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A TECHNIQUE WITH DIMINISHING AND NON-SUMMABLE STEP-SIZE FOR MONOTONE INCLUSION PROBLEMS IN BANACH SPACES

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Abstract. In this paper, an algorithm for approximating zeros of sum of three monotone operators is introduced and its convergence properties are studied in the setting of 2-uniformly convex and uniformly smooth Banach spaces. Unlike the existing algorithms whose step-sizes usually depend on the knowledge of the operator norm or Lipschitz constant, a nice feature of the proposed algorithm is the fact that it requires only a diminishing and non-summable step-size to obtain strong convergence of the iterates to a solution of the problem. Finally, the proposed algorithm is implemented in the setting of a classical Banach space to support the theory established.

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1. Introduction

The following notions will appear frequently in this section. We will first introduce them for familiarity before we introduce the problem. Let E be a real Banach space with dual E^*. A mapping $A : E \to 2^{E^*}$ is called monotone if

$$\langle x - y, u - v\rangle \geq 0,\ u \in Ax,\ v \in Ay,\ \forall x, y \in E$$

and maximal monotone if its has no monotone extension. A mapping $A : E \to E^*$ is called β-strongly monotone if there exists $\beta > 0$ such that for all $x, y \in E$,

$$\langle x - y, Ax - Ay\rangle \geq \beta\|x - y\|^2.$$

It is also called β-cocoercive (or β-inverse strongly monotone) if

$$\langle x - y, Ax - Ay\rangle \geq \beta\|Ax - Ay\|^2.$$

It is called α-Lipschitz if there exists $\alpha > 0$ such that

$$\|Ax - Ay\| \leq \alpha\|x - y\|.$$

Monotone maps were studied by Minty [31], Zarantonello [42], Deepho et al. [23], Chidume et al. [15, 12], Muangchoo et al. [32] and many other authors in Hilbert spaces and more general Banach spaces. These mappings have caught the interest of many authors largely because they are useful in real-world applications, especially when it comes to solving convex optimization problems (see, e.g., [3, 4, 7, 18, 27, 28, 40, 41]).

Now, let $L : E \to 2^{E^*}$ be a set-valued map and $M, N : E \to E^*$ be single-valued maps. Consider the following inclusion problem:

$$\text{find}\ \ x \in E\ \ \text{such that}\ \ 0 \in (L + M + N)x. \tag{1.1}$$

The variational inclusion problem (1.1) popularly known as monotone inclusion problem, when the operators involved are monotone was first studied by Davis and Yin [21] in the setting of real Hilbert spaces, to the best of our knowledge (see [1, 2]). In theory, one may wonder why the interest in problem (1.1) since it can be redefined as $A := L + M + N$ and thus the problem is equivalent to finding a zero of A which the *proximal point algorithm (PPA)* and its variants have been used to solve such cases (see, e.g., [11, 14, 16, 17, 25, 33, 34]). We recall that the PPA of Martinet [30] involving a maximal monotone operator A generates its iterates by solving the recursive equation:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = (I + \lambda_n A)^{-1}x_n, \end{cases}$$

where $\lambda_n > 0$, H is a real Hilbert space and I is the identity mapping on H. However, evaluating the resolvent of A, $(I + \lambda A)^{-1}$ can be challenging in practice especially when A is nonlinear. This challenge is what led to the introduction of problem (1.1). In the literature, several splitting methods have been proposed by many authors to overcome this challenge. The idea is to split A as sum of operators with so that the linear part of A can be used to compute the resolvent easily and other properties of the remaining operators can be exploited independently (see, e.g., [3, 5, 13, 22, 24, 26, 36, 43, 44]).

Davis and Yin [21] studied problem (1.1) due to its numerous applications in solving problems arising from optimization. In particular, they studied the following 3-objective optimization problem which is to find $x \in H$ such that

$$\min_x f(x) + g(x) + h(Bx), \tag{1.2}$$

where f, g and h are proper closed and convex functions, h is $\frac{1}{\beta}$-Lipschitz differentiable and B is a linear mapping. Davis and Yin [21] gave several interesting applications of problem (1.1). In fact, models arising from image inpainting which has to do with reconstructing missing regions in an image appear naturally as the 3-objective minimization problem (see, e.g., [20, 37]). In [21], Davis and Yin recast problem (1.2) to fit in the setting of problem (1.1) by setting $L = \partial f,\ M = \partial g$ and $N = \nabla(hB)$. Then, they introduced the following algorithm for solving problem (1.1) and established a weak convergence result:

$$\begin{cases} x_0 \in H, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \end{cases} \tag{1.3}$$

where $T = J_\lambda^L\big(2J_\lambda^M - I - \lambda_n N J_\lambda^M\big) + I - J_\lambda^M$, $J_{\lambda_n}^L = (I - \lambda L)^{-1}$, L and M are maximal monotone, N is β-cocoercive, $\{\lambda\} \subset (0, 2\beta)$, $\{\alpha_n\} \subset \Big(0, \frac{4\beta-\lambda}{2\beta}\Big)$.

Recently, using the idea of Tseng [38], Malistky and Tam [29] proposed a simple algorithm for solving problem (1.1) and established weak convergence result. Their algorithm is the following:

$$\begin{cases} x_0, x_1 \in H, \\ x_{n+1} = J_\lambda^L\big(x_n - 2\lambda M x_n + \lambda M x_{n-1} + \lambda N x_n\big), \end{cases} \tag{1.4}$$

where L is maximal monotone, M is monotone and l_1-Lipschitz, N is monotone and l_2-cocoercive and $\lambda \in \big(0, \frac{2}{4l_1 + l_2}\big)$.

Remark 1.1. The algorithm of Malistky and Tam [29] requires only one computation of the resolvent operator J_λ per iteration, which reduces the computational cost of implementing the algorithm. On the other hand, the method of Davis and Yin [21] requires the computation of the resolvent twice per iteration. In addition to this, one of the shortcomings of these methods is that the control parameters depend on the knowledge of the Lipschitz constant, which is difficult to compute. In most cases, estimations of the constants are used to implement the algorithm, which affects their performance.

Question A. Can an iteration process be developed that will address the shortcomings of algorithms (1.3) and (1.4) mentioned in Remark 1.1.

This question was answered in the affirmative by Hieu et al. [39] in the setting of Hilbert spaces. They introduced the following algorithm:

$$\begin{cases} x_0, x_1 \in H, \\ x_{n+1} = J^L_{\lambda_n}\big(x_n - \lambda_n M x_n - \lambda_{n-1}(Mx_n - Mx_{n-1}) - \lambda_n N x_n\big), \end{cases} \tag{1.5}$$

where L is maximal monotone, M is α-strongly monotone and l_1-Lipschitz continuous and N is β-cocoercive, $\{\lambda_n\} \subset (0,\infty)$ such that $\lim_{n\to\infty} \lambda_n = 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$. Then $\{x_n\}$ converges strongly to a solution of problem (1.1).

Motivated by Question A and the results of Hieu et al. [39], it is our purpose in this paper to provide an affirmative answer to Question A in the setting of real Banach spaces. Furthermore, we will provide some numerical illustrations to compare the performance of the algorithms of Davis and Yin [21], Malistky and Tam [29] and our proposed algorithm in the setting of Hilbert spaces. Furthermore, we will give a numerical illustration in the setting of the classical Banach space $\ell_{1.5}$ to support the theory we established. Finally, our proposed method extend and generalize many iterative techniques for approximating zeros of sum of two monotone operators in the setting of real Banach spaces.

2. Preliminaries

The following definitions and lemmas will be needed in the proof of our main theorem.

Definition 2.1. Let E be a strictly convex and smooth real Banach space. For $p > 1$, define $J_p : E \to 2^{E^*}$ by

$$J_p(x) := \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\|x^*\|,\ \|x^*\| = \|x\|^{p-1}\}.$$

J_p is called the *generalized duality map on E.* If $p = 2$, then $J_2 := J$ is called the *normalized duality map* and is denoted by J.

In a real Hilbert space H, J is the identity map on H. It is easy to see from the definition that

$$J_p(x) = \|x\|^{p-2} Jx \quad \text{and} \quad \langle x, J_p(x) \rangle = \|x\|^p,\ \forall x \in E.$$

It is well known that if E is smooth, then J is single-valued and if E is strictly convex, J is one-to-one, and J is surjective if E is reflexive.

The next definition is for the Lyapunov functional ϕ introduced by Alber and Ryazantseva [8]. It is useful for estimations involving J and its inverse J^{-1} on smooth Banach space.

Definition 2.2. Let E be a real Banach space that is smooth and $\phi : E\times E \to \mathbb{R}$ be a map. The Lyapunov functional ϕ is defined by

$$\phi(x, y) := \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2,\ \forall\, x, y \in X.$$

Observe that if E is a real Hilbert space, (2.1) reduces to

$$\phi(x, y) = \|x - y\|^2, \forall x, y \in E.$$

The next definition is for the resolvent operator in the setting of a real Banach space.

Definition 2.3. Let E be a reflexive, strictly convex and smooth real Banach space and let $B : E \to 2^{E^*}$ be a maximal monotone operator. Then for any $\lambda > 0$ and $u \in E$, there exists a unique element $u_\lambda \in E$ such that $Ju \in Ju_\lambda + \lambda Bu_\lambda$. The element u_λ is called the resolvent of B and it is denoted by $J^B_\lambda u$. Alternatively, $J^B_\lambda = \big(J + \lambda B)^{-1}J$ for all $\lambda > 0$.

It is easy to verify that $B^{-1}0 = F(J^B_\lambda)$ for all $\lambda > 0$, where $F(J^B_\lambda)$ denotes the set of fixed points of J^B_λ.

The next two lemmas will play a central role in establishing the strong convergence of the sequence generated by our proposed algorithm.

Lemma 2.4. ([9]) *Let E be a uniformly convex and smooth real Banach space. Then the following inequalities holds:*

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle,\ \forall x, y, z \in E.$$

Lemma 2.5. ([9]) *Let E be a 2-uniformly convex real Banach space. Then there exists $\kappa > 0$ such that*

$$\frac{1}{\kappa}\|x - y\|^2 \le \phi(x, y),\ \forall x, y \in E.$$

3. Main result

Algorithm 3.1. (Three Operator Splitting Algorithm)

Step 0. Choose $x_0, x_1 \in E$ and $\{\lambda_n\} \subset (0, 1)$ such that $\lim\limits_{n\to\infty} \lambda_n = 0$ and $\sum\limits_{n=0}^{\infty} \lambda_n = \infty$.

Step 1. Having $x_{n-1},\ x_n$, compute

$$x_{n+1}=\big(J+\lambda_n L\big)^{-1}\big(Jx_n-\lambda_n Mx_n-\lambda_{n-1}(Mx_n-Mx_{n-1})-\lambda_n Nx_n\big). \quad (3.1)$$

Step 2. If $\max\{\|x_{n+1}-x_n\|,\ \|x_n-x_{n-1}\|\}<\epsilon$ for any $\epsilon>0$, STOP else set $n=n+1$ and return to Step 1.

Theorem 3.2. *Let E be a 2-uniformly convex and uniformly smooth real Banach space. Let $L: E\to 2^{E^*}$ be a maximal monotone operator, $M: E\to E^*$ be an η-strongly monotone and γ-Lipschitz operator and $N: E\to E^*$ be μ-inverse strongly monotone. Let $\{x_n\}$ be a sequence in E generated by Algorithm* 3.1. *Then $\{x_n\}$ converges strongly to a solution of problem* (1.1).

Proof. Let x^* be a solution of problem (1.1). Observe that from (3.1), we have

$$Jx_n-\lambda_n Mx_n-\lambda_{n-1}(Mx_n-Mx_{n-1})-\lambda_n Nx_n\in Jx_{n+1}+\lambda_n Lx_{n+1}.$$

Set

$$\begin{aligned} w_n &= Jx_n-Jx_{n+1}-\lambda_n Mx_n-\lambda_{n-1}(Mx_n-Mx_{n-1})-\lambda_n Nx_n\\ &\in \lambda_n Lx_{n+1}. \end{aligned}$$

Furthermore, since x^* is a solution, we get

$$w^*=-\lambda_n Mx^*-\lambda_n Nx^*\in\lambda_n Lx^*.$$

Therefore, by the monotonicity of L, we have that

$$\langle w_n-w^*, x_{n+1}-x^*\rangle\geq 0.$$

That is

$$\begin{aligned} &\langle Jx_n-Jx_{n+1}, x_{n+1}-x^*\rangle-\lambda_n\langle Mx_n-Mx^*, x_{n+1}-x^*\rangle\\ &-\lambda_{n-1}\langle Mx_n-Mx_{n-1}, x_{n+1}-x^*\rangle-\lambda_n\langle Nx_n-Nx^*, x_{n+1}-x^*\rangle\geq 0. \end{aligned} \quad (3.2)$$

We estimate the first three terms of inequality (3.2) above as follows. By Lemma 2.4,

$$2\langle Jx_{n+1}-Jx_n, x^*-x_{n+1}\rangle=\phi(x^*,x_n)-\phi(x^*,x_{n+1})-\phi(x_{n+1},x_n). \quad (3.3)$$

Also, using the η-strong monotonicity of M, we get

$$\begin{aligned} \langle Mx_n-Mx^*, x_{n+1}-x^*\rangle &= \langle Mx_{n+1}-Mx^*, x_{n+1}-x^*\rangle \\ &\quad +\langle Mx_n-Mx_{n+1}, x_{n+1}-x^*\rangle\\ &\geq \eta\|x_{n+1}-x^*\|^2+\langle Mx_n-Mx_{n+1}, x_{n+1}-x^*\rangle. \end{aligned} \quad (3.4)$$

Furthermore,

$$\begin{aligned} \langle Mx_n-Mx_{n-1}, x^*-x_{n+1}\rangle &= \langle Mx_n-Mx_{n-1}, x^*-x_n\rangle\\ &\quad +\langle Mx_n-Mx_{n-1}, x_n-x_{n+1}\rangle. \end{aligned} \quad (3.5)$$

Substituting, equations (3.3) and (3.5), inequality (3.4) in inequality (3.2), we get

$$\begin{aligned}0 \leq\ & \phi(x^*, x_n) - \phi(x^*, x_{n+1}) - \phi(x_{n+1}, x_n) - 2\lambda_n \eta \|x_{n+1} - x^*\|^2 \\ & - 2\lambda_n \langle Mx_n - Mx_{n+1}, x_{n+1} - x^* \rangle + 2\lambda_{n-1} \langle Mx_n - Mx_{n-1}, x^* - x_n \rangle \\ & + 2\lambda_{n-1} \langle Mx_n - Mx_{n-1}, x_n - x_{n+1} \rangle - 2\lambda_n \langle Nx_n - Nx^*, x_{n+1} - x^* \rangle.\end{aligned}$$

Thus

$$\begin{aligned}2\lambda_n \eta \|x_{n+1} - x^*\|^2 \leq\ & \phi(x^*, x_n) - \phi(x^*, x_{n+1}) - \phi(x_{n+1}, x_n) \\ & - 2\lambda_n \langle Mx_n - Mx_{n+1}, x_{n+1} - x^* \rangle \\ & + 2\lambda_{n-1} \langle Mx_{n-1} - Mx_n, x_n - x^* \rangle \\ & + 2\gamma \lambda_{n-1} \|x_n - x_{n-1}\| \|x_{n+1} - x_n\| \\ & - 2\lambda_n \langle Nx_n - Nx^*, x_{n+1} - x^* \rangle. \end{aligned} \tag{3.6}$$

Next, we estimate the last term in inequality (3.6). Now,

$$\begin{aligned}2\langle Nx_n - Nx^*, x_{n+1} - x^* \rangle =\ & 2\langle Nx_n - Nx^*, x_{n+1} - x_n \rangle \\ & + 2\langle Nx_n - Nx^*, x_n - x^* \rangle \\ \geq\ & -2\|Nx_n - Nx^*\| \|x_{n+1} - x_n\| \\ & + 2\mu \|Nx_n - Nx^*\|^2 \\ \geq\ & -2\mu \|Nx_n - Nx^*\|^2 - \frac{1}{2\mu} \|x_{n+1} - x_n\|^2 \\ & + 2\mu \|Nx_n - Nx^*\|^2 \\ =\ & -\frac{1}{2\mu} \|x_{n+1} - x_n\|^2.\end{aligned}$$

Substituting this inequality in (3.6) and using Lemma 2.5, we get

$$\begin{aligned}2\lambda_n \eta \|x_{n+1} - x^*\|^2 \leq\ & \phi(x^*, x_n) - \phi(x^*, x_{n+1}) - \phi(x_{n+1}, x_n) \\ & - 2\lambda_n \langle Mx_n - Mx_{n+1}, x_{n+1} - x^* \rangle \\ & + 2\lambda_{n-1} \langle Mx_{n-1} - Mx_n, x_n - x^* \rangle + \gamma \lambda_{n-1} \|x_n - x_{n-1}\|^2 \\ & + \gamma \lambda_{n-1} \|x_{n+1} - x_n\|^2 + \frac{\lambda_n}{2\mu} \|x_{n+1} - x_n\|^2 \\ \leq\ & \phi(x^*, x_n) + \gamma \lambda_{n-1} \|x_n - x_{n-1}\|^2 \\ & + 2\lambda_{n-1} \langle Mx_{n-1} - Mx_n, x_n - x^* \rangle - \phi(x^*, x_{n+1}) \\ & - \Big(\frac{1}{\kappa} - \gamma \lambda_{n-1} - \frac{\lambda_n}{2\mu}\Big) \|x_{n+1} - x_n\|^2 \\ & - 2\lambda_n \langle Mx_n - Mx_{n+1}, x_{n+1} - x^* \rangle. \end{aligned} \tag{3.7}$$

Let $\Theta_n = \phi(x^*, x_n) + \gamma\lambda_{n-1}\|x_n - x_{n-1}\|^2 + 2\lambda_{n-1}\langle Mx_{n-1} - Mx_n, x_n - x^*\rangle$. Then inequality (3.7) can rewritten as

$$2\lambda_n\eta\|x_{n+1} - x^*\|^2 + \Big(\frac{1}{\kappa} - \gamma\lambda_{n-1} - \frac{\lambda_n}{2\mu} - \lambda_n\gamma\Big)\|x_{n+1} - x_n\|^2 \leq \Theta_n - \Theta_{n+1}. \tag{3.8}$$

Observe that

$$\begin{aligned}\Theta_n &= \phi(x^*, x_n) + \gamma\lambda_{n-1}\|x_n - x_{n-1}\|^2 + 2\lambda_{n-1}\langle Mx_{n-1} - Mx_n, x_n - x^*\rangle \\ &\geq \phi(x^*, x_n) + \gamma\lambda_{n-1}\|x_n - x_{n-1}\|^2 - 2\gamma\lambda_{n-1}\|x_{n-1} - x_n\|\|x_n - x^*\| \\ &\geq \phi(x^*, x_n) + \gamma\lambda_{n-1}\|x_n - x_{n-1}\|^2 \\ &\quad - \gamma\lambda_{n-1}\|x_{n-1} - x_n\|^2 - \gamma\lambda_{n-1}\|x_n - x^*\|^2 \\ &\geq \Big(\frac{1}{\kappa} - \gamma\lambda_{n-1}\Big)\|x_n - x^*\|^2. \end{aligned} \tag{3.9}$$

Let $\theta \in (0, \frac{1}{\kappa})$ be fixed. Since $\lambda_n \to 0$ as $n \to \infty$,

$$\lim_{n\to\infty}\Big(\frac{1}{\kappa} - \gamma\lambda_{n-1} - \frac{\lambda_n}{2\mu} - \gamma\lambda_n\Big) = \frac{1}{\kappa} > \theta.$$

Thus, there exists $n_0 \geq 1$ such that

$$\Big(\frac{1}{\kappa} - \gamma\lambda_{n-1} - \frac{\lambda_n}{2\mu} - \gamma\lambda_n\Big) \geq \theta, \ \forall n \geq n_0.$$

In addition,

$$\frac{1}{\kappa} - \gamma\lambda_{n-1} \geq \theta, \ \forall n \geq n_0.$$

Thus, $\{\Theta_n\}$ is nonnegative. Hence,

$$2\lambda_n\eta\|x_{n+1} - x^*\|^2 + \theta\|x_{n+1} - x_n\|^2 \leq \Theta_n - \Theta_{n+1}, \ \forall\ n \geq n_0.$$

Thus, the sequence $\{\Theta_n\}$ is non-increasing. Consequently, the limit of $\{\Theta_n\}$ exists. Thus, from (3.9), we can conclude that $\{x_n\}$ is bounded.

Next, we show that $\{x_n\}$ converges strongly to a solution of problem (1.1). From (3.8), taking the finite sum of both sides, we have

$$\sum_{k=n_0}^{N}\big(2\lambda_k\gamma\|x_{k+1} - x^*\|^2 + \epsilon\|x_{k+1} - x_k\|^2\big) \leq \Theta_{n_0} - \Theta_{k+1} \leq \Theta_{n_0}.$$

Thus,

$$\sum_{k=n_0}^{\infty}\big(2\lambda_k\gamma\|x_{k+1} - x^*\|^2 + \epsilon\|x_{k+1} - x_k\|^2\big) \leq \Theta_{n_0} - \lim_{k\to\infty}\Theta_{k+1} \leq \Theta_{n_0}.$$

This implies that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\|^2 = 0 \quad \text{and} \quad \sum_{n=n_0}^{\infty} \lambda_n \|x_{n+1} - x^*\|^2 < \infty. \tag{3.10}$$

It follows from inequality (3.10) and the fact that $\sum_{n=n_0}^{\infty} \lambda_n = \infty$ that

$$\liminf_{n\to\infty} \|x_{n+1} - x^*\|^2 = 0$$

and thus

$$\liminf_{n\to\infty} \phi(x^*, x_{n+1}) = 0.$$

We recall that

$$\begin{aligned}\Theta_{n+1} = \phi(x^*, x_{n+1}) &+ \lambda_n \gamma \|x_{n+1} - x_n\|^2 \\ &+ 2\lambda_n \langle Mx_n - Mx_{n+1} - x_{n+1} - x^* \rangle.\end{aligned}$$

Using equation (3.10), the boundedness of $\{x_n\}$, the Lipschitz continuity of M and the fact that $\lambda_n \to 0$, we have

$$\lim_{n\to\infty} \big(\lambda_n \gamma \|x_{n+1} - x_n\|^2 + 2\lambda_n \langle Mx_n - Mx_{n+1} - x_{n+1} - x^* \rangle\big) = 0.$$

This implies that

$$\lim_{n\to\infty} \Theta_{n+1} = \lim_{n\to\infty} \phi(x^*, x_{n+1}).$$

Therefore,

$$\lim_{n\to\infty} \|x_{n+1} - x^*\| = 0.$$

This completes the proof. □

Remark 3.3. Theorem 3.2 extends and generalizes many established results in a Banach space that are 2-uniformly convex and uniformly smooth in the literature. It extends:

(1) Theorem 3.1 of Bello et al. [10]. In the sense that their *weak* convergence result can be modified to obtain strong convergence by just setting $N \equiv 0$ in Algorithm 3.1. Furthermore, the dependency of their step-size on the Lipschitz constant of one of the operators can be dispensed with by using the non-summable and diminishing step size we used in Algorithm 3.1 and using our method of proof in Theorem 3.2.

(2) Algorithm 3.3 of Shehu [35], Algorithm 1 of Cholamjiak et al. [19], Algorithm 3.12 of Adamu et al. [6] and other Tseng-type algorithms in the literature. In the sense that the number of function evaluation in the algorithm can be reduced and by just setting $N \equiv 0$ in our proposed Algorithm 3.1 and using our idea of proof the dependency of the step-size on the Lipschitz constant can be dispensed with.

4. Numerical illustrations and applications

In this section, we will give two numerical examples to compare the performance of our proposed algorithm and that of Davis and Yin [21], Malistky and Tam [29] in solving problem (1.1).

Example 4.1. Let A be an $n \times n$ symmetric and positive definite matrix (spdm). Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $Lx := Ax$. Then L is maximal monotone. Let $M : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $Mx := Ax + b$, $b \in \mathbb{R}^n$. Then, M is γ-strongly monotone. Let $N : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $Nx = Ax$. Then N is μ-cocoercive. To implement Algorithms (1.3), (1.4) and (3.1), we will use a particular spdm to define L, M and N on $\mathbb{R}^3$. In algorithms (1.3), (1.4) and (3.1), set

$$
\begin{aligned}
Lx &= \begin{pmatrix} 3 & -2 & 0 \\ -1 & 4 & -2 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \\
Mx &= \begin{pmatrix} 3 & -2 & 0 \\ -1 & 4 & -2 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \\
Nx &= \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.
\end{aligned}
$$

It is not difficult to verify that the coefficient matrices are symmetric and positive definite. Furthermore, it is easy to verify that L is maximal monotone, M is 1-strongly monotone and 5-Lipschitz continuous (where 5 is an estimated value of the Lipschitz constant) and thus, M is maximal monotone. In addition, N is 2-cocoercive and therefore, it is $\frac{1}{2}$-Lipschitz continuous. Moreover, the solution is $x^* = (-0.14, -0.05, -0.35)^T$.

In Algorithm (1.3), we set $\lambda \in (0, 4)$ to be the sequence $\lambda_n = \frac{1}{n+1}$, because we observe using this choice the algorithm gives a better approximation and $\{\alpha_n\} \subset \left(0, \frac{8-\lambda}{4}\right)$ to be $\frac{2n}{n+1}$. In Algorithm (1.4), we set $\lambda \in (0, 0.17)$; to be $\lambda = 0.01$ finally, in our proposed algorithm, we choose $\lambda_n = \frac{1}{n+1}$. Clearly, these parameters satisfy the hypothesis of Algorithms (1.3), (1.4) and (3.1), respectively. To test the robustness of the algorithms, we vary the starting points as follows:

Test 1: $x_0 = (1, 1, 0)^T$ and $x_1 = (-2, 0.5, 1)^T$;

Test 2: $x_0 = (0, 0, 0)^T$ and $x_1 = (0.5, 0.6, -0.7)^T$;

Test 3: $x_0 = (-1, 3, -5)^T$ and $x_1 = (0, -2, 4)^T$;

Test 4: $x_0 = (\frac{2}{3}, \frac{3}{5}, \frac{5}{7})^T$ and $x_1 = (1, 2, 3)^T$.

Setting maximum number of iterations $n = 300$, the results obtained from the simulations are reported in Table 1 and Figures 1 and 2.

TABLE 1. Numerical results with different starting points for Example 4.1

Numerical Results of $\|x_n - x^*\|$ for Example 4.1												
	Algorithm (1.3)				Algorithm (1.4)				Algorithm (3.1)			
n	Test 1	Test 2	Test 3	Test 4	Test 1	Test 2	Test 3	Test 4	Test 1	Test 2	Test 3	Test 4
1	2.36	0.97	4.77	4.09	2.36	0.97	4.77	4.09	2.36	0.97	4.77	4.09
2	0.95	0.57	2.05	1.81	2.34	0.96	4.73	4.07	5.01	1.79	1.61	2.86
3	0.51	0.51	0.67	0.74	2.31	0.95	4.66	4.01	6.97	2.79	2.62	3.26
4	0.41	0.41	0.41	0.45	2.27	0.94	4.58	4.04	8.25	3.61	4.03	1.91
5	0.32	0.32	0.32	0.33	2.24	0.93	4.51	3.98	8.81	4.28	5.22	1.56
10	0.16	0.16	0.16	0.16	2.10	0.87	4.17	3.84	4.08	2.19	2.87	0.51
30	0.057	0.057	0.057	0.057	1.61	0.69	3.13	3.31	3.9E-4	1.4E-4	8.7E-5	6.2E-4
50	0.034	0.034	0.034	0.034	1.23	0.54	2.41	2.85	1.1E-5	7.7E-5	7.3E-5	1.3E-4
100	0.017	0.017	0.017	0.017	0.63	0.32	1.37	1.94	7.2E-5	7.1E-5	7.1E-5	7.3E-5
150	0.011	0.011	0.011	0.011	0.31	0.23	0.82	1.31	7.1E-5	7.1E-5	7.1E-5	7.2E-5
200	8.7E-3	8.7E-3	8.7E-3	8.7E-3	0.17	0.21	0.48	0.86	7.1E-5	7.1E-5	7.1E-5	7.1E-5
300	5.8E-3	5.8E-3	5.8E-3	5.8E-3	0.18	0.24	0.14	0.32	7.1E-5	7.1E-5	7.1E-5	7.1E-5

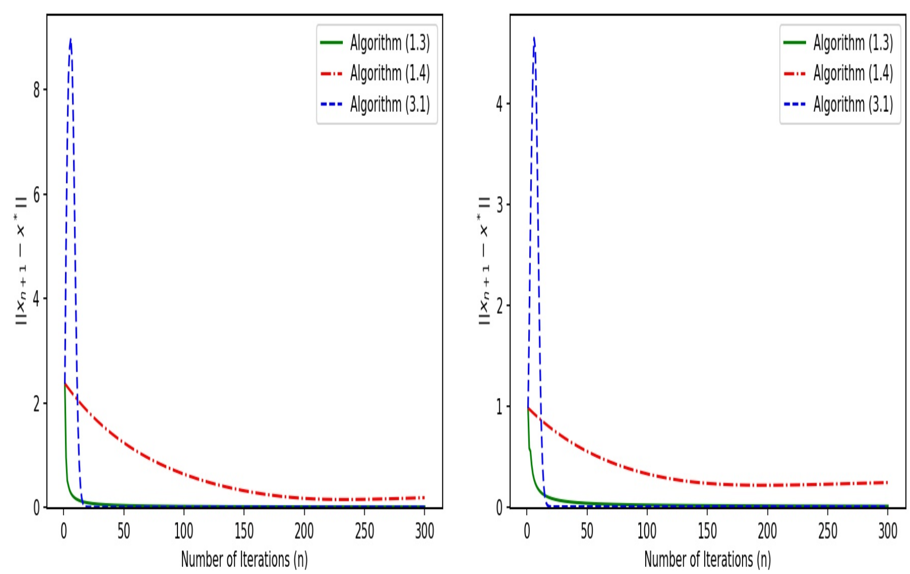

FIGURE 1. Graphical Simulations of Tests 1 and 2 for Algorithms (1.3), (1.4) and (3.1)

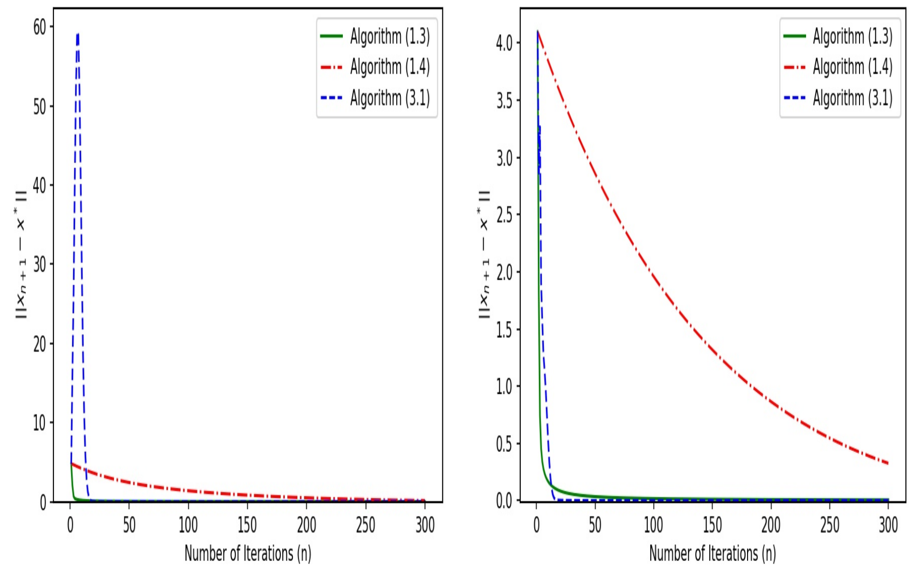

FIGURE 2. Graphical Simulations of Tests 3 and 4 for Algorithms (1.3), (1.4) and (3.1)

In the next example we are going to implement Algorithm 3.1 in the setting of a classical Banach space. Since the theorems of Davis and Yin [21], Malistky and Tam [29] we established in Hilbert spaces, we will not compare the performance of our algorithm with their algorithm in this example.

Example 4.2. In this example, we are going to implement Algorithm (3.1) on the subspace ℓ_p^0 of ℓ_p which consist of finitely many nonzero elements. We recall that

$$\ell_p = \Big\{\{x_n\} \subset \mathbb{R} : \sum_{i=1}^{\infty} |x_i| < \infty\Big\},$$

$$\ell_p^0 = \Big\{\{x_n\} \in \ell_p : \{x_n\} = \{x_1, x_2, \cdots, x_k, 0, 0, 0, \cdots\}\Big\}.$$

We also recall that for $1 < p \leq 2$, ℓ_p spaces are 2-uniformly convex and uniformly smooth. Let $p = 1.5$, $k = 4$. Consider $\ell_{1.5}^o$ with dual space ℓ_3^0. Its is well known that if $1 < p < q < \infty$, $\ell_p \subset \ell_q$. Thus, $\ell_{1.5}^0 \subset \ell_3^0$. Following Alber [8] the duality $J_{1.5}$ map and its inverse J_3 on these subspaces are

$$J_{1.5}(x) = \|x\|_{\ell_{1.5}}^{0.5} y \in \ell_3^0,$$
$$y = \{|x_1|^{-0.5}x_1, |x_2|^{-0.5}x_2, |x_3|^{-0.5}x_3, |x_4|^{-0.5}x_4, 0, 0, \cdots\},$$
$$x = \{x_1, x_2, x_3, x_4, 0, 0, \cdots\}$$

and

$$J_3(x) = \|x\|_{\ell_3}^{-1} y \in \ell_{1.5}^0,$$
$$y = \{|x_1|x_1, |x_2|x_2, |x_3|x_3, |x_4|x_4, 0, 0, \cdots\},$$
$$x = \{x_1, x_2, x_3, x_4, 0, 0, \cdots\},$$

where

$$\|x\|_{\ell_{1.5}} = \Big(\sum_{i=1}^{4} |x_i|^{1.5}\Big)^{\frac{1}{1.5}}$$

and

$$\|x\|_{\ell_3} = \Big(\sum_{i=1}^{4} |x_i|^3\Big)^{\frac{1}{3}}.$$

Remark 4.3. Observe that if $x_i = 0$, $i \in \{1, 2, 3, 4\}$, $J_{1.5}$ is NOT well-defined.

In MATLAB, we constructed a function that returns 0 when $x = \{0, 0, \cdots\}$ and it returns 1 when $x_i = 0$ in computing $|x_i|^{-0.5}$. The following is obtained for testing the function:

$$J_{1.5}(\{1, 0, 3, -0.5, 0, 0, \cdots\}) = \{1.8710, 0, 3.2407, -1.3230, 0, 0, \cdots\}.$$

This new function we constructed took care of the problem raised in Remark 4.3. Now, we are ready to implement Algorithm (3.1) on $\ell_{1.5}^0$.

In Algorithm (3.1), let $L, M, N : \ell_{1.5}^0 \to \ell_3^0$ be defined by

$$Lx = 2J_{1.5}(x),$$

$$Mx = 2J_{1.5}(x)$$

and

$$Nx = \{2|x_1|, 2^2|x_2|, 2^3|x_3|, 2^4|x_4|, 0, 0, \cdots\}.$$

Setting $\lambda_n = \frac{1}{n+1}$ and maximum number of iterations $n = 500$. To test the robustness of the algorithms, we vary the starting points as follows:

Test 1: $x_0 = \{1, 0, 3, -0.5, 0, 0, \cdots\}$ and $x_1 = \{2, 3, 0, 1, 0, 0, \cdots\}$;

Test 2: $x_0 = \{-0.1, -0.2, 0.3, 0.4, 0, 0, \cdots\}$ and $x_1 = \{2, 4, 6, 8, 0, 0, \cdots\}$.

The results of the numerical simulations are presented in Table 2 and Figure 3 below:

Table 2. Numerical results with different starting points for Example 4.2

Numerical Results of $\|x_{n+1} - x_n\|_{\ell_3}$ for Example 4.2		
	Test 1	Test 2
n	$\|x_{n+1} - x_n\|_{\ell_3}$	$\|x_{n+1} - x_n\|_{\ell_3}$
1	5.5936	12.8774
2	12.9783	56.0761
50	1.85E-3	2.07E-3
100	2.43E-4	2.73E-4
150	7.32E-5	8.17E-5
200	3.15E-5	3.46E-5
250	1.71E-5	1.79E-5
300	1.09E-5	1.06E-5
350	7.75E-6	7.09E-6
400	5.89E-6	5.09E-6
450	4.72E-6	3.87E-6
500	3.93E-6	3.08E-6

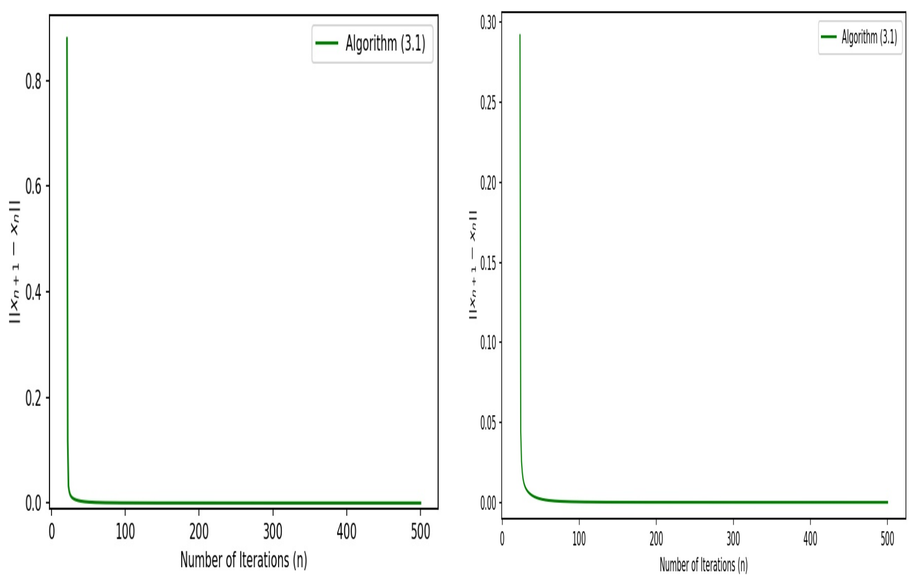

Figure 3. Graphical Simulations of Tests 1 and 2 for Algorithm (1.3)

Conclusion: This paper presents an algorithm with diminishing and non-summable step-size for approximating zeros of sum of three monotone operators in the setting of a real Banach space. A nice and interesting feature of the proposed algorithm is the fact that the step-size does not depend on the knowledge of Lipschitz or cocoercive constant of any of the operators involved. The fact that the approach used in dispensing this dependency does not follow the well-known approaches in the literature made the method of proof of convergence new, technical and innovative. To the best of our knowledge, this is the first paper that considered the inclusion problem (1.1) in the setting of Banach spaces.

Furthermore, numerical illustrations are presented to support the theory established in the paper. Finally, the proposed method extends and generalizes several methods established in the literature for approximating zeros of sum of two monotone operators as seen in Remark 3.3.

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