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EXISTENCE RESULTS FOR p-LAPLACIAN PROBLEMS INVOLVING SINGULAR CYLINDRICAL POTENTIAL

Zeid Ibrahim Almuhiameed

Department of Mathematics, College of Science, Qassim University,
BO 6644, Buraidah: 51452, Kingdom of Saudi Arabia
e-mail: z.almuhiameed21@yahoo.com

Abstract. In this paper, we establish the existence of at least two distinct solutions to a p-Laplacian problems involving critical exponents and singular cylindrical potential, by using the Nehari manifold.

1. Introduction

In this paper, we consider the multiplicity results of nontrivial solutions of the following problem $(\mathcal{P}_{\lambda,\mu})$

$$\begin{cases} -\Delta_p u - \mu |y|^{-p} |u|^{p-2} u = \quad h |y|^{-s} |u|^{p^*(s)-2} u + \lambda f |u|^{q-2} u \text{ in } \mathbb{R}^N, \ y \neq 0, \\ u \in \mathcal{D}_1^p \left(\mathcal{R}^N\right), \end{cases}$$

where $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)$, $1 < p < k$, k and N are integers with $N > p$, $2 < k < N$, $\mathcal{R}^N = \mathcal{R}^k \times \mathcal{R}^{N-k}$, the point $x \in \mathcal{R}^N$ can be written as $x = (y,z) \in \mathcal{R}^k \times \mathcal{R}^{N-k}$, $-\infty < \mu < \bar{\mu}_{k,p} := \left((k-p)/p\right)^p$, $0 < s < p$, $p^*(s) = p(N-s)/(N-p)$ is the critical Hardy-Sobolev exponent, $1 < q \leq p^* = pN/(N-p)$ is the critical Sobolev exponent, $f \in L^\infty\left(\mathcal{R}^N\right)$, h is a bounded positive function on $\mathcal{R}^k$ and λ is a parameter that we will specify later.

In the recent mathematical literature a great deal of work has been devoted to the study of nonlinear elliptic equations. The main motivation of this

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study is the fact that such equations arise in the search for solitary waves of nonlinear evolution equations of the Schrödinger or Klein-Gordon type. Roughly speaking, a solitary wave is a non-singular solution which travels as a localized packet in such a way that the physical quantities corresponding to the invariances of the equation are finite and conserved in time. Accordingly, a solitary wave preserves intrinsic properties of particles such as the energy, the angular momentum and the charge, whose finiteness is strictly related to the finiteness of the L^2-norm. Owing to their particle-like behaviour, solitary waves can be regarded as a model for extended particles and they arise in many problems of mathematical physics, such as classical and quantum field theory, nonlinear optics fluid mechanics and plasma physics.

When $k = N$, $\mu = 0$ and $p = 2$. The fact that the number of nontrivial solutions of equation $(\mathcal{P}_{\lambda,\mu})$ is affected by the nonlinearity terms has been the focus of a great deal of research in recent years. If the weight functions $f \equiv h \equiv 1$, the authors Ambrosetti-Brezis-Cerami [1] have investigated equation $(\mathcal{P}_{\lambda,\mu})$. They found that there exists $\mu_0 > 0$ such that equation $(\mathcal{P}_{1,\mu})$ admits at least two positive solutions for $0 < \mu < \mu_0$, has a positive solution for $\mu = \mu_0$ and no positive solution exists for $\mu > \mu_0$. For more general results, were done by Cao etal. [8], de Figueiredo-Grossez-Ubilla [10], Filippucci et al. [11], Li et al. [14], Wu [17] and the references therein.

In case $1 < k < N$, equations with cylindrical potentials were also studied by many people [2] , [3] , [4] , [5] , [6] , [12] , [13] , [15] , and [19]. For instance, in [18], Xuan studied the multiple weak solutions for p-Laplace equation with singularity and cylindrical symmetry in bounded domains. However, they only considered the equation with sole critical Hardy-Sobolev term.

Let $\mathcal{D}_1^p\left(\mathcal{R}^N\right)$ be the space defined as the completion of $\mathcal{C}_c^\infty\left(\mathcal{R}^N\right)$ with respect to the norm

$$\|\nabla u\|_p = \left(\int_{\mathcal{R}^N} |\nabla u|^p \, dx\right)^{\frac{1}{p}}.$$

Clearly, the problem $(\mathcal{P}_{\lambda,\mu})$ is related to the following Hardy-Sobolev type inequality with cylindrical weight which first proved in [5]

$$\int_{\mathcal{R}^N} |\nabla u|^p \, dx \geq C \int_{\mathcal{R}^N} |y|^{-s} |u|^{p^*(s)} \, dx, \text{ for all } u \in \mathcal{D}_1^p\left(\mathcal{R}^N\right), \tag{1.1}$$

where $C > 0$, $1 < p < k, 2 < k < N$, $x = (y, z) \in \mathcal{R}^k \times \mathcal{R}^{N-k}, 0 < s < p$, $p^*(s) = p(N-s)/(N-p), p^* = pN/(N-p)$, $1 < q < p$. In particular, for $s = p$ and $1 < p < k$, we have Hardy type inequality:

$$\int_{\mathcal{R}^N} |\nabla u|^p \, dx \geq \bar{\mu}_{k,p} \int_{\mathcal{R}^N} |y|^{-p} |u|^p \, dx, \text{ for all } u \in \mathcal{D}_1^p\left(\mathcal{R}^N\right), \tag{1.2}$$

the constant $\bar{\mu}_{k,p} := ((k-p)/p)^p$ is sharp but not achieved [5].

When $\mu < \bar{\mu}_{k,p}$, Hardy type inequality implies that the norm

$$\|u\| = \|u\|_{\mu,p} = \left(\int_{\mathcal{R}^N} \left(|\nabla u|^p - \mu |y|^{-p} |u|^p\right) dx\right)^{1/p}$$

is well defined in $\mathcal{D}_1^p\left(\mathcal{R}^N\right)$ and $\|.\|$ is equivalent to $\|\nabla .\|_p$; since the following inequalities hold:

$$\left(1 - \left(\max(\mu, 0)/\bar{\mu}_{k,p}\right)\right)^{1/p} \|\nabla u\|_p \leq \|u\| \leq \left(1 - \left(\min(\mu, 0)/\bar{\mu}_{k,p}\right)\right)^{1/p} \|\nabla u\|_p$$

for all $u \in \mathcal{D}_1^p\left(\mathcal{R}^N\right)$.

Since our approach is variational, we define the functional J_λ on $\mathcal{D}_1^p\left(\mathcal{R}^N\right)$ by

$$J_\lambda(u) := (1/p)\|u\|^p - (1/p^*(s))\, P(u) - (\lambda/q)\, Q(u)$$

with

$$P(u) := \int_{\mathcal{R}^N} |y|^{-s}\, h\, |u|^{p^*(s)}\, dx, \quad Q(u) := \int_{\mathcal{R}^N} f\, |u|^q\, dx.$$

Let

$$S = S_{(\mu,N,p,0)} := \inf_{u \in \mathcal{D}_1^p(\mathcal{R}^N) \setminus \{0\}} \frac{\|u\|^p}{\left(\int_{\mathcal{R}^N} |u|^{p^*}\, dx\right)^{p/p^*}} \tag{1.3}$$

and

$$\tilde{S} = S_{(\mu,N,p,s)} := \inf_{u \in \mathcal{D}_1^p(\mathcal{R}^N) \setminus \{0\}} \frac{\|u\|^p}{\left(\int_{\mathcal{R}^N} |y|^{-s} |u|^{p^*(s)}\, dx\right)^{p/p^*(s)}}, \tag{1.4}$$

where $0 < s < p$. Then, from [5], $\tilde{S}$ is achieved.

Throughout this work, we consider the following assumption

$$(H) \qquad \lim_{|y| \longrightarrow 0} h(y) = \lim_{|y| \longrightarrow \infty} h(y) = h_0 > 0,\ h(y) \geq h_0,\ y \in \mathcal{R}^k.$$

In our work, we research the critical points as the minimizers of the energy functional associated to the problem $(\mathcal{P}_{\lambda,\mu})$ on the constraint defined by the Nehari manifold, which are solutions of our problem.

Let Λ_0 be positive number

$$\Lambda_0 := L(p,q)\left(\tilde{S}\right)^{p^*(s)(p^*-p)/p(p-p^*(s))} (S)^{-p^*/q},$$

where

$$L(p,q) := \left[\left(\frac{p - p^*(s)}{(q - p^*(s))\,|f|_\infty}\right)\right]\left[|h|_\infty \left(\frac{p^*(s) - q}{(p-q)}\right)\right]^{(p^*-p)/(p^*(s)-p)}$$

and $|f(x)|_\infty = \sup_{x \in \mathcal{R}^N} |f(x)|$, $|h(y)|_\infty = \sup_{y \in \mathcal{R}^k} |h(y)|$.

This paper is organized as follows. In Section 2, we give some preliminaries and Section 3 is devoted to the proofs of main theorems.

2. Preliminaries

Definition 2.1. Let $c \in \mathcal{R}$, $J_\lambda \in C^1(E, \mathcal{R})$.

(1) $\{u_n\}_n$ is a Palais-Smale sequence at level c (in short $(PS)_c$) in E for J_λ if
$$J_\lambda(u_n) = c + o_n(1) \text{ and } J_\lambda^{'}(u_n) = o_n(1),$$
where $o_n(1)$ tends to 0 as n goes at infinity.

(2) We say that J_λ satisfies the $(PS)_c$ condition if any $(PS)_c$ sequence in E for J_λ has a convergent subsequence.

Nehari manifold: It is well known that J_λ is of class C^1 in $\mathcal{D}_1^p(\mathcal{R}^N)$ and the solutions of $(\mathcal{P}_{\lambda,\mu})$ are the critical points of J_λ which is not bounded below on $\mathcal{D}_1^p(\mathcal{R}^N)$. Consider the following Nehari manifold:
$$\mathcal{N} = \left\{u \in \mathcal{H}\setminus\{0\} : \ \left\langle J_\lambda^{'}(u), u\right\rangle = 0\right\}.$$
Thus, $u \in \mathcal{N}$ if and only if
$$\|u\|^p - P(u) - \lambda Q(u) = 0. \tag{2.1}$$

Note that $\mathcal{N}$ contains every nontrivial solution of the problem $(\mathcal{P}_{\lambda,\mu})$. Moreover, we have the following results.

Lemma 2.2. *J_λ is coercive and bounded from below on $\mathcal{N}$.*

Proof. If $u \in \mathcal{N}$, then by (2.1) and the Sobolev inequality, we deduce that
$$\begin{aligned} J_\lambda(u) &= ((p^*(s) - p)/pp^*(s))\|u\|^p - \lambda((p^*(s) - q)/qp^*(s))Q(u) \\ &\geq ((p^*(s) - p)/pp^*(s))\|u\|^p \\ &\quad - \lambda((p^*(s) - q)/qp^*(s))|f|_\infty S^{(q/p^*)}\|u\|^q. \end{aligned} \tag{2.2}$$
Thus, J_λ is coercive and bounded from below on $\mathcal{N}$. □

Define
$$\phi(u) = \left\langle J_\lambda^{'}(u), u\right\rangle.$$
Then, for $u \in \mathcal{N}$,
$$\begin{aligned} \left\langle \phi^{'}(u), u\right\rangle &= p\|u\|^p - p^*(s)P(u) - \lambda q Q(u) \\ &= (p - q)\|u\|^p - (p^*(s) - q)P(u) \\ &= \lambda(p^*(s) - q)Q(u) - (p^*(s) - p)\|u\|^p. \end{aligned} \tag{2.3}$$

Now, we split $\mathcal{N}$ in three parts:

$$\begin{aligned}\mathcal{N}^{+} &= \left\{u \in \mathcal{N} : \left\langle \phi'(u), u\right\rangle > 0\right\},\\ \mathcal{N}^{0} &= \left\{u \in \mathcal{N} : \left\langle \phi'(u), u\right\rangle = 0\right\},\\ \mathcal{N}^{-} &= \left\{u \in \mathcal{N} : \left\langle \phi'(u), u\right\rangle < 0\right\}.\end{aligned}$$

We have the following results.

Lemma 2.3. *Suppose that u_0 is a local minimizer for J_λ on $\mathcal{N}$. Then, if $u_0 \notin \mathcal{N}^0$, u_0 is a critical point of J_λ.*

Proof. If u_0 is a local minimizer for J_λ on $\mathcal{N}$, then u_0 is a solution of the optimization problem

$$\min_{\{u/\ \phi(u)=0\}} J_\lambda(u).$$

Hence, there exists a Lagrange multipliers $\theta \in \mathcal{R}$ such that

$$J'_\lambda(u_0) = \theta\phi'(u_0) \text{ in } \left(\mathcal{D}_1^p\left(\mathcal{R}^N\right)\right)'.$$

Thus,

$$\left\langle J'_\lambda(u_0), u_0\right\rangle = \theta\left\langle \phi'(u_0), u_0\right\rangle.$$

But $\left\langle \phi'(u_0), u_0\right\rangle \neq 0$, since $u_0 \notin \mathcal{N}^0$. Hence $\theta = 0$. This completes the proof. □

Lemma 2.4. *There exists a positive number Λ_0 such that for all λ verifying*

$$0 < \lambda < \Lambda_0,$$

we have $\mathcal{N}^0 = \emptyset$.

Proof. Let us reason by contradiction. Suppose $\mathcal{N}^0 \neq \emptyset$ such that $0 < \lambda < \Lambda_0$. Then, by (2.3) and for $u \in \mathcal{N}^0$, we have

$$\begin{aligned}\|u\|^p &= (p^*(s) - q)/(p - q)\, P(u) \qquad (2.4)\\ &= \lambda\left((p^*(s) - q)/(p^*(s) - p)\right) Q(u).\end{aligned}$$

Moreover, by the Holder inequality and the Sobolev embedding theorem, we obtain

$$\|u\| \geq (S)^{p^*/q(p^*-p)}\left[(p - p^*(s))/\lambda\,(q - p^*(s))\,|f|_\infty\right]^{-1/(p^*-p)} \qquad (2.5)$$

and

$$\|u\| \leq \left[h_0\left((p^*(s) - q)/(p - q)\right)\right]^{1/(p-p^*(s))}\left(\tilde{S}\right)^{-p^*(s)/p(p--p^*(s))}. \qquad (2.6)$$

From (2.5) and (2.6), we obtain $\lambda \geq \Lambda_0$, which contradicts an hypothesis. □

Since $\mathcal{N}^0 = \emptyset$, $\mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^-$. Define

$$c := \inf_{u\in\mathcal{N}} J_\lambda(u), \; c^+ := \inf_{u\in\mathcal{N}^+} J_\lambda(u) \text{ and } c^- := \inf_{u\in\mathcal{N}^-} J_\lambda(u).$$

For the sequel, we need the following Lemma.

Lemma 2.5. (i) *For all λ such that $0 < \lambda < \Lambda_0$, one has $c \le c^+ < 0$.*

(ii) *There exists $\Lambda_1 > 0$ such that for all $0 < \lambda < \Lambda_2 = \min(\Lambda_0, \Lambda_1)$, one has*

$$\begin{aligned} c^- > C_0 &= C_0\left(\lambda,\ p, q,\ S, \tilde{S},\ p^*(s), h_0,\ |f|_\infty\right) \\ &= \left(\frac{(p^*(s)-p)}{pp^*(s)}\right)\left[\frac{(p-q)}{(p^*(s)-q)h_0}\right]^{-p/(p-p^*(s))}\left(\tilde{S}\right)^{p^*(s)/(p^*(s)-p)} \\ &\quad - \lambda\left(\frac{(p^*(s)-q)}{q(p^*(s))}\right)|f|_\infty (S)^{q/p^*}. \end{aligned}$$

Proof. (i) Let $u \in \mathcal{N}^+$. By (2.3), we have

$$[(p-q)/(p^*(s)-q)]\,\|u\|^p > P(u)$$

and so

$$\begin{aligned} J_\lambda(u) &= (-(p-q)/pq)\|u\|^p + ((p^*(s)-q)/q(p^*(s)))P(u) \\ &< -[((p-q)/pq) + ((p-q)/q(p^*(s)))]\|u\|^p. \end{aligned}$$

We conclude that $c \le c^+ < 0$.

(ii) Let $u \in \mathcal{N}^-$. By (2.3), we get

$$[(p-q)/(p^*(s)-q)]\,\|u\|^p < P(u).$$

Moreover, by (H) and Sobolev embedding theorem, we have

$$P(u) \le \left(\tilde{S}\right)^{-p^*(s)/p}\left|h^+\right|_\infty \|u\|^{p^*(s)}.$$

This implies

$$\|u\| > \left(\tilde{S}\right)^{p^*(s)/p(p^*(s)-p)}\left[\frac{(p-q)}{(p^*(s)-q)|h^+|_\infty}\right]^{-1/(p-p^*(s))} \quad \text{for all } u \in \mathcal{N}^-. \tag{2.7}$$

By (2.2), we get

$$\begin{aligned} J_\lambda(u) \;\ge\; &\left(\frac{(p^*(s)-p)}{pp^*(s)}\right)\left[\frac{(p-q)}{(p^*(s)-q)h_0}\right]^{-p/(p-p^*(s))}\left(\tilde{S}\right)^{p^*(s)/(p^*(s)-p)} \\ &-\lambda\left(\frac{(p^*(s)-q)}{q(p^*(s))}\right)|f|_\infty (S)^{q/p^*}. \end{aligned}$$

Thus, for all λ such that $0 < \lambda < \Lambda_2 = \min(\Lambda_0, \Lambda_1)$ with

$$\Lambda_1 = \left(\frac{(p^*(s)-p)}{pp^*(s)}\right)\left[\frac{(p-q)\left(\tilde{S}\right)^{\frac{p^*(s)}{p}}}{(p^*(s)-q)\,h_0}\right]^{\frac{-p}{(p-p^*(s))}}\left[\left(\frac{(p^*(s)-q)\,|f|_\infty}{q\,(p^*(s))}\right)(S)^{-q/p^*}\right]^{-1},$$

we have $J_\lambda(u) \geq C_0$. □

For each $u \in \mathcal{D}_1^p\left(\mathcal{R}^N\right)$, we write

$$t_m := t_{\max}(u) = \left[\frac{\|u\|^p}{(p^*(s)-q)\,P(u)}\right]^{1/(p^*(s)-p)} > 0.$$

Lemma 2.6. *Let λ be a real parameter such that $0 < \lambda < \Lambda_0$. Then for each $u \in \mathcal{D}_1^p\left(\mathcal{R}^N\right)$, we have the followings:*

(i) *If $Q(u) \leq 0$, then there exists a unique $t^- > t_m$ such that $t^- u \in \mathcal{N}^-$ and*

$$J_\lambda\left(t^- u\right) = \sup_{t\geq 0}(tu).$$

(ii) *If $Q(u) > 0$, then there exist unique t^+ and t^- such that $0 < t^+ < t_m < t^-$, $(t^+u) \in \mathcal{N}^+$, $(t^-u) \in \mathcal{N}^-$,*

$$J_\lambda\left(t^+u\right) = \inf_{0\leq t\leq t_m} J_\lambda(tu) \quad \text{and} \quad J_\lambda\left(t^-u\right) = \sup_{t\geq 0} J_\lambda(tu).$$

Proof. With minor modifications, we refer to [7]. □

Proposition 2.7. ([7])

(i) *For all λ such that $0 < \lambda < \Lambda_0$, there exists a $(PS)_{c^+}$ sequence in $\mathcal{N}^+$.*
(ii) *For all λ such that $0 < \lambda < \Lambda_2 = \min(\Lambda_0, \Lambda_1)$, there exists a $(PS)_{c^-}$ sequence in $\mathcal{N}^-$.*

3. Main results

Now, taking as a starting point the work of Tarantello [16], we establish the existence of a local minimum for J_λ on $\mathcal{N}^+$.

Proposition 3.1. *For all λ such that $0 < \lambda < \Lambda_0$, the functional J_λ has a minimizer $u_0^+ \in \mathcal{N}^+$ and it satisfies:*

(i) $J_\lambda\left(u_0^+\right) = c = c^+$,
(ii) *u_0^+ is a nontrivial solution of $(\mathcal{P}_{\lambda,\mu})$.*

Proof. If $0 < \lambda < \Lambda_0$, then by Proposition 2.7-(i) there exists a $(PS)_{c^+}$ sequence $\{u_n\}_n$ in $\mathcal{N}^+$, thus it bounded by Lemma 2.2. Then, there exists $u_0^+ \in \mathcal{D}_1^p\left(\mathcal{R}^N\right)$ and we can extract a subsequence which will denoted by $\{u_n\}_n$ such that

$$\begin{aligned} u_n &\rightharpoonup u_0^+ \text{ weakly in } \mathcal{D}_1^p\left(\mathcal{R}^N\right), \\ u_n &\rightharpoonup u_0^+ \text{ weakly in } \left(L^{p^*(s)}\left(\mathcal{R}^N, |y|^{-s}\right)\right), \\ u_n &\to u_0^+ \text{ strongly in } L^q\left(\mathcal{R}^N\right), \\ u_n &\to u_0^+ \text{ a.e in } \mathcal{R}^N. \end{aligned} \tag{3.1}$$

Thus, by (3.1), u_0^+ is a weak nontrivial solution of $(\mathcal{P}_{\lambda,\mu})$. Now, we show that u_n converges to u_0^+ strongly in $\mathcal{D}_1^p\left(\mathcal{R}^N\right)$. Suppose otherwise. By the lower semi-continuity of the norm, then $\left\|u_0^+\right\| < \liminf_{n\longrightarrow\infty} \|u_n\|$ and we obtain

$$\begin{aligned} c &\leq J_\lambda\left(u_0^+\right) \\ &= \left(\left(p^*(s) - p\right)/p\left(p^*(s)\right)\right)\left\|u_0^+\right\|^p - \left(\left(p^*(s) - q\right)/q\left(p^*(s)\right)\right)Q\left(u_0^+\right) \\ &< \liminf_{n\longrightarrow\infty} J\left(u_n\right) = c. \end{aligned}$$

This is a contradiction. Therefore, u_n converge to u_0^+ strongly in $\mathcal{D}_1^p\left(\mathcal{R}^N\right)$. Moreover, we have $u_0^+ \in \mathcal{N}^+$. If not, then by Lemma 2.6, there are two numbers t_0^+ and t_0^-, uniquely defined so that $\left(t_0^+ u_0^+\right) \in \mathcal{N}^+$ and $\left(t^- u_0^+\right) \in \mathcal{N}^-$. In particular, we have $t_0^+ < t_0^- = 1$. Since

$$\frac{d}{dt} J_\lambda\left(tu_0^+\right)_{\rfloor t=t_0^+} = 0 \quad \text{and} \quad \frac{d^2}{dt^2} J_\lambda\left(tu_0^+\right)_{\rfloor t=t_0^+} > 0,$$

there exists $t_0^+ < t^- \leq t_0^-$ such that $J_\lambda\left(t_0^+ u_0^+\right) < J_\lambda\left(t^- u_0^+\right)$. By Lemma 2.6, we get

$$J_\lambda\left(t_0^+ u_0^+\right) < J_\lambda\left(t^- u_0^+\right) < J_\lambda\left(t_0^- u_0^+\right) = J_\lambda\left(u_0^+\right),$$

which contradicts the fact that $J_\lambda\left(u_0^+\right) = c^+$. □

Theorem 3.2. *Let* $f \in L^\infty\left(\mathcal{R}^N\right)$. *Assume that* $1 < p < k$, $N > p$, $2 < k < N$, $0 < \mu < \bar{\mu}_{k,p} := \left((k-p)/p\right)^p$, $0 < s < p$, $1 < q < p$, (H) *satisfied and* λ *verifying* $0 < \lambda < \Lambda_0$. *Then the equation* $(\mathcal{P}_{\lambda,\mu})$ *has at least one nontrivial solution.*

Proof. Since $J_\lambda\left(u_0^+\right) = J_\lambda\left(\left|u_0^+\right|\right)$ and $\left|u_0^+\right| \in \mathcal{N}^+$, by Lemma 2.3, we may assume that u_0^+ is a nontrivial nonnegative solution of $(\mathcal{P}_{\lambda,\mu})$. By the Harnack inequality, we conclude that $u_0^+ > 0$ and $v_0^+ > 0$, see for example [10]. □

Next, we establish the existence of a local minimum for J_λ on $\mathcal{N}^-$. For this, we require the following lemma.

Lemma 3.3. *For all* λ *such that* $0 < \lambda < \Lambda_2 = \min(\Lambda_0, \Lambda_1)$, *the functional* J_λ *has a minimizer* u_0^- *in* $\mathcal{N}^-$ *and it satisfies:*

(i) $J_\lambda\left(u_0^-\right) = c^- > 0$,

(ii) u_0^- *is a nontrivial solution of* $(\mathcal{P}_{\lambda,\mu})$ *in* $\mathcal{D}_1^p\left(\mathcal{R}^N\right)$.

Proof. If $0 < \lambda < \Lambda_2 = \min(\Lambda_0, \Lambda_1)$, then by Proposition 2.7-(ii) there exists a $(PS)_{c^-}$ sequence $\{u_n\}_n$ in $\mathcal{N}^-$, thus it bounded by Lemma 2.2. Then, there exists $u_0^- \in \mathcal{D}_1^p\left(\mathcal{R}^N\right)$ and we can extract a subsequence which will denoted by $\{u_n\}_n$ such that

$$\begin{aligned} u_n &\rightharpoonup u_0^- \text{ weakly in } \mathcal{D}_1^p\left(\mathcal{R}^N\right), \\ u_n &\rightharpoonup u_0^- \text{ weakly in } L^{p^*(s)}\left(\mathcal{R}^N, |y|^{-s}\right), \\ u_n &\to u_0^- \text{ strongly in } L^q\left(\mathcal{R}^N\right), \\ u_n &\to u_0^- \text{ a.e in } \mathcal{R}^N. \end{aligned}$$

This implies

$$P(u_n) \to P\left(u_0^-\right) \text{ as } n \text{ goes to } \infty.$$

Moreover, by (H) and (2.3) we obtain

$$P(u_n) > (p-q)/(p^*(s)-q)\|u_n\|^p. \tag{3.2}$$

By (2.5) and (3.2) there exists a positive number

$$C_1 := \left[(p-q)/(p^*(s)-q)\right]^{p^*(s)/(p^*(s)-p)}\left(\tilde{S}\right)^{p^*(s)/(p^*(s)-p)}$$

such that

$$P(u_n) > C_1. \tag{3.3}$$

This implies that

$$P\left(u_0^-\right) \geq C_1.$$

Now, we prove that $\{u_n\}_n$ converges to u_0^- strongly in $\mathcal{D}_1^p\left(\mathcal{R}^N\right)$. Suppose otherwise. Then, $\left\|u_0^-\right\| < \liminf_{n\longrightarrow\infty} \|u_n\|$. By Lemma 2.6, there is a unique t_0^- such that $\left(t_0^- u_0^-\right) \in \mathcal{N}^-$. Since

$$u_n \in \mathcal{N}^-, \ J_\lambda(u_n) \geq J_\lambda(t u_n) \text{ for all } t \geq 0,$$

we have

$$J_\lambda\left(t_0^- u_0^-\right) < \lim_{n\longrightarrow\infty} J_\lambda\left(t_0^- u_n\right) \leq \lim_{n\longrightarrow\infty} J_\lambda(u_n) = c^-$$

and this is a contradiction. Hence,

$$u_n \to u_0^- \text{ strongly in } \mathcal{D}_1^p\left(\mathcal{R}^N\right).$$

Thus,

$$J_\lambda(u_n) \text{ converges to } J_\lambda\left(u_0^-\right) = c^- \text{ as } n \text{ tends to } +\infty.$$

Since $J_\lambda\left(u_0^-\right) = J_\lambda\left(\left|u_0^-\right|\right)$ and $u_0^- \in \mathcal{N}^-$, then by (3.3) and Lemma 2.3, we may assume that u_0^- is a nontrivial nonnegative solution of $(\mathcal{P}_{\lambda,\mu})$. By the maximum principle, we conclude that $u_0^- > 0$ and $v_0^- > 0$. □

Theorem 3.4. *In addition to the assumptions of the Theorem* 3.2 *and* (H) *satisfied there exists a positive real* Λ_1 *such that, if* λ *satisfy* $0 < \lambda < \Lambda_2 = \min(\Lambda_0, \Lambda_1)$*, then* $(\mathcal{P}_{\lambda,\mu})$ *has at least two nontrivial solutions.*

Proof. Now, we complete the proof of Theorem 3.4. By Propositions 3.1 and Lemma 3.3, we obtain that $(\mathcal{P}_{\lambda,\mu})$ has two positive solutions $u_0^+ \in \mathcal{N}^+$ and $u_0^- \in \mathcal{N}^-$. Since $\mathcal{N}^+ \cap \mathcal{N}^- = \emptyset$, this implies that u_0^+ and u_0^- are distinct. □

Corollary 3.5. *Let* $s = 0$, $N = 2p$ *and* $a < S_0^{-2}$. *Then there exists* $\widetilde{\Lambda}_* > 0$ *such that problem* $(\mathcal{P}_\lambda)$ *has at least two nontrivial solutions for any* $\lambda \in \left(0,\ \widetilde{\Lambda}_*\right)$.

Conclusion. In our work, we have searched the critical points as the minimizers of the energy functional associated to the problem $(\mathcal{P}_{\lambda,\mu})$ on the constraint defined by the Nehari manifold, which are solutions of our problem. In the sections 3 and 4 , we have proved the existence of at least two positive solutions by using a Nehari and sub-Nehari manifold.

The advantage in the variational methods and the constraint of the Nehari manifold is to ensure the multiplicity of solutions and to overcome the problem of lack of compactness.

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