\le \langle \overrightarrow{f(Py_{n_j})x^*}, \overrightarrow{x_{n_j}x^*} \rangle + \frac{(1-\alpha_{n_j})}{\alpha_{n_j}}(c_{n_j n_j} + e_{n_j}) d(x_{n_j}, x^*) \\
&= \langle \overrightarrow{f(Py_{n_j})f(x^*)}, \overrightarrow{x_{n_j}x^*} \rangle + \langle \overrightarrow{f(x^*)x^*}, \overrightarrow{x_{n_j}x^*} \rangle \\
&\quad + \frac{(1-\alpha_{n_j})}{\alpha_{n_j}}(c_{n_j n_j} + e_{n_j}) d(x_{n_j}, x^*) \\
&\le d(f(Py_{n_j}), f(x^*)) d(x_{n_j}, x^*) + \langle \overrightarrow{f(x^*)x^*}, \overrightarrow{x_{n_j}x^*} \rangle \\
&\quad + \frac{(1-\alpha_{n_j})}{\alpha_{n_j}}(c_{n_j n_j} + e_{n_j}) d(x_{n_j}, x^*) \\
&\le \alpha(d(x_{n_j}, x^*) + e_{n_j}) d(x_{n_j}, x^*) + \langle \overrightarrow{f(x^*)x^*}, \overrightarrow{x_{n_j}x^*} \rangle \\
&\quad + \frac{(1-\alpha_{n_j})}{\alpha_{n_j}}(c_{n_j n_j} + e_{n_j}) d(x_{n_j}, x^*)
\end{aligned}$$

and

$$\begin{aligned}
(1-\alpha) d^2(x_{n_j}, x^*) &\le \alpha e_{n_j} d(x_{n_j}, x^*) + \langle \overrightarrow{f(x^*)x^*}, \overrightarrow{x_{n_j}x^*} \rangle \\
&\quad + \frac{(1-\alpha_{n_j})}{\alpha_{n_j}}(c_{n_j n_j} + e_{n_j}) d(x_{n_j}, x^*).
\end{aligned}$$

Hence

$$\begin{aligned}
d^2(x_{n_j}, x^*) &\le \left(\frac{\alpha}{1-\alpha} e_{n_j} + \frac{1-\alpha_{n_j}}{1-\alpha} \cdot \frac{c_{n_j n_j} + e_{n_j}}{\alpha_{n_j}} \right) d(x_{n_j}, x^*) \\
&\quad + \frac{1}{1-\alpha} \langle \overrightarrow{f(x^*)x^*}, \overrightarrow{x_{n_j}x^*} \rangle. \qquad (3.8)
\end{aligned}$$

Since $\{x_{n_j}\}$ is $\triangle$-convergent to x^*, by Lemma 2.6, we have

$$\limsup_{n\to\infty} \langle \overrightarrow{f(x^*)x^*}, \overrightarrow{x_{n_j}x^*} \rangle \le 0.$$

It follows from (3.8) that $\{x_{n_j}\}$ converges strongly to x^*. Since

$$d(y_{n_j}, x^*) \le d(y_{n_j}, x_{n_j}) + d(x_{n_j}, x^*) = e_{n_j} + d(x_{n_j}, x^*),$$

$\{y_{n_j}\}$ converges strongly to x^*.

II-2. Next, we show that x^* solves the variational inequality (3.2).

Let $q \in \mathcal{F}$. Since

$$\begin{aligned}
d^2(Py_{n_j}, q) &\le d^2(y_{n_j}, q) \\
&\le (d(y_{n_j}, x_{n_j}) + d(x_{n_j}, x_{n_j+1}) + d(x_{n_j+1}, q))^2 \\
&\le 2d^2(y_{n_j}, x_{n_j}) + 2d^2(x_{n_j}, x_{n_j+1}) + d^2(x_{n_j+1}, q) \\
&\quad + 2\{d(y_{n_j}, x_{n_j})d(x_{n_j}, x_{n_j+1}) + d(x_{n_j}, x_{n_j+1})d(x_{n_j+1}, q) \\
&\quad + d(x_{n_j+1}, q)d(y_{n_j}, x_{n_j})\}, \qquad (3.9)
\end{aligned}$$

applying Lemma 2.2 and (3.9),

$$\begin{aligned}
d^2(x_{n_j+1}, q) &= d^2(\alpha_{n_j} f(Py_{n_j}) \oplus (1-\alpha_{n_j})T_{n_j}^{n_j}(Py_{n_j}), q) \\
&\le \alpha_{n_j} d^2(f(Py_{n_j}), q) + (1-\alpha_{n_j})d^2(T_{n_j}^{n_j}(Py_{n_j}), q) \\
&\quad - \alpha_{n_j}(1-\alpha_{n_j})d^2(f(Py_{n_j}), T_{n_j}^{n_j}(Py_{n_j})) \\
&\le \alpha_{n_j} d^2(f(Py_{n_j}), q) \\
&\quad + (1-\alpha_{n_j})[c_{n_j n_j}^2 + d^2(Py_{n_j}, q) + 2c_{n_j n_j} d(Py_{n_j}, q)] \\
&\quad - \alpha_{n_j}(1-\alpha_{n_j})d^2(f(Py_{n_j}), T_{n_j}^{n_j}(Py_{n_j})) \\
&\le \alpha_{n_j} d^2(f(Py_{n_j}), q) \\
&\quad + (1-\alpha_{n_j})[c_{n_j n_j}^2 + 2e_{n_j}^2 + 2d^2(x_{n_j}, x_{n_j+1}) + d^2(x_{n_j+1}, q) \\
&\quad + 2\{c_{n_j n_j} d(x_{n_j}, x_{n_j+1}) + d(x_{n_j}, x_{n_j+1})d(x_{n_j+1}, q) \\
&\quad + c_{n_j n_j} d(x_{n_j+1}, q)\} + 2c_{n_j n_j} d(Py_{n_j}, q)] \\
&\quad - \alpha_{n_j}(1-\alpha_{n_j})d^2(f(Py_{n_j}), T_{n_j}^{n_j}(Py_{n_j})).
\end{aligned}$$

So,

$$\begin{aligned}
&\alpha_{n_j}(1-\alpha_{n_j})d^2(f(Py_{n_j}), T_{n_j}^{n_j}(Py_{n_j})) + \alpha_{n_j} d^2(x_{n_j+1}, q) \\
&\quad \le \alpha_{n_j} d^2(f(Py_{n_j}), q) + (1-\alpha_{n_j})[c_{n_j n_j}^2 + 2e_{n_j}^2 + 2d^2(x_{n_j}, x_{n_j+1}) \\
&\qquad + 2\{c_{n_j n_j} d(x_{n_j}, x_{n_j+1}) + d(x_{n_j}, x_{n_j+1})d(x_{n_j+1}, q) \\
&\qquad + c_{n_j n_j} d(x_{n_j+1}, q)\} + 2c_{n_j n_j} d(Py_{n_j}, q)],
\end{aligned}$$

we have

$$\begin{aligned}
&(1-\alpha_{n_j})d^2(f(Py_{n_j}), T_{n_j}^{n_j}(Py_{n_j})) + d^2(x_{n_j+1}, q) \\
&\quad \le d^2(f(Py_{n_j}), q) + \frac{1-\alpha_{n_j}}{\alpha_{n_j}}[c_{n_j n_j}^2 + 2e_{n_j}^2 + 2d^2(x_{n_j}, x_{n_j+1}) \\
&\qquad + 2\{c_{n_j n_j} d(x_{n_j}, x_{n_j+1}) + d(x_{n_j}, x_{n_j+1})d(x_{n_j+1}, q) \\
&\qquad + c_{n_j n_j} d(x_{n_j+1}, q)\} + 2c_{n_j n_j} d(Py_{n_j}, q)]. \qquad (3.10)
\end{aligned}$$

Since $x_{n_j} \to x^*$ and (3.5), we have $T_{n_j}^{n_j}(Py_{n_j}) \to x^*$. Take limit on both sides in (3.10), from assumptions and continuity of the metric distance d, we obtain

$$d^2(f(x^*), x^*) + d^2(x^*, q) \leq d^2(f(x^*), q).$$

Hence

$$\begin{aligned} 0 &\leq \frac{1}{2}[d^2(x^*, x^*) + d^2(f(x^*), q) - d^2(x^*, q) - d^2(f(x^*), x^*)] \\ &= \langle \overrightarrow{x^* f(x^*)}, \overrightarrow{qx^*} \rangle, \quad \forall\, q \in \mathcal{F}, \end{aligned}$$

that is, x^* solves the inequality (3.2).

III. Finally, we will show the uniqueness of the solution of the variational inequality of Equation (3.2). Assume there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which $\triangle$-converges to ω by the same argument. We know that $\omega \in \mathcal{F}$ and solves the variational inequality of Equation (3.2), that is,

$$\langle \overrightarrow{x^* f(x^*)}, \overrightarrow{x^* \omega} \rangle \leq 0 \tag{3.11}$$

and

$$\langle \overrightarrow{\omega f(\omega)}, \overrightarrow{\omega x^*} \rangle \leq 0. \tag{3.12}$$

From (3.11) and (3.12), we can obtain

$$\begin{aligned} 0 &\geq \langle \overrightarrow{x^* f(x^*)}, \overrightarrow{x^* \omega} \rangle - \langle \overrightarrow{\omega f(\omega)}, \overrightarrow{x^* \omega} \rangle \\ &= \langle \overrightarrow{x^* f(\omega)}, \overrightarrow{x^* \omega} \rangle + \langle \overrightarrow{f(\omega) f(x^*)}, \overrightarrow{x^* \omega} \rangle - \langle \overrightarrow{\omega x^*}, \overrightarrow{x^* \omega} \rangle - \langle \overrightarrow{x^* f(\omega)}, \overrightarrow{x^* \omega} \rangle \\ &= \langle \overrightarrow{x^* \omega}, \overrightarrow{x^* \omega} \rangle - \langle \overrightarrow{f(\omega) f(x^*)}, \overrightarrow{\omega x^*} \rangle \\ &\geq \langle \overrightarrow{x^* \omega}, \overrightarrow{x^* \omega} \rangle - d(f(\omega), f(x^*)) d(\omega, x^*) \\ &\geq d^2(x^*, \omega) - \alpha d^2(\omega, x^*) \\ &= (1-\alpha) d^2(x^*, \omega). \end{aligned}$$

Since $0 < \alpha < 1$, we have

$$d(x^*, \omega) = 0,$$

so

$$x^* = \omega.$$

Hence $\{x_n\}$ converges strongly to x^*, which solves the variational inequality of Equation (3.2). □

If we have $P = I$ (Identity mapping), we get the following result.

Corollary 3.2. *Suppose that C is a closed and convex subset of a complete $CAT(0)$ space (X, d) and $\{T_n\}$ is a family of asymptotically nonexpansive type self-mappings on C such that $\mathcal{F} := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{\alpha_n\} \subset [0,1]$, $\{e_n\} \subset [0,\infty)$ and $\{y_n\} \subset X$ be sequences such that the modified viscosity inexact Mann iteration $\{x_n\}$ is generated by*

$$\begin{aligned} x_{n+1} &= \alpha_n f(y_n) \oplus (1-\alpha_n) T_n^n(y_n), \\ d(y_n, x_n) &\leq e_n, \\ x_0 &\in C, \end{aligned}$$

where f is given contraction mapping. Suppose $\sum_{n=1}^{\infty} e_n < \infty$ and $\{\alpha_n\} \subset [a,b]$ with $a, b \in (0,1)$. Then we have the following statements.

(i) *Let $\{T_n\}$ be a sequence of asymptotically nonexpansive type mappings such that the condition (3.1) is satisfied. Set*

$$c_{ni} = \max\{0, \sup_{x,y\in C}(d(T_i^n x, T_i^n y) - d(x,y))\}.$$

If $\sum_{n=1}^{\infty} c_{nn} < \infty$, $\sum_{n=1}^{\infty} \alpha_n < \infty$, $d(x_n, x_{n+1}) = o(\alpha_n)$ and $\lim_{n\to\infty} \frac{c_{nn}}{\alpha_n} = \lim_{n\to\infty} \frac{e_n}{\alpha_n} = 0$, then $\{x_n\}$ is convergent to $q \in \mathcal{F}$;

(ii) *let $e_n \equiv 0$ and $\{T_n\}$ be a sequence of asymptotically quasi-nonexpansive type mappings such that the condition (3.1) is satisfied. Set*

$$c_{ni} = \max\{0, \sup\{(d(T_i^n x, T_i^n p) - d(x,p)) : x \in C, p \in \mathcal{F}\}.$$

If $\sum_{n=1}^{\infty} c_{nn} < \infty$, $\sum_{n=1}^{\infty} \alpha_n < \infty$, $d(x_n, x_{n+1}) = o(\alpha_n)$ and $\lim_{n\to\infty} \frac{c_{nn}}{\alpha_n} = \lim_{n\to\infty} \frac{e_n}{\alpha_n} = 0$, then $\{x_n\}$ is convergent to $q \in \mathcal{F}$. Moreover,

$$x^* = \mathcal{F} \cap F(f),$$

which is equivalent to the following variational inequality

$$\langle \overrightarrow{x^* f(x^*)}, \overrightarrow{x x^*} \rangle \geq 0, \quad \forall\, x \in \mathcal{F} = \bigcap_{n=1}^{\infty} F(T_n), \tag{3.13}$$

where $F(f) = \{x^* \in C : x^* = f(x^*)\}$.

If we have $f = I$(: Identity mapping), we get the following result.

Corollary 3.3. *Suppose that C is a closed and convex subset of a complete $CAT(0)$ space (X, d) and $\{T_n\}$ is a family of asymptotically nonexpansive type self-mappings on C such that $\mathcal{F} := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{\alpha_n\} \subset [0,1]$, $\{e_n\} \subset [0,\infty)$ and $\{y_n\} \subset X$ be sequences such that the modified inexact Mann* *iteration* $\{x_n\}$ *is generated by*

$$\begin{aligned} x_{n+1} &= \alpha_n P(y_n) \oplus (1-\alpha_n)T_n^n(Py_n), \\ d(y_n, x_n) &\leq e_n, \\ x_0 &\in C, \end{aligned}$$

where f *is given contraction mapping. Suppose* $\sum_{n=1}^{\infty} e_n < \infty$ *and* $\{\alpha_n\} \subset [a,b]$ *with* $a, b \in (0,1)$*. Then we have the following statements.*

(i) *Let* $\{T_n\}$ *be a sequence of asymptotically nonexpansive type mappings such that the condition* (3.1) *is satisfied. Set*

$$c_{ni} = \max\{0, \sup_{x,y\in C}(d(T_i^n x, T_i^n y) - d(x,y))\}.$$

If $\sum_{n=1}^{\infty} c_{nn} < \infty$, $\sum_{n=1}^{\infty} \alpha_n < \infty$, $d(x_n, x_{n+1}) = o(\alpha_n)$ *and* $\lim_{n\to\infty} \frac{c_{nn}}{\alpha_n} = \lim_{n\to\infty} \frac{e_n}{\alpha_n} = 0$, *then* $\{x_n\}$ *is convergent to* $q \in \mathcal{F}$*;*

(ii) *Let* $e_n \equiv 0$ *and* $\{T_n\}$ *be a sequence of asymptotically quasi-nonexpansive type mappings such that the condition* (3.1) *is satisfied. Set*

$$c_{ni} = \max\{0, \sup\{(d(T_i^n x, T_i^n p) - d(x,p)) : x \in C, p \in \mathcal{F}\}.$$

If $\sum_{n=1}^{\infty} c_{nn} < \infty$, $\sum_{n=1}^{\infty} \alpha_n < \infty$, $d(x_n, x_{n+1}) = o(\alpha_n)$ *and* $\lim_{n\to\infty} \frac{c_{nn}}{\alpha_n} = \lim_{n\to\infty} \frac{e_n}{\alpha_n} = 0$, *then* $\{x_n\}$ *is convergent to* $q \in \mathcal{F}$*.*

Remark 3.4. Corollary 3.2 and 3.3 are generalization and improvement of the results of [34] and [29], respectively.

4. Open problem

For a real number κ, a $CAT(\kappa)$ space is defined by a geodesic metric space whose geodesic triangle is sufficiently thinner than the corresponding triangle in a model space with curvature κ.

For $\kappa = 0$, the 2-dimensional model space $M_\kappa^2 = M_0^2$ is the Euclidean space $\mathbb{R}^2$ with the metric induced from the Euclidean norm. For $\kappa > 0$, M_κ^2 is the 2-dimensional sphere $\frac{1}{\sqrt{\kappa}}\mathbb{S}^2$ whose metric is length of a minimal great arc joining each two points. For $\kappa < 0$, M_κ^2 is the 2-dimensional hyperbolic space $\frac{1}{\sqrt{-\kappa}}\mathbb{H}^2$ with the metric defined by a usual hyperbolic distance. For more details about the properties of $CAT(\kappa)$ spaces (see, [4], [13], [22]).

Open Problem. It will be interesting to obtain a generalization of both Theorem 3.1, Corollary 3.2 and Corollary 3.3 to $CAT(\kappa)$ space.

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