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(p,q)-ANALOGUE OF THE NATURAL TRANSFORM WITH APPLICATIONS

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Abstract. The natural transform is represented by two (p,q)-analogues, and their comparative characteristics are established. To resolve some (p,q)-difference and functional equations, applications are carried out.

1. Introduction

For many decades, the integral transforms play a precious role in solving many differential and integral equations. Using an appropriate integral transform helps to reduce differential and integral operators, from a considered

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domain into multiplication operators in another domain. Solving the deduced problem in the new domain, and then applying the inverse transform serve to invert the manipulated solution back to the required solution of the problem in its original domain.

The most popular integral transforms have been contributed largely by Laplace, Fourier, Mellin, Hankel and Sumudu. The Laplace transform is of great importance among these transforms.

In recent years, the theory of q-integral transforms has advanced quickly [12]. Since the so-called q-Jackson integral [16] was precisely defined by Jackson [15]. Many authors, including Purohit and Kalla [22], Fitouhi and Bettaibi [8, 9], Abdi [5], Hahn [13, 14], Albayrak [30], Yadav and Purohit [32], Al-Omari [4, 6], Ucar and Albayrak [31], Albayrak et al. [1, 2], Al-Omari et al. [3], and others, have examined the q-analogues.

Our goal is to provide insights on (p,q)-calculus theory. However, we confine ourselves to discussing a few (p,q)-analogues of the natural transform, an integral transform, and to calculate the appropriate transform parameters. The traditional theory of the natural transform and the traditional theory of Laplace and Sumudu are closely related two of the most well-known integral transforms [10, 17].

According to [5, 7], the Natural transform $f(x)$ is defined as

$$N_{+}f(x) = \frac{1}{u}\int_{0}^{\infty} e^{-sx/u} f(x)dx, \tag{1.1}$$

where $x \in R_{+}$ on the set of the functions.

$$A = \{f(x) : \exists M, \tau_1, \tau_2 > 0,\ \ |f(x)| < Me^{x/\tau_i},\ \text{ if }\ x \in (-1)^i \times [0,\infty)\},$$

where s and u are transform variables.

In [4], the authors have defined the q-analogue of natural transform by the q-Jackson [16] integrals as follows:

$$_{q}N(f)(u;v) = \frac{1}{(1-q)u}\int_{0}^{\infty} f(t)e_q\left(\frac{-vt}{u}\right)d_q t. \tag{1.2}$$

The (p,q)-shifted factorial is based on the concept of twin-basic number $[n]_{(p,q)} = \frac{(p^n-q^n)}{(p-q)}$. The basic number occurs in the theory of two parameter quantum algebras and has also been introduced in combinatorics by Jagannathan et al. [24]. Several properties of this number were studied briefly in [23]. Around the same time as [23], Brodimas et al. [11] and Arik et al. [20] also independently introduced the (p,q)-number in the physics literature, but in a very much less detailed manner. The (p, q)-identities thus derived, with doubling of the number of parameters, offer more choices for applications.

It has been observed that many of the q-results can be generalized directly to (p, q)-results. If we have the (p, q)-results, the q-results can be obtained more easily by mere substitutions for the parameters instead of any limiting process as required in the usual q-theory [19]. This also provides a new look for the q-identities. The q-deformed algebra [18, 21] and their generalization to (p, q)-analogue [7, 23, 29] have attracted much attention of the researchers to increase the accessibility of different dimensions of (p, q)-analogue algebra. The main reason is that these topics stand for real life problems, in mathematics and physics, later to the theory of quantum calculus.

In the present paper, the authors attention is towards defining the (p, q)-analogue of natural transform with applications.

2. Preliminaries

We give some definitions and their properties for our main results.

The twin basic number is a natural generalization of the q-number, that is,

$$[n]_{p,q}! = [n]_q \quad \text{as} \quad p \longrightarrow 1.$$

The (p, q)-factorial is defined by Sadjang [25] as follows:

$$[n]_{p,q}! = \prod_{k=1}^{n} [k]_{p,q}!, \ n \geq 1, \ [0]_{p,q}! = 1.$$

The (p, q)-binomial coefficients are defined as:

$$\binom{n}{k}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}, \ 0 \leq k \leq n.$$

Note that as $p \longrightarrow 1$, the (p, q)-binomial coefficient reduce to the q-binomial coefficient. It is clear by definition that

$$\binom{n}{k}_{p,q} = \binom{n}{n-k}_{p,q}.$$

Definition 2.1. Let f be an arbitrary function and a be a real number, then the (p, q)-integral of f is defined by [25]

$$\int_0^a f(x) d_{p,q}x = (p-q)a \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f\left(\frac{q^k a}{p^{k+1}}\right) \quad \text{if} \quad |\frac{p}{q}| > 1.$$

Definition 2.2. The improper (p,q)-integral of $f(x)$ on $[0,\infty]$ is defined by [25] as follows:

$$\int_0^\infty f(x)d_{p,q}x = (p-q)\sum_{j=-\infty}^{\infty}\frac{q^j}{p^{j+1}}f\left(\frac{q^j}{p^{j+1}}\right), \quad 0<\frac{q}{p}<1.$$

Let f be a function defined on the set of the complex numbers.

Definition 2.3. The (p,q)-derivative of the function f is defined by [25] as

$$D_{p,q}f(x) = \frac{f(px)-f(qx)}{(p-q)x}, \quad x\neq 0$$

and $(D_{p,q}f)(0)=f'(0)$, provided that f is differentiable at 0.

Proposition 2.4. *Sadjang [25] defined the (p,q)-derivative and it fulfills the following product and quotient rules:*

$$D_{p,q}\Big(f(x)g(x)\Big) = f(px)D_{p,q}g(x)+g(qx)D_{p,q}f(x)$$

or

$$D_{p,q}\Big(f(x)g(x)\Big) = g(px)D_{p,q}f(x)+f(qx)D_{p,q}g(x)$$

and

$$D_{p,q}\left(\frac{f(x)}{g(x)}\right) = \frac{g(qx)D_{p,q}f(x)-f(qx)D_{p,q}g(x)}{g(px)g(qx)}$$

or

$$D_{p,q}\left(\frac{f(x)}{g(x)}\right) = \frac{g(px)D_{p,q}f(x)-f(px)D_{p,q}g(x)}{g(px)g(qx)}.$$

The following proposition is derived by Sadjang [25] as follows:

Proposition 2.5. *Suppose that $f(x)$ and $g(x)$ are two functions whose ordinary derivatives exist in a neighborhood of $x=0$. a and b are two real numbers such that $a<b$, then*

$$\int_a^b f(px)(D_{p,q}g(x))d_{p,q}x = f(b)g(b)-f(a)g(a)-\int_a^b g(qx)(D_{p,q}f(x))d_{p,q}x.$$

Like in the q-case, there are many definitions of the (p,q)-exponential function. The following two (p,q)-analogues of the exponential function (see [28]) will be frequently used throughout this paper:

$$e_{p,q}(x) = \sum_{n=0}^{\infty}\frac{p^{\binom{n}{2}}}{[n]_{p,q}!}x^n \tag{2.1}$$

and

$$E_{p,q}(x)=\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}}{[n]_{p,q}!}x^n. \tag{2.2}$$

The next two propositions given in [28] gave the n-th derivative of the (p,q)-exponential functions. These formulas are very important for the computations the (p,q)-natural transforms of some functions in the next sections.

Proposition 2.6. *Let λ be a complex number, then the following relations hold.*

$$D_{p,q}e_{p,q}(\lambda x)=\lambda e_{p,q}(\lambda px)$$

and

$$D_{p,q}E_{p,q}(\lambda x)=\lambda E_{p,q}(\lambda qx).$$

Proposition 2.7. *Let λ be a complex number and n be a nonnegative integer, then the following relations hold.*

$$D^n_{p,q}e_{p,q}(\lambda x)=\lambda^n p^{\binom{n}{2}}e_{p,q}(\lambda p^n x)$$

and

$$D^n_{p,q}E_{p,q}(\lambda x)=\lambda^n q^{\binom{n}{2}}E_{p,q}(\lambda q^n x).$$

The (p,q)-cosine and the (p,q)-sine functions are defined in [28] as follows:

$$\cos_{p,q}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n p^{\binom{2n}{2}}}{[2n]_{p,q}!}x^{2n}$$

and

$$\sin_{p,q}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n p^{\binom{2n+1}{2}}}{[2n+1]_{p,q}!}x^{2n+1}.$$

Also, the (p,q)-Cosine and the (p,q)-Sine functions are defined in [28] in the following manner:

$$\mathrm{Cos}_{p,q}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{2n}{2}}}{[2n]_{p,q}!}x^{2n}$$

and

$$\mathrm{Sin}_{p,q}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{2n+1}{2}}}{[2n+1]_{p,q}!}x^{2n+1}.$$

Sadjang [28] defined the hyperbolic (p,q)-cosine and the hyperbolic (p,q)-sine functions as follows:

$$\cosh_{p,q}(x) = \frac{e_{p,q}(x) + e_{p,q}(-x)}{2} = \sum_{n=0}^{\infty} \frac{p^{\binom{2n}{2}}}{[2n]_{p,q}!} x^{2n},$$

$$\sinh_{p,q}(x) = \frac{e_{p,q}(x) - e_{p,q}(-x)}{2} = \sum_{n=0}^{\infty} \frac{p^{\binom{2n+1}{2}}}{[2n+1]_{p,q}!} x^{2n+1},$$

$$\mathrm{Cosh}_{p,q}(x) = \frac{E_{p,q}(x) + E_{p,q}(-x)}{2} = \sum_{n=0}^{\infty} \frac{q^{\binom{2n}{2}}}{[2n]_{p,q}!} x^{2n},$$

$$\mathrm{Sinh}_{p,q}(x) = \frac{E_{p,q}(x) - E_{p,q}(-x)}{2} = \sum_{n=0}^{\infty} \frac{q^{\binom{2n+1}{2}}}{[2n+1]_{p,q}!} x^{2n+1}.$$

Definition 2.8. Sadjang [26] defined the (p, q)-Gamma function of the first kind by

$$\Gamma_{p,q}(z) = p^{\frac{z(z-1)}{2}} \int_0^{\infty} t^{z-1} E_{p,q}(-qt) d_{p,q}t,$$

where $0 < q < p$.

Definition 2.9. ([26]) Let z be a complex number such that $\Gamma_{p,q}(z + 1)$ and $\Gamma_{p,q}(z)$ exist. Then

$$\Gamma_{p,q}(z + 1) = [z]_{p,q} \Gamma_{p,q}(z).$$

If n is a nonnegative integer, it follows from above that

$$\Gamma_{p,q}(n + 1) = [n]_{p,q}!.$$

Definition 2.10. Sadjang [26] defined the (p, q)-Gamma function of the second kind by

$$\gamma_{p,q}(z) = q^{\frac{z(z-1)}{2}} \int_0^{\infty} t^{z-1} e_{p,q}(-pt) d_{p,q}t, \ \ Re(z) > 0,$$

where $0 < q < p$.

Proposition 2.11. ([26]) *Let z be a complex number such that $\gamma_{p,q}(z+1)$ and $\gamma_{p,q}(z)$ exist. Then*

$$\gamma_{p,q}(z + 1) = [z]_{p,q} \gamma_{p,q}(z).$$

Moreover, if n is a nonnegative integer, it follows from above that

$$\gamma_{p,q}(n + 1) = [n]_{p,q}!.$$

3. (p,q)-analogue of natural transform

Al-Omari [4] defined the q-analogue of Natural transform of first and second types as follows:

Let $\hat{A}$ and $\check{A}$ be defined by

$$\hat{A}=\{f(t) : \exists M, \tau_1, \tau_2 > 0, |f(t)|<ME_q(|t|/\tau_j), \text{ if } t \in (-1)^j \times [0,\infty), j=1,2\}$$

and

$$\check{A}=\{f(t) : \exists M, \tau_1, \tau_2 > 0, |f(t)|<Me_q(|t|/\tau_j), \text{ if } t \in (-1)^j \times [0,\infty), j=1,2\}$$

respectively. Then the q-analogues of the natural transforms respectively are as follows:

$$N_q(f)(u;v) = \frac{1}{(1-q)u}\int_0^{u/v} f(t)E_q\left(q\frac{v}{u}t\right)d_qt \tag{3.1}$$

and

$$_qN(f)(u;v) = \frac{1}{(1-q)u}\int_0^{\infty} f(t)e_q\left(\frac{-v}{u}t\right)d_qt \tag{3.2}$$

provided both the integrals exist.

Where the q-exponential functions E_q and e_q are defined by

$$e_q(t) = \sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n} = \frac{1}{(t;q)_\infty},\ |t|<1$$

and

$$E_q(t) = \sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}} t^n}{(q;q)_n} = (t;q)_\infty,\ t \in C,$$

and the symbol $(a;q)_n$ denotes the q-pocchammer symbol and defined as

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^n),\ n \geq 1 \text{ and } (a;q)_0 = 1.$$

For a given function $f(t)$, Sadjang [28] defined (p,q)-Laplace transform of the first kind as:

$$F_1(s) = L_{p,q}f(t)(s) = \int_0^{\infty} f(t)E_{p,q}(-qts)dp, qt,\ s>0.$$

Also for $\alpha > -1$, Sadjang [28] defined the (p,q)-Laplace transform of first kind of the power function as:

$$L_{p,q}(t^\alpha) = \frac{\Gamma_{p,q}(\alpha+1)}{p^{\frac{\alpha(\alpha+1)}{2}}s^{\alpha+1}},\ s>0.$$

For a given function $f(t)$, Sadjang [28] defined (p,q)-Laplace transform of the second kind as:

$$F_2(s) =_{p,q} L\{f(t)\}(s) = \int_0^\infty f(t) e_{p,q}(-pts) dp, qt, \ s > 0.$$

Also for $\alpha > -1$, Sadjang [28] defined the (p,q)-Laplace transform of second kind of the power function as:

$$\mathbb{L}_{p,q}(t^\alpha) = \frac{\gamma_{p,q}(\alpha+1)}{q^{\frac{\alpha(\alpha+1)}{2}} s^{\alpha+1}}, \ s > 0.$$

The (p,q)-Sumudu transform of first kind for a given function $f(t)$ is defined by Sadjang [27] as follows:

$$\begin{aligned} G_1(s) = S_{p,q} f(t)(s) &= \frac{1}{s} \int_0^\infty f(t) E_{p,q}\left(\frac{-qt}{s}\right) dp, qt \\ &= \int_0^\infty f(st) E_{p,q}(-qt) dp, qt, \ s > 0. \end{aligned}$$

Sadjang [27] also defined the (p,q)-Sumudu transform of first kind of the power function as:

$$S_{p,q}(t^\alpha) = \frac{\Gamma_{p,q}(\alpha+1)}{p^{\frac{\alpha(\alpha+1)}{2}}} s^\alpha, \ s > 0.$$

The (p,q)-Sumudu transform of second kind for a given function $f(t)$ is defined by Sadjang [27] as follows:

$$\begin{aligned} G_2(s) = \mathbb{S}_{p,q} f(t)(s) &= \frac{1}{s} \int_0^\infty f(t) e_{p,q}\left(\frac{-pt}{s}\right) dp, qt \\ &= \int_0^\infty f(st) e_{p,q}(-qt) dp, qt, \ s > 0. \end{aligned}$$

Sadjang [27] also defined the (p,q)-Sumudu transform of second kind of the power function as:

$$\mathbb{S}_{p,q}(t^\alpha) = \frac{\gamma_{p,q}(\alpha+1)}{q^{\frac{\alpha(\alpha+1)}{2}}} s^\alpha, \ s > 0.$$

Now, the following two transforms which may be regraded as (p,q)-extensions of the Natural transforms are introduced and their relative properties are studied.

3.1. (p,q)**-natural transform of first kind.**

Definition 3.1. The (p,q)-natural transform of first kind is defined as

$$N_{p,q}(f)(u;v) = \frac{v}{u}\int_0^{\infty} E_{p,q}\left(\frac{-qvt}{u}\right) f(t) d_{p,q}t$$
$$= \int_0^{\infty} E_{p,q}(-qt) f\left(\frac{u}{v}t\right) d_{p,q}t. \tag{3.3}$$

The following propositions arise from the definition.

Proposition 3.2. *The (p,q)-natural transform of the first kind and the (p,q)-Sumudu and Laplace transform of the first kind are related in the following way:*

$$N_{p,q}(f)(u;1) = S_{p,q}(f)(u) \tag{3.4}$$

and

$$N_{p,q}(f)(1;v) = L_{p,q}(f)(v). \tag{3.5}$$

Proof. The proof follows from easily from the definitions. □

Theorem 3.3. (Linearity, Scaling) *If f and g are two functions for which the (p,q)-natural transform $N_{p,q}$ exists, α and β are two complex numbers. The following relations apply*

$$N_{p,q}\{\alpha f(t) + \beta g(t)\}(u;v) = \alpha N_{p,q}\{f(t)\}(u;v) + \beta N_{p,q}\{g(t)\}(u;v), \tag{3.6}$$
$$N_{p,q}\{f(\alpha t)\}(u;v) = N_{p,q}\{f(t)\}(\alpha u;v). \tag{3.7}$$

Proof. The linearity is obvious. For the scaling property, we write:

$$N_{p,q}\{f(\alpha t)\}(u;v) = \frac{v}{u}\int_0^{\infty} E_{p,q}(\frac{-qvt}{u}) f(\alpha t) d_{p,q}t$$
$$= \frac{v}{\alpha u}\int_0^{\infty} E_{p,q}(\frac{-qvt}{\alpha u}) f(t) d_{p,q}t$$
$$= N_{p,q}\{f(t)\}(\alpha u;v).$$

□

Theorem 3.4. (Transform of the first derivative) *If f and $D_{p,q}f$ have (p,q)-natural transform, then*

$$N_{p,q}\{(D_{p,q}f)(t)\}(u;v) = \frac{v}{u} N_{p,q}\{f(t)\}(pu;v) - \frac{v}{u} f(0). \tag{3.8}$$

Proof. Using the definition of $N_{p,q}$ and the formula of the (p,q)-integration by parts, it follows that

$$\begin{aligned}
N_{p,q}\{(D_{p,q}f)(t)\}(u;v) &= \frac{v}{u}\int_0^{\infty} E_{p,q}(\frac{-qvt}{u})D_{p,q}f(t)d_{p,q}t \\
&= \frac{v}{u}\{[E_{p,q}(\frac{-vt}{u})f(t)]_0^{\infty} - \int_0^{\infty} f(pt)D_{p,q}E_{p,q}(\frac{-vt}{u})d_{p,q}t\} \\
&= \frac{v}{u}\{-f(0) + \frac{v}{u}\int_0^{\infty} f(pt)E_{p,q}(\frac{-qvt}{u})d_{p,q}t\} \\
&= \frac{v}{u}\{-f(0) + \frac{v}{pu}\int_0^{\infty} f(t)E_{p,q}(\frac{-qvt}{pu})d_{p,q}t\} \\
&= \frac{v}{u}N_{p,q}\{f(t)\}(pu;v) - \frac{v}{u}f(0).
\end{aligned}$$

□

The theorem can be extended to a n-th derivative as follows:

Theorem 3.5. (Transform of the n-th derivative) *Let n be a nonnegative integer. If for each $k \in \{0,1,2,...,n\}, D_{p,q}^k$ has a (p,q)-natural transform. Then the following equation applies:*

$$N_{p,q}\{D_{p,q}^n f(t)\}(u;v) = \frac{v^n}{p^{\binom{n}{2}}u^n}N_{p,q}\{f(t)\}(p^n u;v) - \sum_{k=0}^{n-1}\frac{v^{n-k}}{p^{\binom{n-k}{2}}u^{n-k}}(D_{p,q}^k f)(0). \tag{3.9}$$

Proof. The theorem is obvious for $n = 1$, see for instance Theorem 3.4. Let $n \geq 1$, assume that

$$N_{p,q}\{D_{p,q}^n f(t)\}(u;v) = \frac{v^n}{p^{\binom{n}{2}}u^n}N_{p,q}\{f(t)\}(p^n u;v) - \sum_{k=0}^{n-1}\frac{v^{n-k}}{p^{\binom{n-k}{2}}u^{n-k}}(D_{p,q}^k f)(0).$$

We need to prove it is true for $n+1$ also. Then, using Theorem 3.4, with $g = D_{p,q}^n f$, we have

$$\begin{aligned}
N_{p,q}\{D_{p,q}^{n+1} f(t)\}(u;v) &= \frac{v}{u}N_{p,q}\{D_{p,q}^n f(t)\}(pu;v) - \frac{v}{u}(D_{p,q}^n f)(0) \\
&= \frac{v}{u}\Bigg\{\frac{1}{p^{\binom{n}{2}}(\frac{pu}{v})^n}N_{p,q}\{f(t)\}(p^n(pu;v)) \\
&\quad - \sum_{k=0}^{n-1}\frac{1}{p^{\binom{n-k}{2}}(\frac{pu}{v})^{n-k}}(D_{p,q}^k f)(0)\Bigg\} - \frac{v}{u}(D_{p,q}^n f)(0)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{v^{n+1}}{p^{\binom{n+1}{2}} u^{n+1}} N_{p,q}\{f(t)\}(p^{n+1}(u;v)) \\
&\quad - \sum_{k=0}^{n-1} \frac{v^{n+1-k}}{p^{\binom{n+1-k}{2}} u^{n+1-k}} (D_{p,q}^k f)(0) - \frac{v}{u}(D_{p,q}^n f)(0) \\
&= \frac{v^{n+1}}{p^{\binom{n+1}{2}} u^{n+1}} N_{p,q}\{f(t)\}(p^{n+1}(u;v)) \\
&\quad - \sum_{k=0}^{n} \frac{v^{n+1-k}}{p^{\binom{n+1-k}{2}} u^{n+1-k}} (D_{p,q}^k f)(0).
\end{aligned}
$$

So the theorem is proved. □

Theorem 3.6. (Transform of the (p,q)-integral) *Let f be a function which is (p,q)-integrable over $(0,+\infty)$. Define $F(t)=\int_0^t f(x)d_{p,q}x$, then the following formula applies*

$$N_{p,q}\{F(pt)\}(u;v) = \frac{u}{v} N_{p,q}\{f(t)\}(u;v). \tag{3.10}$$

Proof. By definition of $N_{p,q}$ and the use of (p,q)-integration by parts, we have

$$
\begin{aligned}
N_{p,q}\{F(pt)\}(u;v) &= \frac{u}{v}\int_0^\infty E_{p,q}(\frac{-qvt}{u})F(pt)d_{p,q}t \\
&= -\int_0^\infty D_{p,q}E_{p,q}\left(\frac{-vt}{u}\right)F(pt)d_{p,q}t \\
&= -\left\{\left[E_{p,q}\left(\frac{-vt}{u}\right)F(t)\right]_0^\infty - \int_0^\infty E_{p,q}\left(\frac{-qvt}{u}\right)D_{p,q}F(pt)d_{p,q}t\right\} \\
&= 0 + \int_0^\infty E_{p,q}\left(\frac{-qvt}{u}\right)f(t)d_{p,q}t \\
&= \frac{u}{v}N_{p,q}\{f(t)\}(u;v).
\end{aligned}
$$

Note that if we replace t by tp^{-1} in (3.10), and using the scaling property (3.7), then we have

$$
\begin{aligned}
N_{p,q}\{F(t)\}(u;v) &= \frac{u}{v}N_{p,q}\{f(tp^{-1})\}(u;v) \\
&= \frac{u}{v}N_{p,q}\{f(t)\}(up^{-1};v).
\end{aligned} \tag{3.11}
$$

□

3.2. (p,q)**-natural transform of second kind.** Whereas in the previous sections we introduce the (p,q)-Natural transform of the first kind and prove some of its important properties. In this section, we introduce the (p,q)-natural transform of the second kind. The main difference is at the level of the (p,q)-exponential used in the definition. The motivation of the next definition comes from the fact that when we transform the big (p,q)-exponential, the result remains in term of a series which we cannot simplify.

Definition 3.7. The (p,q)-natural transform of second kind is defined as

$$\begin{aligned} {}_{p,q}N(f)(u;v) &= \frac{v}{u}\int_0^\infty e_{p,q}\left(\frac{-pvt}{u}\right)f(t)d_{p,q}t \\ &= \int_0^\infty e_{p,q}(-pt)f\left(\frac{u}{v}t\right)d_{p,q}t. \end{aligned} \tag{3.12}$$

The results concerning ${}_{p,q}N$ are proved in the same way as the one of $N_{p,q}$. So we will only give them here and refer the reader to the previous subsection.

Theorem 3.8. *The (p,q)-natural transform of the second kind and the (p,q)-Sumudu and Laplace transform of the second kind are related in the following way:*

$$ {}_{p,q}N(f)(u;1) = \mathbb{S}_{p,q}(f)(u) \tag{3.13}$$

and

$$ {}_{p,q}N(f)(1;v) = \mathbb{L}_{p,q}(f)(v). \tag{3.14}$$

Theorem 3.9. (Linearity, Scaling) *If f and g are two functions for which the (p,q)-natural transform ${}_{p,q}N$ exists, α and β are two complex numbers. The following relations apply:*

$$ {}_{p,q}N\{\alpha f(t)+\beta g(t)\}(u;v) = \alpha_{p,q}N\{f(t)\}(u;v) + \beta_{p,q}N\{g(t)\}(u;v), \tag{3.15}$$

$$ {}_{p,q}N\{f(\alpha t)\}(u;v) =_{p,q} N\{f(t)\}(\alpha u;v). \tag{3.16}$$

Proof. The linearity is obvious. For the scaling property, we write:

$$\begin{aligned} {}_{p,q}N\{f(\alpha t)\}(u;v) &= \frac{v}{u}\int_0^\infty e_{p,q}(\frac{-pvt}{u})f(\alpha t)d_{p,q}t \\ &= \frac{v}{\alpha u}\int_0^\infty e_{p,q}(\frac{-pvt}{\alpha u})f(t)d_{p,q}t \\ &=_{p,q} N\{f(t)\}(\alpha u;v). \end{aligned}$$

□

Theorem 3.10. (Transform of the first derivative) *If f and $D_{p,q}f$ have (p,q)-natural transform, then*

$$ {}_{p,q}N\{(D_{p,q}f)(t)\}(u;v) = \frac{v}{u}\,{}_{p,q}N\{f(t)\}(qu;v) - \frac{v}{u}f(0). \tag{3.17}$$

Proof. Using the definition of ${}_{p,q}N$ and the formula of the (p,q)-integration by parts, it follows that

$$\begin{aligned}
&{}_{p,q}N\{(D_{p,q}f)(t)\}(u;v)\\
&= \frac{v}{u}\int_0^\infty e_{p,q}\left(\frac{-pvt}{u}\right)D_{p,q}f(t)d_{p,q}t\\
&= \frac{v}{u}\{[e_{p,q}(\frac{-vt}{u})f(t)]_0^\infty - \int_0^\infty f(qt)D_{p,q}e_{p,q}\left(\frac{-vt}{u}\right)d_{p,q}t\}\\
&= \frac{v}{u}\{-f(0) + \frac{v}{u}\int_0^\infty f(qt)e_{p,q}\left(\frac{-pvt}{u}\right)d_{p,q}t\}\\
&= \frac{v}{u}\{-f(0) + \frac{v}{qu}\int_0^\infty f(t)e_{p,q}\left(\frac{-pvt}{pu}\right)d_{p,q}t\}\\
&= \frac{v}{u}\ {}_{p,q}N\{f(t)\}(qu;v) - \frac{v}{u}f(0).
\end{aligned}$$

□

The theorem can be extended to a n-th derivative as follows:

Theorem 3.11. (Transform of the n-th derivative) *Let n be a nonnegative integer. If for each $k \in \{0,1,2,...,n\}$, $D^k_{p,q}$ has a (p,q)-natural transform, then the following equation applies:*

$$\begin{aligned}
&{}_{p,q}N\{D^n_{p,q}f(t)\}(u;v)\\
&= \frac{v^n}{q^{\binom{n}{2}}u^n}\ {}_{p,q}N\{f(t)\}(q^n u;v) - \sum_{k=0}^{n-1}\frac{v^{n-k}}{q^{\binom{n-k}{2}}u^{n-k}}(D^k_{p,q}f)(0).
\end{aligned} \tag{3.18}$$

Proof. The Theorem is obvious for $n=1$, see for instance Theorem 3.10. Let $n \geq 1$, assume that

$$\begin{aligned}
&{}_{p,q}N\{D^n_{p,q}f(t)\}(u;v)\\
&= \frac{v^n}{q^{\binom{n}{2}}u^n}\ {}_{p,q}N\{f(t)\}(q^n u;v) - \sum_{k=0}^{n-1}\frac{v^{n-k}}{q^{\binom{n-k}{2}}u^{n-k}}(D^k_{p,q}f)(0).
\end{aligned}$$

We need to prove it is true for $n+1$ also. Then, using Theorem 3.10, with $g = D^n_{p,q}f$, we have

$$
\begin{aligned}
&{}_{p,q}N\{D_{p,q}^{n+1}f(t)\}(u;v)\\
&= \frac{v}{u}\ {}_{p,q}N\{D_{p,q}^{n}f(t)\}(qu;v) - \frac{v}{u}(D_{p,q}^{n}f)(0)\\
&= \frac{v}{u}\left\{ \frac{1}{q^{\binom{n}{2}}(\frac{qu}{v})^{n}}\ {}_{p,q}N\{f(t)\}(q^{n}(qu;v)) \right.\\
&\quad \left. - \sum_{k=0}^{n-1} \frac{1}{q^{\binom{n-k}{2}}(\frac{qu}{v})^{n-k}}(D_{p,q}^{k}f)(0) \right\} - \frac{v}{u}(D_{p,q}^{n}f)(0)\\
&= \frac{v^{n+1}}{q^{\binom{n+1}{2}}u^{n+1}}\ {}_{p,q}N\{f(t)\}(q^{n+1}(u;v))\\
&\quad - \sum_{k=0}^{n-1} \frac{v^{n+1-k}}{q^{\binom{n+1-k}{2}}u^{n+1-k}}(D_{p,q}^{k}f)(0) - \frac{v}{u}(D_{p,q}^{n}f)(0)\\
&= \frac{v^{n+1}}{q^{\binom{n+1}{2}}u^{n+1}}\ {}_{p,q}N\{f(t)\}(q^{n+1}(u;v))\\
&\quad - \sum_{k=0}^{n} \frac{v^{n+1-k}}{p^{\binom{n+1-k}{2}}u^{n+1-k}}(D_{p,q}^{k}f)(0).
\end{aligned}
$$

So the theorem is proved. □

Theorem 3.12. (Transform of the (p,q)-integral) *Let f be a function which is (p,q)-integrable over $(0,+\infty)$. Define $F(t)=\int_0^t f(x)d_{p,q}x$, then the following formula applies*

$$
{}_{p,q}N\{F(qt)\}(u;v) = \frac{u}{v}\ {}_{p,q}N\{f(t)\}(u;v). \tag{3.19}
$$

Proof. By definition of ${}_{p,q}N$ and the use of (p,q)-integration by parts, we have

$$
\begin{aligned}
&{}_{p,q}N\{F(qt)\}(u;v)\\
&= \frac{u}{v}\int_0^{\infty} e_{p,q}(\frac{-pvt}{u})F(qt)d_{p,q}t\\
&= -\int_0^{\infty} D_{p,q}e_{p,q}\left(\frac{-vt}{u}\right)F(qt)d_{p,q}t\\
&= -\left\{\left[e_{p,q}\left(\frac{-vt}{u}\right)F(t)\right]_0^{\infty} - \int_0^{\infty} e_{p,q}\left(\frac{-pvt}{u}\right)D_{p,q}F(pt)d_{p,q}t\right\}\\
&= 0 + \int_0^{\infty} e_{p,q}\left(\frac{-qvt}{u}\right)f(t)d_{p,q}t\\
&= \frac{u}{v}\ {}_{p,q}N\{f(t)\}(u;v).
\end{aligned}
$$

Note that if we replace t by tq^{-1} in (3.19), and using the scaling property (3.16), then we have

$$\begin{aligned} {}_{p,q}N\{F(t)\}(u;v) &= \frac{u}{v}\ {}_{p,q}N\{f(tq^{-1})\}(u;v) \\ &= \frac{u}{v}\ {}_{p,q}N\{f(t)\}(uq^{-1};v). \end{aligned}$$

□

4. Table of some transforms

In the following table, $f(t)$ is the original function,

$$N_1(u;v){=}N_{p,q}\{f(t)\}(u;v)$$

and

$$N_2(u;v){=}{}_{p,q}N\{f(t)\}(u;v).$$

$f(t)$	$N_1(u;v)$	$N_2(u;v)$
1	1	1
t^α	$\frac{\Gamma_{p,q}(\alpha+1)}{p^{\binom{\alpha+1}{2}}}\left(\frac{u}{v}\right)^\alpha$	$\frac{\gamma_{p,q}(\alpha+1)}{q^{\binom{\alpha+1}{2}}}\left(\frac{u}{v}\right)^\alpha$
$e_{p,q}(\alpha t)$	$\frac{pv}{pv-\alpha u}$	Infinite sum (4.1)
$cos_{p,q}(\alpha t)$	$\frac{p^2v^2}{p^2v^2+\alpha^2u^2}$	Infinite sum
$sin_{p,q}(\alpha t)$	$\frac{\alpha puv}{p^2v^2+\alpha^2u^2}$	Infinite sum
$E_{p,q}(\alpha t)$	Infinite sum (4.2)	$\frac{qv}{qv-\alpha u}$
$cosh_{p,q}(\alpha t)$	$\frac{p^2v^2}{p^2v^2-\alpha^2u^2}$	Infinite sum
$sinh_{p,q}(\alpha t)$	$\frac{\alpha puv}{p^2v^2-\alpha^2u^2}$	Infinite sum
$Cos_{p,q}(\alpha t)$	Infinite sum	$\frac{q^2v^2}{q^2v^2+\alpha^2u^2}$
$Sin_{p,q}(\alpha t)$	Infinite sum	$\frac{\alpha quv}{q^2v^2+\alpha^2u^2}$
$Cosh_{p,q}(\alpha t)$	Infinite sum	$\frac{q^2v^2}{q^2v^2-\alpha^2u^2}$
$Sinh_{p,q}(\alpha t)$	Infinite sum	$\frac{\alpha quv}{q^2v^2-\alpha^2u^2}$

Proposition 4.1. *The following formula apply*

$$N_{p,q}\{E_{p,q}(\alpha t)\}(u;v) = \frac{u}{v}\sum_{n=0}^{\infty}\left(\frac{\alpha u}{pv}\right)^n\left(\frac{q}{p}\right)^{\binom{n}{2}} \tag{4.1}$$

and

$${}_{p,q}N\{e_{p,q}(\alpha t)\}(u;v) = \frac{u}{v}\sum_{n=0}^{\infty}\left(\frac{\alpha u}{qv}\right)^n\left(\frac{p}{q}\right)^{\binom{n}{2}}. \tag{4.2}$$

Proof. First we will prove (4.1),

$$\begin{aligned} N_{p,q}\{E_{p,q}(\alpha t)\}(u;v) &= \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{[n]_{p,q}!}\alpha^n N_{p,q}\{t^n\}(u;v) \\ &= \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{[n]_{p,q}!}\alpha^n \frac{[n]_{p,q}!}{p^{\binom{n+1}{2}}}\left(\frac{u}{v}\right)^n \\ &= \frac{u}{v}\sum_{n=0}^{\infty}\left(\frac{\alpha u}{pv}\right)^n\left(\frac{q}{p}\right)^{\binom{n}{2}}. \end{aligned}$$

That is the end of proof. Continuing the same steps (4.2) can be obtained. □

5. Some applications

Since the (p,q)-Sumudu transforms are used to solve (p,q)-difference equations and the Laplace and Z-transforms are frequently used to solve differential and difference equations, respectively, it is anticipated that the (p,q)-natural transforms will play a similar function. The underlying concept never changes. If the natural transform is used to solve the (p,q)-Cauchy problem, a solution may be shown to be produced.

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