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SOLVABILITY FOR A CLASS OF FDES WITH SOME (e_1, e_2, θ)-NONLOCAL ANTI PERIODIC CONDITIONS AND ANOTHER CLASS OF KDV BURGER EQUATION TYPE

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Abstract. In this paper, we work two different problems. First, we investigate a new class of fractional differential equations involving Caputo sequential derivative with some $(e_1, e_2, \theta)-$periodic conditions. The existence and uniqueness of solutions are proven. The stability of solutions is also discussed. The second part includes studying traveling wave solutions of a conformable fractional Korteweg-de Vries-Burger (KdV Burger) equation through the Tanh method. Graphs of some of the waves are plotted and discussed, and a conclusion follows.

1. Introduction

Fractional Differential Equations (FDEs) generalize the classical integer-order differential equations to non-integer orders, allowing for the modeling of systems with memory and long-range dependence. FDEs have applications in

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many fields, including physics, engineering, biology, and finance. Analytical and numerical methods for solving FDEs are an active area of research due to their complexity and nonlocality, see for instance the papers [4, 5]. In this sense, scientists are especially interested in the problems with boundary values of FDEs, see reference [19].

In particular, we mention the periodic affine boundary problem that describes physical phenomena [31]. Also, the investigation into the existence and uniqueness of solutions to FDEs is still receiving great attention from mathematicians, through the approach of Caputo and Riemann-Louiville. Readers may be referred to references [9, 13, 14]. In this aspect, the stability problems of Ulam Hyers have been given much attention by researchers, and many stability problems of various FDEs have been investigated, see [11, 30].

In [12], Gouari et al. have studied the following three-sequential fractional problem of Duffing type:

$$\begin{cases} D^{\alpha}(D^{\beta}(D^{\delta}y(t))) + f(t,y(t),D^{p}y(t)) + g(t,y(t),I^{q}y(t)) + h(t,y(t)) = l(t), \\ y(0) = \xi \in \mathbb{R},\ \ y(1) = \displaystyle\int_0^{\eta} y(s)ds,\ \ 0<\eta<1, \\ I^{e}y(\theta) = D^{\delta}y(1),\ \ 0<u<1, \\ 0<\alpha,\beta,\delta,p\leq 1, \quad q>0, \quad t\in J, \end{cases}$$

where $J=[0,1]$, $D^{\alpha}, D^{\beta}, D^{\delta}, D^{p}$ are derivatives of Caputo, I^{q} denotes the Riemann-Liouville fractional integral of order q, and $f, g: J\times\mathbb{R}^2\to\mathbb{R}$ are two given functions, $h: J\times\mathbb{R}\to\mathbb{R}$ is another given function and l is defined on J. The authors have proved the existence and uniqueness of solutions by application of Banach contraction principle. Then, by means of Schaefer fixed point theorem, they have studied the existence of at least one solution for the problem.

Also in [29], the authors have been concerned with the following Duffing-type problem:

$$\begin{cases} D^{\gamma}D^{\beta}D^{\alpha}z(t)+kf\left(t,D^{\alpha}z(t)\right)+g\left(t,z(t),D^{p}z(t)\right)+h\left(t,z(t),J^{q}(z(t))\right)=L(t), \\ z(0)=A_1\in\mathbb{R}, D^{\alpha}z(0)=A_2\in\mathbb{R}, J^{\alpha}z(1)=A_3\in\mathbb{R}, \\ 0\leq p<\alpha\leq 1, 0\leq\beta,\gamma\leq 1, 1<\alpha+\beta\leq 2, 1<\beta+\gamma\leq 2, t\in I, \end{cases}$$

where $I:=[0,1]$, the derivatives of the problem are in the sense of Caputo, J^{q} is the Riemann-Liouville integral with $q\geq 0, f: I\times\mathbb{R}^2\to\mathbb{R}, g: I\times\mathbb{R}^3\to\mathbb{R}, h: I\times\mathbb{R}^3\to\mathbb{R}$ and $L: I\to\mathbb{R}$ are four given functions. Then, in [10], Gao et al. have investigated the following sequential FDE with affine periodic boundary conditions:

$$\begin{cases} \left({}^C D^{\beta} + \lambda {}^C D^{\alpha}\right) z(t) = g(t, z(t)), \quad t \in [0, T], \\ z(T) = az(0), \\ z^{'}(T) = az^{'}(0), \end{cases}$$

where ${}^C D^p$ expresses the Caputo fractional derivative, $p \in \alpha, \beta$ with $0 < \alpha < 1 < \beta < 2$, $\beta = \alpha + 1$, $\lambda, a \in \mathbb{R}$ with $g(t, z) : [0, T] \times C([0, T]; \mathbb{R}) \to \mathbb{R}$ is a continuous function.

Also in [24], the author has been concerned with the study of the uniqueness, Ulam-Hyers stability and Ulam-Hyers-Rassias stability of solutions for the following sequential fractional pantograph equation:

$$\begin{cases} \left[D^{\alpha} + kD^{\beta}\right] u(t) = \phi\left(t, u(t), u(\eta t), D^{\beta} u(\eta t)\right), \ t \in [0, T], \\ u(0) = f(u), \quad u(T) = \theta, \theta \in \mathbb{R}, \end{cases}$$

where $k \in \mathbb{R}^+$, $0 < \eta < 1$, $1 < \alpha \leq 2$, $0 < \beta \leq 1$, D^{α}, D^{β} are the Caputo type fractional derivatives, $\phi : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $f : C([0, T], \mathbb{R}) \to \mathbb{R}$ are given continuous functions.

Very recently, Abdelnebi and Dahmani [2] have studied the existence, uniqueness, and stability of solutions for the following Van der Pol-Duffing (VdPD) jerk equation:

$$\begin{cases} D^{\alpha}\left(D^{2-\beta} + \lambda D^{\alpha}\right) x(t) + k_1 f_1\left(t, x(t), D^{\alpha} x(t)\right) + k_2 f_2\left(t, x(t), J^p x(t)\right) = h(t), \\ x(1) = 0, \quad D^{1-(\alpha-\beta)} D^{\alpha-\beta} x(1) = A^* \in \mathbb{R}, \quad x(T) = 0, \\ 0 \leq \beta < \alpha \leq 1, \quad 0 \leq \alpha + \beta < 1, \quad 0 < p, \quad t \in I, \end{cases}$$

where $D^{\alpha}, D^{2-\beta}$ are the Caputo-Hadamard fractional derivatives, J^p is the Hadamard fractional integral $I = [1, T]$, k_1, k_2 are real constants and the functions f_1, f_2 and h are continuous.

In [6], the authors have investigated the following problem:

$$\begin{cases} {}^c D^{\alpha_1} {}^c D^{\alpha_2} {}^c D^{\alpha_3} [{}^c D^{\alpha_4} u(t) - \lambda f(t) u(t)] = g(t, u(t), {}^c D^{\alpha_2} u(t), {}^c D^{\alpha_3} u(t), {}^c D^{\alpha_4} u(t)), \\ u(0)) = 0, \ \ u(1) = a_1, \\ {}^c D^{\alpha_4} u(0) = a_2, \\ {}^c D^{\alpha_4} u(1) = 0, \\ t \in J = [0, 1], \end{cases}$$

where ${}^c D^{\alpha_i}$, are Caputo fractional derivatives, $0 < \alpha_i \leq 1$, $i = 1, .., 4$, $\alpha_2 < \alpha_4$, $\alpha_3 < \alpha_4$, $\lambda > 0$, $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ and $g : [0, 1] \times \mathbb{R}^4 \to \mathbb{R}$ are continuous.

In the present paper, we shall be concerned with two different problems:

In the first part, we study the following problem:

$$\begin{cases} D^{\alpha}D^{\beta}D^{\gamma}s(t)+\lambda D^{\alpha}D^{\beta}s(t)=m(t,s(t),D^{\delta}s(t))+n(t,s(t),I^{p}s(t))+r(t,s(t))+l(t), \\ t\in[0,1], \\ s(1)=e_1 s(0), \\ D^{\gamma}s(1)=e_2 D^{\gamma}s(0), \\ D^{\gamma}s(0)+\lambda s(0)=\theta, \\ 0<\alpha,\beta\leq 1, \quad 0<\delta<\gamma\leq 1, \quad e_1,e_2,p\in\mathbb{R}_{+}^{*},\lambda\in\mathbb{R}, \end{cases} \tag{1.1}$$

where $J:=[0,1]$, the functions m,n,r and l will be specified later, the operators $D^{\alpha},D^{\beta},D^{\gamma}$ and D^{δ} are the derivatives in the sense of Caputo.

In the second part of our paper, we will use the Tanh method to find new traveling wave solutions for an evolution equation of KdV Burgers type with time and space conformable fractional derivative. The problem is the following:

$$\mathbb{T}_t^{\alpha}u+\nu(u\mathbb{T}_x^{\beta}u)+\eta\mathbb{T}_x^{2\beta}u+\mu\mathbb{T}_x^{3\beta}u=0, \tag{1.2}$$

where $\mathbb{T}_x^{\beta},\mathbb{T}_t^{\alpha}$ are the conformable fractional derivative with $0<\alpha,\beta\leq 1$.

To motivate the second part, we note that traveling waves are observed in many areas of sciences and applications. Many powerful numerical methods have been implemented to obtain solutions of partial FDFs, such as the exp-function method [3, 16, 25], the (G'/G) method [34], and the Tanh method. This method is one of the most effective algebraic methods for finding exact solutions to nonlinear differential equations. It was presented by Malfliet [20] and then modified and extended by Wazwaz [32] for the computation of exact traveling wave solutions.

For the above-motivating method and to cite some of the papers that have motivated the present part, we begin by the reference [26], where, Rakah et al. have been concerned with finding traveling wave solutions for the following evolution problem [18]:

$$T_t^{2\alpha}u+T_x^{\beta}(G(u)T_x^{3\beta}u)+T_x^{\beta}(H(u)T_x^{\beta}u)=F(u),$$

where T_x^{β},T_t^{α} are the conformable fractional derivatives with $0<\alpha,\beta\leq 1$ and f,G,H are some given functions.

Also in [21], the authors have investigated the 3D-fractional Wazwaz-Benjam in-Bona-Mahony equation equations that involve some sequential conformable fractional derivatives. Several solutions containing hyperbolic and trigonometric function solutions have also been obtained by applying the modified extended Tanh method.

Very recently in [8], Dahmani et al. have presented an $(n+1)$-dimensional extended Tanh function method to investigate nonlinear conformable fractional differential equations using Khalil conformable approach; in fact, they have presented new traveling wave solutions for the $(1+3)$-dimensional conformable time and space fractional Burgers equation.

Then, in [28] the authors have obtained new exact solutions to the following nonlinear fractional Klein-Gordon equation via extended tanh-function method with conformable fractional derivative:

$$D_{tt}^{2\alpha}u(x,t) + D_{xx}^{2\alpha}u(x,t) + pu(x,t) - qu^2(x,t) = 0.$$

In [33], the author has been concerned with the $(\frac{G}{G'})$ expansion method to find the exact solutions of nonlinear fractional partial differential equations with the modified Riemann-Liouville derivative by Jumarie [17]:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \omega u \frac{\partial^\beta u}{\partial x^\beta} + \eta \frac{\partial^{2\beta} u}{\partial x^{2\beta}} + \nu \frac{\partial^{3\beta} u}{\partial x^{3\beta}} = 0, \quad t > 0,\ 0 < \alpha, \beta \leq 1.$$

In this sense, the two authors of the paper [15] have presented an implementation of Petrov-Galerkin technique for solutions of the following time-fractional KdV Burger equation:

$$D_t^\alpha u + \varepsilon u u_x - v u_{xx} + \mu u_{xxx} = 0,$$

where ε, v and μ are constants and α represents the order of fractional derivative. In this numerical technique, the fractional derivative has been discretized by the Grnwald-Letnikov derivative.

One way to connect the two parts of the present paper is by using the existence and uniqueness result for FDEs involving Caputo derivatives to analyze the numerical solutions obtained using the Tanh method with the conformable fractional derivative approach on the KdV Burger equation. Specifically, the existence and uniqueness result can provide theoretical guarantees for the accuracy and convergence of the numerical solutions obtained using the Tanh method with conformable derivatives, while the numerical results obtained from the Tanh method can be used to validate and test the theoretical results obtained from the existence and uniqueness analysis. Furthermore, the KdV Burger is a well-known model for wave propagation in nonlinear media, and studying its solutions with FDEs and numerical methods can provide insights into the behavior of these waves in realistic physical systems. Therefore, the two studies can complement each other by providing both theoretical and numerical insights into the behavior of solutions of FDEs on the KdV Burger equation.

This paper is organized as follows: In the next Section, we review some definitions and properties of the Caputo derivatives. In Section 3, we prove the main result on the existence and uniqueness of solutions for the proposed class of FDEs as well as the result of its Ulam Hyers stability. In the fourth section, we introduce the time and space KdV Burger equation with conformable fractional derivative and we apply the Tanh method to obtain new traveling wave solutions to "our equation". Finally, in Section 5, we summarize the outcomes of this paper in the conclusion section.

2. Caputo derivatives

In this part, we will introduce some definitions of the integral and partial derivation by Caputo's approach, with some properties related to it. Using these tools, we will address the integral solution of the problem, and this will allow us to discuss the results of the existence of solutions, their uniqueness and stability.

We need to introduce the Caputo derivatives. For more details, we refer to the reference [23].

Definition 2.1. Let $\alpha > 0$ and $f : J \longmapsto \mathbb{R}$ be a continuous function. The Riemann-Liouville integral is defined by:

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau.$$

Definition 2.2. Let us take $f \in C^n(J, \mathbb{R})$ and $n-1 < \alpha \leq n$, so the Caputo derivative is defined by:

$$\begin{aligned} D^{\alpha} f(t) &= I^{n-\alpha} \frac{d^n}{dt^n}(f(t)) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds. \end{aligned}$$

To study (1.1), we need the following two results [23]:

Lemma 2.3. *Let $n \in \mathbb{N}^*$, and $n-1 < \alpha < n$. Then the general solution of $D^{\alpha} y(t) = 0; t \in J$ is:*

$$y(t) = \sum_{i=0}^{n-1} c_i t^i,$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, .., n-1$.

Lemma 2.4. *If $n \in \mathbb{N}^*$, and $n-1 < \alpha < n$, then we have*

$$I^{\alpha} D^{\alpha} y(t) = y(t) + \sum_{i=0}^{n-1} c_i t^i, \quad t \in J$$

and $c_i \in \mathbb{R},\ i = 0, 1, 2, .., n-1$.

Now, prove the following lemma.

Lemma 2.5. *Let* $G \in C(J)$*. Then the problem*

$$\begin{cases} D^{\alpha}D^{\beta}D^{\gamma}s(t) + \lambda D^{\alpha}D^{\beta}s(t) = F(t),\ t \in J, \\ s(1) = e_1 s(0), \\ D^{\gamma}s(1) = e_2 D^{\gamma}s(0), \\ D^{\gamma}s(0) + \lambda s(0) = \theta, \\ 0 < \alpha, \beta \leq 1, \quad 0 < \gamma \leq 1, \quad e_1, e_2, \delta, p \in \mathbb{R}_+^*, \lambda \in \mathbb{R} \end{cases} \tag{2.1}$$

admits the following expression as the integral solution

$$\begin{aligned} s(t) &= I^{\alpha+\beta+\gamma}F(t) - \lambda I^{\gamma}s(t) \\ &- \Bigg[I^{\alpha+\beta}F(1) + \theta(1-e_2) + \lambda(e_2 - e_1)\Upsilon\Bigg(I^{\alpha+\beta+\gamma}F(1) \\ &- \lambda I^{\gamma}s(1) + \frac{\theta}{\Gamma(\gamma+1)} - \frac{\Gamma(\beta+1)}{\Gamma(\beta+\gamma+1)}\Big(I^{\alpha+\beta}F(1) + \theta(1-e_2)\Big)\Bigg)\Bigg] \\ &\times \frac{\Gamma(\beta+1)t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)} + \theta\frac{t^{\gamma}}{\Gamma(\gamma+1)} + \Upsilon\Bigg(I^{\alpha+\beta+\gamma}F(1) - \lambda I^{\gamma}s(1) \\ &+ \frac{\theta}{\Gamma(\gamma+1)} - \frac{\Gamma(\beta+1)}{\Gamma(\beta+\gamma+1)}\Big(I^{\alpha+\beta}F(1) + \theta(1-e_2)\Big)\Bigg), \end{aligned} \tag{2.2}$$

where

$$\Upsilon = \frac{\Gamma(\beta+\gamma+1)}{\lambda\Gamma(\beta+1)(e_2-e_1) + (e_1-1)\Gamma(\beta+\gamma+1)}$$

and

$$(e_2 - e_1)(e_1 - 1) \neq \lambda\Gamma(\beta+\gamma+1)\Gamma(\beta+1).$$

Proof. We use the properties established in Lemma 2.4 to (1.1), so we observe that

$$\begin{aligned} D^{\gamma}s(t) + \lambda s(t) &= I^{\alpha+\beta}F(t) + k_0\frac{t^{\beta}}{\Gamma(\beta+1)} + k_1, \\ s(t) &= I^{\alpha+\beta+\gamma}F(t) - \lambda I^{\gamma}s(t) - k_0\frac{t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)} + k_1\frac{t^{\gamma}}{\Gamma(\gamma+1)} + k_2. \end{aligned} \tag{2.3}$$

Taking into account the initial conditions, we can obtain

$$\begin{aligned} s(1) &= e_1 s(0), \\ D^{\gamma}s(1) &= e_2 D^{\gamma}s(0), \end{aligned}$$

$$D^{\gamma}s(0)+\lambda s(0)=\theta.$$

So, we get

$$\begin{aligned}
k_0 &= I^{\alpha+\beta}F(1)+\theta(1-e_2)+\lambda(e_2-e_1)\Upsilon\Bigg(I^{\alpha+\beta+\gamma}F(1)-\lambda I^{\gamma}s(1)+\frac{\theta}{\Gamma(\gamma+1)}\\
&\quad -\frac{\Gamma(\beta+1)}{\Gamma(\beta+\gamma+1)}\Big(I^{\alpha+\beta}F(1)+\theta(1-e_2)\Big)\Bigg),\\
k_1 &= \theta,\\
k_2 &= \Upsilon\Bigg(I^{\alpha+\beta+\gamma}F(1)-\lambda I^{\gamma}s(1)+\frac{\theta}{\Gamma(\gamma+1)}\\
&\quad -\frac{\Gamma(\beta+1)}{\Gamma(\beta+\gamma+1)}\times\Big(I^{\alpha+\beta}F(1)+\theta(1-e_2)\Big)\Bigg).
\end{aligned}$$

We end the proof of the above lemma. □

In what follows, we apply the theory of fixed point on Banach spaces to study the problem. So, we shall consider the Banach space

$$T:=\{s\in C(J,\mathbb{R}),D^{\delta}s\in C(J,\mathbb{R})\}$$

and

$$\|s\|_T=Max\{\|s\|_\infty\ ,\|D^{\delta}s\|_\infty\},$$

where

$$\|s\|_\infty=\sup_{t\in J}|s(t)|\ ,\|D^{\delta}s\|_\infty=\sup_{t\in J}|D^{\delta}s(t)|.$$

Then, we consider the operator $Z:T\to T$,

$$\begin{aligned}
Zs(t) &= I^{\alpha+\beta+\gamma}F_s^*(t)-\lambda I^{\gamma}s(t)-\Bigg[I^{\alpha+\beta}F_s^*(1)+\theta(1-e_2)+\lambda(e_2-e_1)\\
&\quad\times\Upsilon\Bigg(I^{\alpha+\beta+\gamma}F_s^*(1)-\lambda I^{\gamma}s(1)+\frac{\theta}{\Gamma(\gamma+1)}-\frac{\Gamma(\beta+1)}{\Gamma(\beta+\gamma+1)}\Big(I^{\alpha+\beta}F_s^*(1)\\
&\quad+\theta(1-e_2)\Big)\Bigg)\Bigg]\frac{\Gamma(\beta+1)t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)}+\theta\frac{t^{\gamma}}{\Gamma(\gamma+1)}+\Upsilon\Bigg(I^{\alpha+\beta+\gamma}F_s^*(1)-\lambda I^{\gamma}s(1)\\
&\quad+\frac{\theta}{\Gamma(\gamma+1)}-\frac{\Gamma(\beta+1)}{\Gamma(\beta+\gamma+1)}\Big(I^{\alpha+\beta}F_s^*(1)+\theta(1-e_2)\Big)\Bigg),
\end{aligned}$$

where

$$F_s^*(t)=m(t,s(t),D^{\delta}s(t))+n(t,s(t),I^{p}s(t))+r(t,s(t))+l(t).$$

3. Main results

The following hypotheses are only sufficient and one can replace them with other conditions like the notions of Caratheodory functions or the measurable functions, instead of Lipchitz functions. We impose what follows:

(Q1) The functions m and n defined on $J \times \mathbb{R}^2$, r defined on $J \times \mathbb{R}$ and l defined on J are continuous.

(Q2) There exist nonnegative constants $\vartheta_{m1}, \vartheta_{m2}, \vartheta_{n1}, \vartheta_{n2}$, such that for any t $\in J$, $s_i, s_i^* \in \mathbb{R}$,

$$\begin{aligned}|m(t, s_1, s_2) - m(t, {s_1}^*, {s_2}^*)| &\leq \sum_{i=1}^{2} \vartheta_{mi}|s_i - {s_i}^*|,\\ |n(t, s_1, s_2) - n(t, {s_1}^*, {s_2}^*)| &\leq \sum_{i=1}^{2} \vartheta_{ni}|s_i - {s_i}^*|.\end{aligned}$$

And for any t $\in J$, $s, s' \in \mathbb{R}$,

$$|h(t, s) - h(t, s')| \leq R|s - s'|.$$

It is considered that

$$M := \max(\vartheta_{m1}, \vartheta_{m2}), N := \max(\vartheta_{n1}, \vartheta_{n2}).$$

Also, we put

$$\begin{aligned}\Xi_1 = &\frac{1}{\Gamma(\alpha+\beta+\gamma+1)} + \frac{|\lambda|}{\Gamma(\gamma+1)} + \frac{\Gamma(\beta+1)}{\Gamma(\beta+\gamma+1)}\\ &\times \left(\frac{1}{\Gamma(\alpha+\beta+1)} + |\lambda(e_2 - e_1)|\Delta^*\right) + \Delta^*,\end{aligned}$$

$$\begin{aligned}\Xi_2 = &\frac{1}{\Gamma(\alpha+\beta+\gamma-\delta+1)} + \frac{|\lambda|}{\Gamma(\gamma-\delta+1)} + \frac{\Gamma(\beta+1)}{\Gamma(\beta+\gamma-\delta+1)}\\ &\times \left(\frac{1}{\Gamma(\alpha+\beta+1)} + |\lambda(e_2 - e_1)|\Delta^*\right),\end{aligned}$$

where

$$\Delta^* = |\Upsilon|\left(\frac{1}{\Gamma(\alpha+\beta+\gamma+1)} + \frac{|\lambda|}{\Gamma(\gamma+1)} + \frac{\Gamma(\beta+1)}{\Gamma(\beta+\gamma+1)\Gamma(\alpha+\beta+1)}\right).$$

In the following, we prove the existence and uniqueness of solution by application of Banach contraction principle. Then, we present an example to show the applicability of the main result. Also, we investigate the stability of solutions (the Ulam-Hyers and the generalized Ulam-Hyers stabilities).

3.1. **A unique solution.** We prove the following result:

Theorem 3.1. *Assume that* (Q_1) *and* (Q_2) *are satisfied. Then, the problem* (1.1) *has a unique solution, under the condition that* $\Xi < \chi_{m,n,r}^{-1}$ *such that*

$$\Xi := max\left\{\Xi_1, \Xi_2\right\} \ \ and \ \ \chi_{m,n,r} = R + 2M + N + \frac{N}{\Gamma(p+1)}.$$

Proof. We proceed to prove that Z is a contraction mapping. For $(s, s^{'}) \in X^2$, we can write

$$\begin{aligned}\|Zs - Zs^{'}\|_\infty \leq \chi_{m,n,r}\Bigg[&\frac{1}{\Gamma(\alpha+\beta+\gamma+1)} + \frac{|\lambda|}{\Gamma(\gamma+1)} + \frac{\Gamma(\beta+1)}{\Gamma(\beta+\gamma+1)} \\ &\times\left(\frac{1}{\Gamma(\alpha+\beta+1)} + |\lambda(e_2-e_1)|\Delta^*\right) + \Delta^*\Bigg]\|s - s^{'}\|_T \\ \leq{}& \chi_{m,n,r}\Xi_1\|s - s^{'}\|_T.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}D^\delta Zs(t) = I^{\alpha+\beta+\gamma-\delta}F_s^*(t) - \lambda I^{\gamma-\delta}s(t) - \Bigg[&I^{\alpha+\beta}F_s^*(1) + \theta(1-e_2) + \lambda(e_2-e_1) \\ &\times\Upsilon\Bigg(I^{\alpha+\beta+\gamma}F_s^*(1) - \lambda I^\gamma s(1) + \frac{\theta}{\Gamma(\gamma+1)} - \frac{\Gamma(\beta+1)}{\Gamma(\beta+\gamma+1)} \\ &\times\left(I^{\alpha+\beta}F_s^*(1) + \theta(1-e_2)\right)\Bigg)\Bigg]\frac{\Gamma(\beta+1)t^{\beta+\gamma-\delta}}{\Gamma(\beta+\gamma-\delta+1)} + \theta\frac{t^{\gamma-\delta}}{\Gamma(\gamma-\delta+1)}\end{aligned}$$

and

$$\begin{aligned}\|D^\delta Zs - D^\delta Zs^{'}\|_\infty \leq \chi_{m,n,r}\Bigg[&\frac{1}{\Gamma(\alpha+\beta+\gamma-\delta+1)} + \frac{|\lambda|}{\Gamma(\gamma-\delta+1)} \\ &+ \frac{\Gamma(\beta+1)}{\Gamma(\beta+\gamma-\delta+1)}\left(\frac{1}{\Gamma(\alpha+\beta+1)} + |\lambda(e_2-e_1)|\Delta^*\right)\Bigg] \\ &\times\|s - s^{'}\|_T \\ \leq{}& \chi_{m,n,r}\Xi_2\|s - s^{'}\|_T.\end{aligned}$$

Consequently, we observe that

$$\|Zs - Zs^{'}\|_T \ \ \leq \ \ \chi_{m,n,r}\Xi\|s - s^{'}\|_T.$$

□

3.2. **An illustrative example.**

Example 3.2. We consider the following problem:

$$\begin{cases} D^{\frac{9}{10}} D^{\frac{3}{5}} D^{\frac{4}{5}} s(t) + 2D^{\frac{9}{10}} D^{\frac{3}{5}} s(t) = \left(\frac{1}{14e^{t+3}} \cos(s(t)) + \frac{1}{12} D^{\frac{1}{2}} s(t) + \frac{\cos(t+8)}{3} \right) \\ \quad + \left(\frac{2}{31} s(t) + \frac{\sin(2+t^3)}{\pi(31+t)} + \frac{3}{40} I^{\frac{1}{5}} s(t) \right) + \left(\dfrac{e^2-2}{30} s(t) + \frac{1}{15+t^2} \right) + ln(t^2+1), \\ s(1) = \frac{1}{2} s(0), \\ D^{\frac{4}{5}} s(1) = \frac{2}{3} D^{\frac{4}{5}} s(0), \\ D^{\frac{4}{5}} s(0) + 2s(0) = 3, \end{cases}$$

where we take

$$\begin{aligned} m(t, s_1, s_2) &= \frac{1}{14e^{t+3}} \cos(s_1) + \frac{1}{12} s_2 + \frac{\cos(t+8)}{3}, \\ n(t, s_1, s_2) &= \frac{2}{31} s_1 + \frac{\sin(2+t^3)}{\pi(31+t)} + \frac{3}{40} s_2, \\ r(t, s) &= \frac{e^2-2}{30} s + \frac{1}{15+t^2}, \\ l(t) &= ln(t^2+1) \end{aligned}$$

and

$$\begin{aligned} &\Upsilon_1 = 0.1178, \quad \Upsilon_2 = 0.1219, \\ &\Upsilon = max \{\Upsilon_1, \Upsilon_2\} = 0.1219. \end{aligned}$$

The conditions of Theorem 3.1 hold. Therefore, our example has a unique solution on $[0, 1]$.

3.3. **Ulam type stabilities.**

Definition 3.3. The equation (1.1) has the Ulam Hyers stability if there exists a real number $\Theta > 0$ such that for each $\zeta > 0$, $t \in J$ and for each $s \in T$ solution of the inequality

$$\begin{aligned} \big| D^{\alpha} D^{\beta} D^{\gamma} s(t) &+ \lambda D^{\alpha} D^{\beta} s(t) - m(t, s(t), D^{\delta} s(t)) \\ &- n(t, s(t), I^{p} s(t)) - r(t, s(t)) - l(t) \big| \leq \zeta, \end{aligned} \tag{3.1}$$

there exists $s^* \in T$ a solution of (1.1) such that

$$\|s - s^*\|_T \leq \Theta\zeta.$$

Definition 3.4. The equation (1.1) has the Ulam Hyers stability in the generalized sense if there exists $\varpi \in C(\mathbb{R}^+, \mathbb{R}^+)$; $\varpi(0) = 0$ such that for each

$\zeta > 0$ and for any $s \in T$ solution of (3.1), there exists a solution $s^* \in T$ of (1.1) such that

$$\|s - s^*\|_T < \varpi(\zeta).$$

Now, we have:

Theorem 3.5. *Further, assume that the conditions of Theorem* 3.1 *are satisfied. Then, the problem* (1.1) *is Hyers-Ulam stable.*

Proof. Let $s \in T$ be a solution of (3.1) and let, by Theorem 3.1, $s^* \in T$ be the unique solution of (1.1). By integration of (3.1), we obtain

$$\Bigg| s(t) - I^{\alpha+\beta+\gamma}F_s^*(t) + \lambda I^\gamma s(t) + \Bigg[I^{\alpha+\beta}F_s^*(1) + \theta(1-e_2) + \lambda(e_2 - e_1) \\ \times \Upsilon\Bigg(I^{\alpha+\beta+\gamma}F_s^*(1) - \lambda I^\gamma s(1) + \frac{\theta}{\Gamma(\gamma+1)} - \frac{\Gamma(\beta+1)}{\Gamma(\beta+\gamma+1)}\Big(I^{\alpha+\beta}F_s^*(1) \\ + \theta(1-e_2)\Big)\Bigg)\Bigg] \frac{\Gamma(\beta+1)t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)} - \theta\frac{t^\gamma}{\Gamma(\gamma+1)} - \Upsilon\Bigg(I^{\alpha+\beta+\gamma}F_s^*(1) - \lambda I^\gamma s(1) \\ + \frac{\theta}{\Gamma(\gamma+1)} - \frac{\Gamma(\beta+1)}{\Gamma(\beta+\gamma+1)}\Big(I^{\alpha+\beta}F_s^*(1) + \theta(1-e_2)\Big)\Bigg) \| \leq \frac{\zeta}{\Gamma(\alpha+\beta+\gamma+1)}. \tag{3.2}$$

Using (3.1) and (3.2), we get

$$\|s - s^*\|_\infty \leq \frac{\zeta}{\Gamma(\alpha+\beta+\gamma+1)} + \chi_{m,n,r}\Bigg[\frac{1}{\Gamma(\alpha+\beta+\gamma+1)} + \frac{|\lambda|}{\Gamma(\gamma+1)} \\ + \frac{\Gamma(\beta+1)}{\Gamma(\beta+\gamma+1)}\left(\frac{1}{\Gamma(\alpha+\beta+1)} + |\lambda(e_2 - e_1)|\Delta^* \right) + \Delta^* \Bigg] \\ \times \|s - s'\|_\infty. \tag{3.3}$$

So

$$\|s - s^*\|_\infty \leq \frac{\zeta}{\Gamma(\alpha+\beta+\gamma+1)} + \chi_{m,n,r}\Xi_1\|s - s^*\|_\infty.$$

Therefore, we have

$$\|s - s^*\|_\infty \leq \frac{\zeta}{\Gamma(\alpha+\beta+\gamma+1)(1-\chi_{m,n,r}\Xi_1)} \leq \zeta\ \Theta.$$

On the other hand, we have

$$\|D^\delta(s - s^*)\|_\infty \leq \frac{\zeta}{\Gamma(\alpha+\beta+\gamma-\delta+1)(1-\chi_{m,n,r}\Xi_2)} \leq \zeta\ \Theta^*.$$

Thus,

$$\|s - s^*\|_T \quad \leq \quad \zeta\big(\ \Theta + \Theta^*\big).$$

Hence, (1.1) has the Ulam–Hyers stability. □

Remark 3.6. In the case $\varpi(\zeta) = \zeta\big(\Theta + \Theta^*\big)$, we obtain the generalised Ulam-Hyers stability for (1.1).

4. Time and space conformable fractional KdV Burger Equation

Let us consider the following problem:

$$\mathbb{T}_t^{\alpha} u + \nu(u\mathbb{T}_x^{\beta} u) + \eta\mathbb{T}_x^{2\beta} u + \mu\mathbb{T}_x^{3\beta} u = 0,$$

where, $\mathbb{T}_x^{\beta}, \mathbb{T}_t^{\alpha}$ are the conformable fractional derivative with $0 < \alpha, \beta \leq 1$.

It is to note that when $\alpha = \beta = 1$, the above conformable problem is transformed into the classical nonlinear KdV Burger equation:

$$u_t + \nu(uu_x) + \eta u_{xx} + \mu u_{xxx} = 0. \tag{4.1}$$

To be able to study the above conformable problem, we need to introduce the following preliminaries, see [1, 7, 27].

4.1. Conformable fractional derivatives.

In this subsection, we recall the definition of the conformable derivative and its important properties, as established by Khalil et al. [18].

Definition 4.1. Let $f : (0, \infty) \to \mathbb{R}$. Then, the conformable fractional derivative of order α is defined by

$$\left(\mathbb{T}^{\alpha} f\right)(t) = \frac{\partial^{\alpha} f(t,x)}{\partial t^{\alpha}} = \lim_{\varepsilon \to 0} \left(\frac{f\big(t + \varepsilon t^{1-\alpha}\big) - f(t)}{\varepsilon} \right), \quad t > 0, \quad 0 < \alpha \leq 1.$$

It is to note that when $\alpha = 1$, the above formula is reduced to the standard derivative or order one.

Definition 4.2. The conformable fractional integral of a function $f : (0, \infty) \to \mathbb{R}$ of order α is defined as

$$\left(\mathbb{I}^{\alpha} f\right)(t) = \int_0^t \tau^{\alpha-1} f(\tau)\, d\tau, \qquad 0 < \alpha \leq 1.$$

The following properties are needed.

$$\mathbb{I}^{\alpha}\mathbb{T}^{\alpha} f(t) = f(t) - f(0)$$

and

$$\left(\mathbb{T}^{\alpha} f\right)(t) = t^{1-\alpha} \frac{df(t)}{dt}.$$

4.1.1. **Tanh method.** In this section, we recall the main steps of Tanh method for the case of Khalil derivatives [8].

Let us consider the general case of the equation:

$$\mathcal{F}\left(u, \mathbb{T}_t^{\alpha}u, \mathbb{T}_x^{\beta}u, \mathbb{T}_t^{2\alpha}u, \mathbb{T}_t^{\alpha}(\mathbb{T}_x^{\beta}u), \mathbb{T}_x^{2\beta}u, ...\right) = 0, \tag{4.2}$$

where $\mathbb{T}_t^{\alpha}u$ is the conformable fractional derivative of u of order $\alpha, 0 < \alpha \leq 1$. Then, we consider

$$\xi = \tfrac{k}{\alpha}t^{\alpha} + \tfrac{\omega}{\beta}x^{\beta}, \tag{4.3}$$

where k and ω are constants. So, (4.2) can be easily converted to the following nonlinear ODE:

$$\mathcal{G}\left(U, U', U'', U''', ...\right) = 0. \tag{4.4}$$

We then introduce the variable

$$\psi = \tanh(\xi), \tag{4.5}$$

so, we get

$$\begin{cases} \frac{d}{d\xi} = \left(1-\psi^2\right)\frac{d}{d\psi}, \\ \frac{d^2}{d\xi^2} = -2\psi\left(1-\psi^2\right)\frac{d}{d\psi} + \left(1-\psi^2\right)^2\frac{d^2}{d\psi^2}, \\ \frac{d^3}{d\xi^3} = 2\left(1-\psi^2\right)\left(3\psi^2-1\right)\frac{d}{d\psi} - 6Y\left(1-\psi^2\right)^2\frac{d^2}{d\psi^2} + \left(1-\psi^2\right)^3\frac{d^3}{d\psi^3}. \end{cases} \tag{4.6}$$

Now, we assume that the solution can be expressed in the form

$$u(x,t) = U(\xi) = F(\psi) = \textstyle\sum_{i=0}^{m} a_i\psi^i, \tag{4.7}$$

where m is a positive integer determined by the balancing procedure in the resulting nonlinear ODE in F. Thus, we have an algebraic system of equations from which the constants $k, \omega, a_i (i = 0, \cdots, m)$ are obtained and determine the function U, hence we get the exact solutions of $(3, 2)$.

4.2. **An example.** Consider [22, 35]:

$$\mathbb{T}_t^{\alpha}u + \nu(u\mathbb{T}_x^{\beta}u) + \eta\mathbb{T}_x^{2\beta}u + \mu\mathbb{T}_x^{3\beta}u = 0. \tag{4.8}$$

Using (4.3), to change (4.8) into the following nonlinear ODE

$$kU_{\zeta} + \nu\omega UU_{\zeta} + \eta\omega^2U_{\zeta\zeta} + \mu\omega^3U_{\zeta\zeta\zeta} = 0.$$

Integrating the above equation, we have

$$kU + \tfrac{\nu\omega}{2}U^2 + \eta\omega^2U_{\zeta} + \mu\omega^3U_{\zeta\zeta} = 0. \tag{4.9}$$

Substituting (4.6) and (4.7) into (4.9), we can get

$$kF+\frac{\nu\omega}{2}F^2+\eta\omega^2\left[\left(1-\psi^2\right)\frac{dF}{d\psi}\right]+\mu\omega^3\left[-2\psi\left(1-\psi^2\right)\frac{dF}{d\psi}+\left(1-\psi^2\right)^2\frac{d^2F}{d\psi^2}\right]=0. \tag{4.10}$$

To determine the parameter m we usually balance $\psi^4\frac{d^2F}{d\psi^2}$ with F^2. This in turn gives

$$4+m-2=2m$$

so that $m=2$. This gives the solution in the form

$$F(\psi)=a_0+a_1\psi+a_2\psi^2. \tag{4.11}$$

Substituting (4.11) into (4.10), we can get

$$\begin{aligned}&k(a_0+a_1\psi+a_2\psi^2)+\frac{\nu\omega}{2}(a_0+a_1\psi+a_2\psi^2)^2+\eta\omega^2(1-\psi^2)(a_1+2a_2\psi)\\&+\mu\omega^3\Big[-2\psi\left(1-\psi^2\right)(a_1+2a_2\psi)+2a_2\left(1-\psi^2\right)^2\Big]=0.\end{aligned} \tag{4.12}$$

Then, we have the system:

$$\begin{cases} ka_0+\frac{1}{2}\nu\omega a_0^2+\eta\omega^2a_1+2\mu\omega^3a_1=0,\\ -2\mu\omega^3a_1+2\eta\omega^2a_2+\nu\omega a_0a_1+ka_1=0,\\ ka_2+\nu\omega a_0a_2+\frac{1}{2}\nu\omega a_1^2-\eta\omega^2a_1-4\mu\omega^3a_1-4\mu\omega^3a_2=0,\\ 2\mu\omega^3a_1-2\eta\omega^2a_2+\nu\omega a_1a_2=0,\\ \frac{1}{2}\nu\omega a_2^2+2\mu\omega^3a_1+4\mu\omega^3a_2=0.\end{cases}$$

We solve the algebraic system with the aid of Maple. We obtain traveling wave solutions of (4.8) as follows:

Case 1:

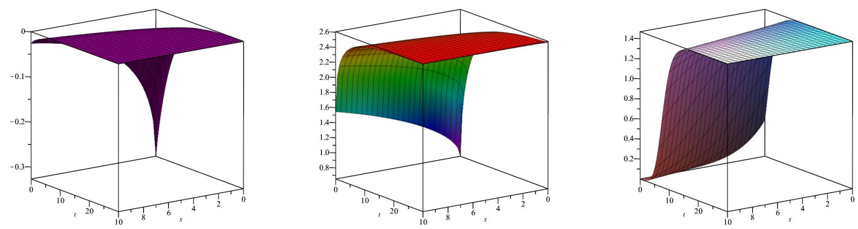
$$a_0=-\frac{4\eta^2}{49\mu\nu},a_1=-\frac{8\eta^2}{49\mu\nu},a_2=-\frac{4\eta^2}{49\mu\nu},k=-\frac{4\eta^3}{343\mu^2},w=-\frac{\eta}{14\mu},$$

$$u(x,t)=-\frac{8\eta^2}{49\mu\nu}-\frac{8\eta^2}{49\mu\nu}\tanh(\xi)-\frac{4\eta^2}{49\mu\nu}\tanh^2(\xi). \tag{4.13}$$

Case 2:

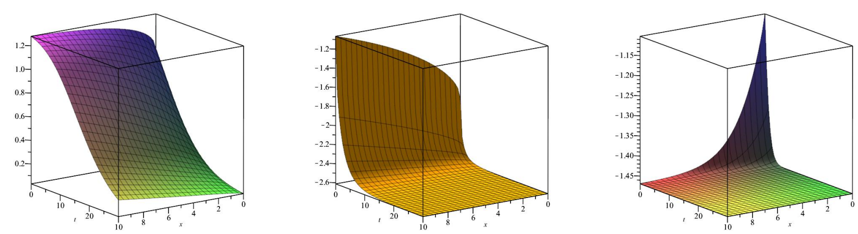
$$a_0=\frac{12\eta^2}{49\mu\nu},a_1=-\frac{8\eta^2}{49\mu\nu},a_2=-\frac{4\eta^2}{49\mu\nu},k=\frac{4\eta^3}{343\mu^2},w=-\frac{\eta}{14\mu},$$

$$u(x,t)=\frac{12\eta^2}{49\mu\nu}-\frac{8\eta^2}{49\mu\nu}\tanh(\xi)-\frac{4\eta^2}{49\mu\nu}\tanh^2(\xi). \tag{4.14}$$



(a) $\nu = 1, \mu = 1,$ $\eta = 2, \alpha = \frac{9}{10}, \beta = \frac{1}{2}.$ **(b)** $\nu = -2, \mu = 9,$ $\eta = -12, \alpha = \frac{1}{2}, \beta = \frac{1}{2}.$ **(c)** $\nu = 2, \mu = -1,$ $\eta = -3, \alpha = \frac{95}{100}, \beta = \frac{9}{10}$

Figure 1. Plots of solution (4.13) with $0 \leq x \leq 10$ and $0 \leq t \leq 30$.



(a) $\nu = 1, \mu = 1,$ $\eta = 2, \alpha = \frac{9}{10}, \beta = \frac{1}{2}.$ **(b)** $\nu = -2, \mu = 9,$ $\eta = -12, \alpha = \frac{1}{2}, \beta = \frac{1}{2}.$ **(c)** $\nu = 2, \mu = -1,$ $\eta = -3, \alpha = \frac{95}{100}, \beta = \frac{9}{10}$

Figure 2. Plots of solution (4.14) with $0 \leq x \leq 10$ and $0 \leq t \leq 30$.

5. Conclusion

We have worked on two different problems. First, we have investigated a class of fractional differential equations involving Caputo sequential derivatives with some periodic conditions. The existence and uniqueness result has been discussed via Banach contraction principle and an illustrative example has been presented to show the applicability of the hypotheses of Theorem 3.1.

The stability of solutions has also been discussed. The second part we have investigated involves studying traveling wave solutions for the time and space conformable fractional KdV Burger equation by applying the Tanh method. The obtained traveling wave solutions have been expressed in terms of hyperbolic tangent functions depending on different parameters. it seems that the applied method is direct, concise and effective; it can be applied to other nonlinear evolution equations with conformable fractional derivatives in time and space. Graphs of the waves are plotted under some particular values of the data of the conformable fractional KdV Burger equation.

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