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EXISTENCE AND APPROXIMATE SOLUTION FOR THE FRACTIONAL VOLTERRA FREDHOLM INTEGRO-DIFFERENTIAL EQUATION INVOLVING ς-HILFER FRACTIONAL DERIVATIVE

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Abstract. In this paper, we are motivated to evaluate and investigate the necessary conditions for the fractional Volterra Fredholm integro-differential equation involving the ς-Hilfer fractional derivative. The given problem is converted into an equivalent fixed point problem by introducing an operator whose fixed points coincide with the solutions to the problem at hand. The existence and uniqueness results for the given problem are derived by applying Krasnoselskii and Banach fixed point theorems respectively. Furthermore, we investigate the convergence of approximated solutions to the same problem using the modified Adomian decomposition method. An example is provided to illustrate our findings.

1. Introduction

Because of their numerous applications in mathematics, biology, physics, finance, engineering, dynamical systems and control theory, fractional differential equations (FDEs) are of great interest, see [7, 13, 20, 25, 28, 33] and the

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references therein. However, because of the complexities of their initial values, several physical interpretations of FDEs are still unknown, so the theory of FDEs is still in its infancy. Nonetheless, because of their numerous practical applications and theoretical significance, these equations have become the most popular topic of discussion among a number of examiners. There has also been shown a significant interest in the study of FDEs by many authors, for instance (see [1, 2, 3, 9, 10, 14, 24, 32]).

Vanterler et al. [16] recently proposed a new type of fractional differential (FD) operator called a ψ-Hilfer fractional operator, which generalises the Hilfer fractional operator [21, 26, 29]. It is important to note that the ψ-Hilfer fractional derivative is defined with respect to another function, and it unifies the various fractional derivative definitions found in the literature.

Additionally, a lot of study has been done using George Adomian's approach of Adomian decomposition to estimate the solution of this type of equation [4] and other numerical methods for more details see [15, 17, 18, 38]. The style and simplicity of the Adomian decomposition approach make it appealing. The answer is given as a series, where each equation may be calculated with ease using Adomian polynomials that are appropriate for nonlinear components (see [4, 5, 6, 19, 27, 29]).

In [37], Wazwaz introduced the method of modified Adomian decomposition (MADM), which entails splitting the 1^{st} term of the series into two 2^{nd} terms, one of which is kept to define the 2^{nd} term of the series. This approach's primary goals are to perform fewer operations and accelerate convergence to the precise solution to the stated problem. For instance, we quote [23] when discussing the application of the MADM.

The goal of the current paper is to discuss the uniqueness and existence of the solution by applying Banach's and Krasnoselskii's fixed point theorems, then we use the MADM for the following $\varsigma-$Hilfer fractional Volterra Fredholm integro-differential equation ($\varsigma-$Hilfer fractional VFIDE)

$$\begin{cases} {}^{H}\mathcal{D}_{0^+}^{\nu_1,\nu_2;\varsigma}\omega(\varkappa) = g(\varkappa) + \Pi_1\omega(\varkappa) + \Pi_2\omega(\varkappa), \ \varkappa \in \hbar = [0,1], \\ \omega(0) = \omega_0 + y(\omega), \end{cases} \tag{1.1}$$

where $0 < \nu < 1$, ${}^{H}\mathcal{D}_{0^+}^{\nu_1,\nu_2;\varsigma}$ is $\varsigma-$Hilfer fractional derivative of order ν_1 and parameter ν_2, $g : \hbar \to \mathbb{R}$, $y : \mathbb{C}(\hbar, \mathbb{R}) \to \mathbb{R}$, χ_1, $\chi_2 : \hbar \times \hbar \to \mathbb{R}$ are continuous functions and $\mathcal{N}_j : \mathbb{R} \to \mathbb{R}$, $j = 1, 2$ are Lipschitz continuous functions. In brief, we set

$$\Pi_1\omega(\varkappa) := \int_0^{\varkappa} \chi_1(\varkappa, \xi)\mathcal{N}_1(\omega(\xi))d\xi$$

and

$$\Pi_2\omega(\varkappa) := \int_0^1 \chi_2(\varkappa,\xi)\mathcal{N}_2(\omega(\xi))d\xi.$$

Numerous authors have used fixed point methods to study some findings on the presence of solutions to $\varsigma-$Hilfer fractional differential equations (see [11, 30, 31]).

In this paper, we establish the existence and uniqueness findings of the $\varsigma-$Hilfer fractional VFIDE (1.1) using a contemporary methodology. We arrive at a few prerequisites that are necessary for fractional integrodifferential equations with nonlocal conditions to have solutions. To acquire a rough solution to, the MADM is utilised. The fixed point theorems of Krasnoselskii and Banach are also used to assess our findings.

The paper is structured as follows. In Section 2, we provide some fundamental findings in relation to the hypotheses and various lemmas used in this paper. In Section 3, we utilise the fixed point theorems of Krasnoselskii and Banach to demonstrate the existence and uniqueness of solutions to the proposed problem. In section 4, We discuss the MADM and prove that the series created by the MADM converges to the precise solution of the $\varsigma-$Hilfer fractional VFIDE. In Section 5, we provide an example to further clarify our findings.

2. Preliminaries

In this section, we setting notations and some introductory facts that will be applied in the proofs of the subsequent results.

Let $\mathbb{C}(\hbar,\mathbb{R})$ and $L(\hbar,\mathbb{R})$ are the Banach spaces of continuous functions and Lebesgue integrable functions from $\hbar$ into $\mathbb{R}$ with the norms

$$\|z\|_\infty = \sup\{|z| : \varkappa \in \hbar\}$$

and

$$\|z\|_L = \int_a^b |z(\varkappa)|\, d\varkappa,$$

respectively.

For $\varepsilon = \nu_1 + 2\nu_2 - \nu_1\nu_2$, $0 < \nu_1 < 1$ and $0 \le \nu_2 \le 1$. Let $\varsigma \in C^1(\hbar,\mathbb{R})$ be an increasing function with $\varsigma'(\varkappa) \neq 0$ for all $\varkappa \in \hbar$.

Definition 2.1. ([25]) Let $\nu_1 > 0$ and $z \in L^1(\hbar,\mathbb{R})$. The ς-RL fractional integral of order ν_1 of a function z is given by

$$\mathcal{I}_{0^+}^{\nu_1;\varsigma} z(\varkappa) = \frac{1}{\Gamma(\nu_1)}\int_a^{\varkappa} \varsigma'(t)(\varsigma(\varkappa)-\varsigma(t))^{\nu_1-1}z(t)dt,$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Definition 2.2. ([36]) The $\varsigma-$Hilfer FD of order ν_1 and parameter ν_2 is defined by

$$ {}^{H}\mathcal{D}_{0^+}^{\nu_1,\nu_2;\varsigma} z(\varkappa) = \mathcal{I}^{\nu_2(n-\nu_1);\varsigma} \left(\frac{1}{\varsigma'(\varkappa)} \frac{d}{d\varkappa} \right)^n \mathcal{I}^{(1-\nu_2)(n-\nu_1);\varsigma} z(\varkappa), $$

where $n-1<\nu_1<n$, $0 \le \nu_2 \le 1$, $\varkappa > a$.

Lemma 2.3. ([25, 36]) *Let* $\nu_1, \eta, \delta > 0$. *Then*

(1) $\mathcal{I}^{\nu_1;\varsigma}\mathcal{I}^{\eta;\varsigma} z(\varkappa) = \mathcal{I}^{\nu_1+\eta;\varsigma} z(\varkappa)$.
(2) $\mathcal{I}^{\nu_1;\varsigma}(\varsigma(\varkappa)-\varsigma(a))^{\delta-1} = \frac{\Gamma(\delta)}{\Gamma(\nu_1+\delta)}(\varsigma(\varkappa)-\varsigma(a))^{\nu_1+\delta-1}$.

We note also that ${}_{H}\mathcal{D}^{\nu_1,\nu_2;\varsigma}(\varsigma(\varkappa)-\varsigma(a))^{\varepsilon-1} = 0$, where $\varepsilon = \nu_1 + \nu_2(n-\nu_1)$.

Lemma 2.4. ([36]) *Let* $z \in L^1(\hbar, \mathbb{R})$, $\nu_1 \in (n-1, n]$ $(n \in \mathbb{N})$ *and* $\nu_2 \in [0,1]$. *Then*

$$ \left(\mathcal{I}^{\nu_1;\varsigma}\ {}_{H}\mathcal{D}^{\nu_1,\nu_2;\varsigma} z\right)(\varkappa) = z(\varkappa) - \sum_{k=0}^{n} \frac{(\varsigma(\varkappa)-\varsigma(a))^{\varepsilon-k}}{\Gamma(\varepsilon-k+1)} z_{\varsigma}^{[n-k]} \times \lim_{\varkappa\to a} \left(\mathcal{I}^{(1-\nu_2)(n-\nu_1);\varsigma} z\right)(a), $$

where $z_{\varsigma}^{[n-k]}(\varkappa) = \left(\frac{1}{\varsigma'(\varkappa)}\frac{d}{d\varkappa}\right)^{[n-k]} z(\varkappa)$.

Here we can suffice to refer to Banach's fixed point theorem and Krasnoselskii's fixed point theorem [34].

3. Existence result via Krasnoselkii's fixed point theorem

By utilizing Krasnoselkii's fixed point theorem, we examine the existence of a solution to the $\varsigma-$Hilfer fractional VFIDE (1.1) in this section.

We start by assuming the following.

(H$_1$): Let $\mathcal{N}_1(\omega(\varkappa))$, $\mathcal{N}_2(\omega(\varkappa))$ can be thought of as continuous nonlinearity terms, and constants exist $\ell_{\mathcal{N}_1}(>0)$ and $\ell_{\mathcal{N}_2}(>0)$ such that

$$ |\mathcal{N}_j(\omega_1(\varkappa)) - \mathcal{N}_j(\omega_2(\varkappa))| \le \ell_{\mathcal{N}_j} |\omega_1 - \omega_2|, \ j=1,2, \ \forall \omega_1, \omega_2 \in \mathbb{R}. $$

(H$_2$): The kernels $\chi_1(\varkappa,\xi)$ and $\chi_1(\varkappa,\xi)$ are continuous on $\hbar \times \hbar$, and there exist two positive constants χ_1^* and χ_2^* in $\hbar \times \hbar$ such that

$$ \chi_j^* = \sup_{\varkappa\in\hbar} \int_0^{\varkappa} |\chi_j(\varkappa,\xi)|\, d\xi < \infty, \ j=1,2. $$

(H$_3$): $g : \hbar \to \mathbb{R}$ is continuous on $\hbar$.

($\mathbf{H}_4$): $y : \mathbb{C}(\hbar, \mathbb{R}) \to \mathbb{R}$ is continuous on $\mathbb{C}(\hbar)$ and there exist constant $0 < \ell_y < 1$ such that

$$|y(\omega_1(\varkappa)) - y(\omega_2(\varkappa))| \leq \ell_y |\omega_1 - \omega_2|, \ \forall \omega_1, \omega_2 \in \mathbb{C}(\hbar, \mathbb{R}), \ \varkappa \in \hbar.$$

The problem (1.1) and the integral equation are equivalent according to the next lemma. Because it resembles a few traditional arguments that are known from the literature, the proof for this lemma is disregarded.

Lemma 3.1. *The function $\omega \in \mathbb{C}(\hbar, \mathbb{R})$ is the $\varkappa-$Hilfer fractional VFIDE's* (1.1) *solution if and only if ω is the integral equation's solution, which given by*

$$\begin{aligned}\omega(\varkappa) &= \omega_0 + y(\omega) + \frac{1}{\Gamma(\nu_1)} \int_0^{\varkappa} \varsigma'(\eta)(\varsigma(\varkappa) - \varsigma(\eta))^{\nu_1 - 1} g(\eta) d\eta \\ &+ \frac{1}{\Gamma(\nu_1)} \int_0^{\varkappa} \varsigma'(\eta)(\varsigma(\varkappa) - \varsigma(\eta))^{\nu_1 - 1} \left\{ \int_0^{\eta} \chi_1(\eta, \zeta) \mathcal{N}_1(\omega(\zeta)) d\zeta \right. \\ &+ \left. \int_0^1 \chi_2(\eta, \zeta) \mathcal{N}_2(\omega(\zeta)) d\zeta \right\} d\eta.\end{aligned}$$

Our first result relates to existence based on the Krasnoselkii's fixed point theorem.

Theorem 3.2. *Assume $(H_1)-(H_4)$ hold. Then the $\varsigma-$Hilfer fractional VFIDE* (1.1) *has at least one solution on $\hbar$ if*

$$\Lambda_1 := \left(\ell_y + \frac{\sum_{j=1}^{2} \ell_{\mathcal{N}_j} \chi_j^*}{\Gamma(\nu_1 + 1)} \right) < 1. \tag{3.1}$$

Proof. Think about the ball

$$\mathcal{S}_\gamma = \{ \omega \in \mathbb{C}(\hbar, \mathbb{R}) : \|\omega\|_\infty \leq \gamma \} \subset \mathbb{C}(\hbar, \mathbb{R}). \tag{3.2}$$

$\mathcal{S}_\gamma$ is clearly a nonempty convex closed subset of $\mathbb{C}(\hbar, \mathbb{R})$. Choose γ in such a way that $\gamma \geq \frac{\Lambda_2}{1 - \Lambda_1}$, where $\Lambda_1 < 1$,

$$\Lambda_2 := \mu_0 + \frac{\mu_g + \sum_{j=1}^{2} \mu_{\mathcal{N}_j} \chi_j^*}{\Gamma(\nu_1 + 1)}, \tag{3.3}$$

$\mu_g := \sup_{\varkappa \in [0,1]} |g(\varkappa)|$, $\mu_0 := |\omega_0| + \mu_y$, $\mu_y = |y(0)|$, $\mu_{\mathcal{N}_1} := |\mathcal{N}_1(0)|$ and $\mu_{\mathcal{N}_2} := |\mathcal{N}_2(0)|$.

According to Lemma 3.1, the equivalent fractional integral equation to $\varsigma-$Hilfer fractional VFIDE (1.1) can be expressed as an operator equation as follows

$$\omega = \mathbb{T}_1 \omega + \mathbb{T}_2 \omega, \quad \omega \in \mathcal{S}_\gamma \subset \mathbb{C}(\hbar, \mathbb{R}), \tag{3.4}$$

where $\mathbb{T}_1$ and $\mathbb{T}_2$ are two operators on $\mathcal{S}_\gamma$ defined by

$$\begin{aligned}(\mathbb{T}_1\omega)(\varkappa) &= \frac{1}{\Gamma(\nu_1)}\int_0^{\varkappa} \varsigma'(\eta)(\varsigma(\varkappa)-\varsigma(\eta))^{\nu_1-1} \\ &\quad \times \left\{\int_0^{\eta} \chi_1(\eta,\zeta)\mathcal{N}_1(\omega(\zeta))d\zeta + \int_0^1 \chi_2(\eta,\zeta)\mathcal{N}_2(\omega(\zeta))d\zeta\right\} d\eta\end{aligned}$$

and

$$(\mathbb{T}_2\omega)(\varkappa) = \omega_0 + y(\omega) + \frac{1}{\Gamma(\nu_1)}\int_0^{\varkappa} \varsigma'(\eta)(\varsigma(\varkappa)-\varsigma(\eta))^{\nu_1-1} g(\eta)d\eta.$$

Now, using the conditions of Theorem 3.2, we obtain the fixed point of the operator equation (3.4) as follows:

Step 1: We demonstrate $\mathbb{T}_1\omega + \mathbb{T}_2\varpi \in \mathcal{S}_\gamma$ for each $\omega, \varpi \in \mathcal{S}_\gamma$.

By (H_1) and for any $\omega, \varpi \in \mathcal{S}_\gamma$, we have

$$\begin{aligned}|\mathcal{N}_j(\omega(\varkappa))| &\leq |\mathcal{N}_j(\omega(\varkappa)) - \mathcal{N}_j(0)| + |\mathcal{N}_j(0)| \\ &\leq \ell_{\mathcal{N}_j}\|\omega\|_\infty + |\mathcal{N}_j(0)| \\ &\leq \ell_{\mathcal{N}_j}\gamma + \mu_{\mathcal{N}_j}, \quad \text{for all } j = 1, 2\end{aligned}$$

and

$$\begin{aligned}|y(\varpi(\varkappa))| &\leq |y(\varpi(\varkappa)) - y(0)| + |y(0)| \\ &\leq \ell_y\|\varpi\|_\infty + |y(0)| \\ &\leq \ell_y\gamma + \mu_y.\end{aligned}$$

Let $\omega, \varpi \in \mathcal{S}_\gamma$. Then

$$\begin{aligned}&|(\mathbb{T}_1\omega)(\varkappa) + (\mathbb{T}_2\varpi)(\varkappa)| \\ &\quad\leq \frac{1}{\Gamma(\nu_1)}\int_0^{\varkappa} \varsigma'(\eta)(\varsigma(\varkappa)-\varsigma(\eta))^{\nu_1-1} \\ &\quad\quad \times \left\{\int_0^{\eta} \chi_1(\eta,\zeta)\mathcal{N}_1(\omega(\zeta))d\zeta + \int_0^1 \chi_2(\eta,\zeta)\mathcal{N}_2(\omega(\zeta))d\zeta\right\} d\eta \\ &\quad\quad + |\omega_0| + |y(\varpi)| + \frac{1}{\Gamma(\nu_1)}\int_0^{\varkappa} \varsigma'(\eta)(\varsigma(\varkappa)-\varsigma(\eta))^{\nu_1-1}|g(\eta)|d\eta \\ &\quad\leq \mu_0 + \ell_y\gamma + \frac{\mu_g + \sum_{j=1}^2\left(\ell_{\mathcal{N}_j}\gamma + \mu_{\mathcal{N}_j}\right)\chi_j^*}{\Gamma(\nu_1+1)}(\varsigma(\varkappa))^{\nu_1},\end{aligned}$$

which implies

$$\begin{aligned}\|\mathbb{T}_1\omega + \mathbb{T}_2\varpi\|_\infty &\leq \mu_0 + \frac{\mu_g + \sum_{j=1}^2 \mu_{\mathcal{N}_j}\chi_j^*}{\Gamma(\nu_1+1)} + \left(\ell_y + \frac{\sum_{j=1}^2 \ell_{\mathcal{N}_j}\chi_j^*}{\Gamma(\nu_1+1)}\right)\gamma \\ &\leq \Lambda_2 + \Lambda_1\gamma \leq \gamma.\end{aligned}$$

Consequently, we have

$$\mathbb{T}_1\omega + \mathbb{T}_2\varpi \in \mathcal{S}_\gamma.$$

Step 2: We demonstrate that $\mathbb{T}_2$ is a contraction on $\mathcal{S}_\gamma$.

Let $\omega, \omega^* \in \mathcal{S}_\gamma$. It follows from ($H_4$) that

$$\begin{aligned}\|\mathbb{T}_2\omega - \mathbb{T}_2\omega^*\|_\infty &= \sup_{\varkappa\in\hbar} |\mathbb{T}_2\omega(\varkappa) - \mathbb{T}_2\omega(\varkappa)| \\ &= \sup_{\varkappa\in\hbar} |y(\omega(\varkappa)) - y(\omega^*(\varkappa))| \\ &\le \ell_y \|\omega - \omega^*\|_\infty .\end{aligned}$$

Since $\ell_y < 1$, $\mathbb{T}_2$ is a contraction mapping.

Step 3: We show that, $\mathbb{T}_1$ is completely continuous on $\mathcal{S}_\gamma$.

First, we show that $\mathbb{T}_1$ is continuous. Let $\{\omega_n\}$ be a sequence such that $\omega_n \to \omega$ in $\mathbb{C}(\hbar, \mathbb{R})$. Then for each $\omega_n, \omega \in \mathbb{C}(\hbar, \mathbb{R})$ and for any $\varkappa \in \hbar$, we have

$$\begin{aligned}&|(\mathbb{T}_1\omega_n)(\varkappa) - (\mathbb{T}_1\omega)(\varkappa)| \\ &\quad\le \frac{1}{\Gamma(\nu_1)}\int_0^\varkappa \varsigma'(\eta)(\varsigma(\varkappa)-\varsigma(\eta))^{\nu_1-1}\left(\int_0^\eta |\chi_1(\eta,\zeta)|\,|\mathcal{N}_1(\omega_n(\zeta)) - \mathcal{N}_1(\omega(\zeta))|\,d\zeta\right. \\ &\qquad \left. + \int_0^1 |\chi_2(\eta,\zeta)|\,|\mathcal{N}_2(\omega_n(\zeta)) - \mathcal{N}_2(\omega(\zeta))|\,d\zeta\right)d\eta \\ &\quad\le \frac{\sum_{j=1}^2 \ell_{\mathcal{N}_j}\chi_j^*}{\Gamma(\nu_1+1)}\|\omega_n - \omega\|_\infty .\end{aligned}$$

Since $\omega_n \to \omega$ as $n \to \infty$, $\|\mathbb{T}_1\omega_n - \mathbb{T}_1\omega\|_\infty \to 0$, as $n \to \infty$. This proves that $\mathbb{T}_1$ is continuous on $\mathbb{C}(\hbar, \mathbb{R})$.

Next, from Step 1, we observe that

$$\begin{aligned}|(\mathbb{T}_1\omega)(\varkappa)| &\le \frac{1}{\Gamma(\nu_1)}\int_0^\varkappa \varsigma'(\eta)(\varsigma(\varkappa)-\varsigma(\eta))^{\nu_1-1}\Big(\int_0^\eta |\chi_1(\eta,\zeta)|\,|\mathcal{N}_1(\omega(\zeta))|\,d\zeta \\ &\qquad + \int_0^1 |\chi_2(\eta,\zeta)|\,|\mathcal{N}_2(\omega(\zeta))|\,d\zeta\Big)d\eta \\ &\le \frac{\sum_{j=1}^2 \left(\ell_{\mathcal{N}_j}\gamma + \mu_{\mathcal{N}_j}\right)\chi_j^*}{\Gamma(\nu_1+1)}(\varsigma(\varkappa))^{\nu_1} .\end{aligned}$$

Thus

$$\|\mathbb{T}_1\omega\|_\infty \le \frac{\sum_{j=1}^2 \left(\ell_{\mathcal{N}_j}\gamma + \mu_{\mathcal{N}_j}\right)\chi_j^*}{\Gamma(\nu_1+1)}.$$

This proves that $(\mathbb{T}_1\mathcal{S}_\gamma)$ is uniformly bounded.

Finally, we show that $(\mathbb{T}_1\mathcal{S}_\gamma)$ is equicontinuous. Let $\omega \in \mathcal{S}_\gamma$. Then for $\varkappa_1, \varkappa_2 \in \hbar$ with $\varkappa_1 \leq \varkappa_2$, we have

$$\begin{aligned}
&|(\mathbb{T}_1\omega)(\varkappa_2) - (\mathbb{T}_1\omega)(\varkappa_1)| \\
&= \left| \frac{1}{\Gamma(\nu_1)} \int_0^{\varkappa_2} \varsigma'(\eta)(\varsigma(\varkappa_2) - \varsigma(\eta))^{\nu_1 - 1} \left(\int_0^\eta |\chi_1(\eta,\zeta)| \, |\mathcal{N}_1(\omega(\zeta))| \, d\zeta \right. \right. \\
&\quad \left. + \int_0^1 |\chi_2(\eta,\zeta)| \, |\mathcal{N}_2(\omega(\zeta))| \, d\zeta \right) d\eta \\
&\quad - \frac{1}{\Gamma(\nu_1)} \int_0^{\varkappa_1} \varsigma'(\eta)(\varsigma(\varkappa_1) - \varsigma(\eta))^{\nu_1 - 1} \left(\int_0^\eta |\chi_1(\eta,\zeta)| \, |\mathcal{N}_1(\omega(\zeta))| \, d\zeta \right. \\
&\quad \left. \left. + \int_0^1 |\chi_2(\eta,\zeta)| \, |\mathcal{N}_2(\omega(\zeta))| \, d\zeta \right) d\eta \right| \\
&\leq \frac{1}{\Gamma(\nu_1)} \left(\int_{\varkappa_1}^{\varkappa_2} \varsigma'(\eta)(\varsigma(\varkappa_2) - \varsigma(\varkappa_1))^{\nu_1 - 1} \int_0^\eta |\chi_1(\eta,\zeta)| \, |\mathcal{N}_1(\omega(\zeta))| \, d\zeta d\eta \right. \\
&\quad \left. + \int_0^{\varkappa_1} \varsigma'(\eta) \left| (\varsigma(\varkappa_2) - \varsigma(\eta))^{\nu_1 - 1} - (\varsigma(\varkappa_1) - \varsigma(\eta))^{\nu_1 - 1} \right| \int_0^\eta |\chi_1(\eta,\zeta)| \, |\mathcal{N}_1(\omega(\zeta))| \, d\zeta d\eta \right) \\
&\quad + \frac{1}{\Gamma(\nu_1)} \left(\int_{\varkappa_1}^{\varkappa_2} \varsigma'(\eta)(\varsigma(\varkappa_2) - \varsigma(\eta))^{\nu_1 - 1} \int_0^\eta |\chi_2(\eta,\zeta)| \, |\mathcal{N}_2(\omega(\zeta))| \, d\zeta d\eta \right. \\
&\quad \left. + \int_0^{\varkappa_1} \varsigma'(\eta) \left| (\varsigma(\varkappa_2) - \varsigma(\eta))^{\nu_1 - 1} - (\varsigma(\varkappa_1) - \varsigma(\eta))^{\nu_1 - 1} \right| \int_0^\eta |\chi_2(\eta,\zeta)| \, |\mathcal{N}_2(\omega(\zeta))| \, d\zeta d\eta \right),
\end{aligned}$$

which implies

$$\begin{aligned}
&|(\mathbb{T}_1\omega)(\varkappa_2) - (\mathbb{T}_1\omega)(\varkappa_1)| \\
&\quad \leq \frac{(\ell_{\mathcal{N}_1}\gamma + \mu_{\mathcal{N}_1})\chi_1^*}{\Gamma(\nu_1)} \left(\int_{\varkappa_1}^{\varkappa_2} \varsigma'(\eta)(\varsigma(\varkappa_2) - \varsigma(\eta))^{\nu_1 - 1} d\eta \right. \\
&\quad\quad \left. + \int_0^{\varkappa_1} \varsigma'(\eta) \left| (\varsigma(\varkappa_2) - \varsigma(\eta))^{\nu_1 - 1} - (\varsigma(\varkappa_1) - \varsigma(\eta))^{\nu_1 - 1} \right| d\eta \right) \\
&\quad\quad + \frac{(\ell_{\mathcal{N}_2}\gamma + \mu_{\mathcal{N}_2})\chi_2^*}{\Gamma(\nu_1)} \left(\int_{\varkappa_1}^{\varkappa_2} \varsigma'(\eta)(\varsigma(\varkappa_2) - \varsigma(\eta))^{\nu_1 - 1} d\eta \right. \\
&\quad\quad \left. + \int_0^{\varkappa_1} \varsigma'(\eta) \left| (\varsigma(\varkappa_2) - \varsigma(\eta))^{\nu_1 - 1} - (\varsigma(\varkappa_1) - \varsigma(\eta))^{\nu_1 - 1} \right| d\eta \right)
\end{aligned}$$

$$\leq \left(\frac{(\ell_{\mathcal{N}_1}\gamma+\mu_{\mathcal{N}_1})\chi_1^*}{\Gamma(\nu_1+1)}+\frac{(\ell_{\mathcal{N}_2}\gamma+\mu_{\mathcal{N}_2})\chi_2^*}{\Gamma(\nu_1+1)}\right)$$
$$\times\left(\frac{(\varsigma(\varkappa_2)-\varsigma(\varkappa_1))^{\nu_1}}{\nu_1}+\frac{\varsigma(\varkappa_1)}{\nu_1}-\frac{\varsigma(\varkappa_2)}{\nu_1}+\frac{(\varsigma(\varkappa_2)-\varsigma(\varkappa_1))^{\nu_1}}{\nu_1}\right)$$
$$\leq \frac{2\sum_{j=1}^{2}\left(\ell_{\mathcal{N}_j}\gamma+\mu_{\mathcal{N}_j}\right)\chi_j^*}{\Gamma(\nu_1+1)}(\varsigma(\varkappa_2)-\varsigma(\varkappa_1))^{\nu_1},$$

which tends to zero as $\varkappa_2-\varkappa_1\to 0$. So, $(\mathbb{T}_1\mathcal{S}_\gamma)$ is equicontinuous. As a result of this, as well as the Arzela-Ascoli theorem, it is concluded that $\mathbb{T}_1:\mathbb{C}(\hbar,\mathbb{R})\to\mathbb{C}(\hbar,\mathbb{R})$ is continuous and completely continuous. An application of Krasnoselskii's fixed point theorem demonstrates that $\mathbb{T}_1$ has a fixed point ω in $\mathcal{S}_\gamma$ which is a solution of the $\varsigma-$Hilfer fractional VFIDE (1.1). □

The second result is based on Banach's fixed point theorem.

Theorem 3.3. *Assume* $(H_1)-(H_4)$ *hold. If*

$$\Lambda_1<1, \tag{3.5}$$

then the $\varsigma-$*Hilfer fractional VFIDE* (1.1) *has a unique solution on* $\hbar$.

Proof. According to Lemma 3.1, the equivalent fractional integral equation to $\varsigma-$Hilfer fractional VFIDE (1.1) can be expressed as operator equation as follows

$$\omega=\Upsilon\omega,\quad \omega\in\mathbb{C}(\hbar,\mathbb{R}),$$

where the operator $\Upsilon:\mathbb{C}(\hbar,\mathbb{R})\to\mathbb{C}(\hbar,\mathbb{R})$ defined by

$$(\Upsilon\omega)(\varkappa)=\omega_0+y(\omega)+\frac{1}{\Gamma(\nu_1)}\int_0^{\varkappa}\varsigma'(\eta)(\varsigma(\varkappa)-\varsigma(\eta))^{\nu_1-1}g(\eta)d\eta$$
$$+\frac{1}{\Gamma(\nu_1)}\int_0^{\varkappa}\varsigma'(\eta)(\varsigma(\varkappa)-\varsigma(\eta))^{\nu_1-1}\left(\int_0^{\eta}\chi_1(\eta,\zeta)\mathcal{N}_1(\omega(\zeta))d\zeta\right.$$
$$\left.+\int_0^{1}\chi_2(\eta,\zeta)\mathcal{N}_2(\omega(\zeta))d\zeta\right)d\eta$$

for all $\varkappa\in\hbar$.

Let $\omega,\omega^{*}\in\mathbb{C}(\hbar,\mathbb{R})$. Then for each $\varkappa\in\hbar$ we have

$$\begin{aligned}
|\Upsilon\omega(\varkappa)-\Upsilon\omega^{*}(\varkappa)| &\le |y(\omega(\varkappa))-y(\omega^{*}(\varkappa))| \\
&\quad+\frac{1}{\Gamma(\nu_1)}\int_0^{\varkappa}\varsigma'(\eta)(\varsigma(\varkappa)-\varsigma(\eta))^{\nu_1-1} \\
&\quad\times\Big(\int_0^{\eta}\chi_1(\eta,\zeta)\left|\mathcal{N}_1(\omega(\zeta))-\mathcal{N}_1(\omega^{*}(\zeta))\right|d\zeta\Big)d\eta \\
&\quad+\frac{1}{\Gamma(\nu_1)}\int_0^{\varkappa}\varsigma'(\eta)(\varsigma(\varkappa)-\varsigma(\eta))^{\nu_1-1} \\
&\quad\times\Big(\int_0^{1}\chi_2(\eta,\zeta)\left|\mathcal{N}_2(\omega(\zeta))-\mathcal{N}_2(\omega^{*}(\zeta))\right|d\zeta\Big)d\eta \\
&\le \ell_y\left\|\omega-\omega^{*}\right\|_{\infty} \\
&\quad+\frac{1}{\Gamma(\nu_1)}\int_0^{\varkappa}\varsigma'(\eta)(\varsigma(\varkappa)-\varsigma(\eta))^{\nu_1-1}\chi_1^{*}\ell_{\mathcal{N}_1}\left\|\omega-\omega^{*}\right\|_{\infty}d\eta \\
&\quad+\frac{1}{\Gamma(\nu_1)}\int_0^{\varkappa}\varsigma'(\eta)(\varsigma(\varkappa)-\varsigma(\eta))^{\nu_1-1}\chi_2^{*}\ell_{\mathcal{N}_2}\left\|\omega-\omega^{*}\right\|_{\infty}d\eta \\
&\le\left(\ell_y+\frac{\chi_1^{*}\ell_{\mathcal{N}_1}+\chi_2^{*}\ell_{\mathcal{N}_2}}{\Gamma(\nu_1+1)}\left(\varsigma(\varkappa)\right)^{\nu_1}\right)\left\|\omega-\omega^{*}\right\|_{\infty},
\end{aligned}$$

which implies

$$\left\|\Upsilon\omega-\Upsilon\omega^{*}\right\|_{\infty}\le\left(\ell_y+\frac{\sum_{j=1}^{2}\ell_{\mathcal{N}_j}\chi_j^{*}}{\Gamma(\nu_1+1)}\right)\left\|\omega-\omega^{*}\right\|_{\infty}.$$

The inequality (3.5) shows that Υ is a contraction mapping on $\mathbb{C}(\hbar,\mathbb{R})$. As a result of Banach's fixed point theorem, Υ has a unique fixed point that is the solution of the $\varsigma-$Hilfer fractional VFIDE (1.1). □

4. Approximate solution

In this section, we present an approximate solution to the $\varsigma-$Hilfer fractional VFIDE (1.1) using the fractional Adomian decomposition technique.

First, we recall the classical Adomian decomposition technique, which yields the solution to our problem as a series

$$\omega=\sum_{n=0}^{\infty}\omega_n \tag{4.1}$$

and the nonlinear terms $\mathcal{N}_1,\mathcal{N}_2$ and y are decomposed as

$$\mathcal{N}_1=\sum_{n=0}^{\infty}A_n,\ \ \mathcal{N}_2=\sum_{n=0}^{\infty}B_n,\ y=\sum_{n=0}^{\infty}D_n, \tag{4.2}$$

where A_n, B_n, D_n are Adomian polynomials for all $n \in \mathbb{N}$, and write

$$\omega = \omega(\lambda) = \sum_{n=0}^{\infty} \lambda^n \omega_n = \omega_0 + \lambda\omega_1 + \lambda^2\omega_2 + \cdots + \lambda^k\omega_k + \cdots, \tag{4.3}$$

$$\mathcal{N}_1 = \mathcal{N}_1(\lambda) = \sum_{n=0}^{\infty} \lambda^n A_n = A_0 + \lambda A_1 + \lambda^2 A_2 + \cdots + \lambda^k A_k + \cdots, \tag{4.4}$$

$$\mathcal{N}_2 = \mathcal{N}_2(\lambda) = \sum_{n=0}^{\infty} \lambda^n B_n = B_0 + \lambda B_1 + \lambda^2 B_2 + \cdots + \lambda^k B_k + \cdots, \tag{4.5}$$

$$y = y(\lambda) = \sum_{n=0}^{\infty} \lambda^n D_n = D_0 + \lambda D_1 + \lambda^2 D_2 + \cdots + \lambda^k D_k + \cdots. \tag{4.6}$$

Using the previous formulas (4.3), (4.4), (4.5) and (4.6), we can conclude that

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left(\mathcal{N}_1 \sum_{j=0}^{\infty} \lambda^j \omega_j\right)\right]_{\lambda=0},$$

$$B_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left(\mathcal{N}_2 \sum_{j=0}^{\infty} \lambda^j \omega_j\right)\right]_{\lambda=0}$$

and

$$D_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left(y \sum_{j=0}^{\infty} \lambda^j \omega_j\right)\right]_{\lambda=0},$$

where ω_0, ω_1, $\omega_2, ...$ are repeatedly specified by

$$\begin{cases} \omega_0(\varkappa) = \omega_0 + \mathcal{I}_{0^+}^{\nu_1;\varsigma}\left(g(\varkappa)\right), \\ \omega_{k+1}(\varkappa) = D_k + \mathcal{I}_{0^+}^{\nu_1;\varsigma}\left(\int_0^{\varkappa} \chi_1(\varkappa,\xi) A_k d\xi\right) \\ \qquad +\mathcal{I}_{0^+}^{\nu_1;\varsigma}\left(\int_0^1 \chi_2(\varkappa,\xi) B_k d\xi\right), \quad k \geq 1. \end{cases} \tag{4.7}$$

Now, we use the modified Adomian decomposition method, and the scheme (4.7) yields

$$\begin{cases} \omega_0(\varkappa) = \omega_0 + R_1(\varkappa), \\ \omega_1(\varkappa) = R_2(\varkappa) + D_0 + \mathcal{I}_{0^+}^{\nu_1;\varsigma}\left(\int_0^{\varkappa} \chi_1(\varkappa,\xi) A_0 d\xi\right) \\ \qquad +\mathcal{I}_{0^+}^{\nu_1;\varsigma}\left(\int_0^1 \chi_2(\varkappa,\xi) B_0 d\xi\right), \\ \omega_{k+1}(\varkappa) = D_k + \mathcal{I}_{0^+}^{\nu_1;\varsigma}\left(\int_0^{\varkappa} \chi_1(\varkappa,\xi) A_k d\xi\right) \\ \qquad +\mathcal{I}_{0^+}^{\nu_1;\varsigma}\left(\int_0^1 \chi_2(\varkappa,\xi) B_k d\xi\right), \quad k \geq 1. \end{cases} \tag{4.8}$$

Now, we will investigate the convergence theorem of the solution based on the MADM.

Theorem 4.1. *Assume that* $(H_1)-(H_4)$ *and* (3.1) *are satisfied, if the solution* $\omega(\varkappa) = \sum_{j=0}^{\infty} \omega_j(\varkappa)$ *and* $\|\omega\|_\infty < \infty$ *is convergent, then it converges to the exact solution of the* $\varsigma-$*Hilfer fractional VFIDE* (1.1).

Proof. We omit the proof because it is similar to some works found in the literature, see [8]. □

5. An example

Example 5.1. Consider an integro-differential equation with $\varsigma-$Hilfer fractional derivative

$$\begin{cases} {}^{C}\mathcal{D}_{0^+}^{\nu_1,\nu_2;\varsigma}\omega(\varkappa) = \frac{2}{\sqrt{\pi}}\left(\frac{4\varkappa^{\frac{3}{2}}}{\Gamma(6)} + \varkappa^{\frac{1}{2}}\right) + \frac{\varkappa^3}{\Gamma(7)} + \frac{\varkappa}{\Gamma(8)} \\ +\frac{1}{4}\int_0^{\varkappa}(1+\varkappa-\eta)\omega(\eta)d\eta + \frac{5}{18}\int_0^1 e^{\eta-\varkappa}\omega^2(\eta)d\eta \end{cases} \tag{5.1}$$

with the nonlocal condition

$$\omega(0) = \frac{1}{4}\omega(\frac{1}{3}), \tag{5.2}$$

where $\nu_1 = \frac{1}{4}$, $\nu_2 = \frac{1}{3}$, $\varsigma(\varkappa) = \varkappa$, $\omega_0 = 0$, $y(\omega(\varkappa)) = \frac{1}{4}\omega(\frac{1}{3})$,

$$\begin{aligned} g(\varkappa) &= \frac{2}{\sqrt{\pi}}\left(\frac{4\varkappa^{\frac{3}{2}}}{\Gamma(6)} + \varkappa^{\frac{1}{2}}\right) + \frac{\varkappa^3}{\Gamma(7)} + \frac{\varkappa}{\Gamma(8)}, \\ \chi_1(\varkappa,\xi) &= \frac{1}{4}(1+\varkappa-\xi),\ \chi_2(\varkappa,\xi) = \frac{5}{18}e^{\xi-\varkappa}. \end{aligned}$$

Clearly, $\ell_{\mathcal{N}_1} = \ell_{\mathcal{N}_2} = 1$, $\ell_y = \frac{1}{4}$,

$$\begin{aligned} \mu_g : &= \sup_{\varkappa\in[0,1]} |g(\varkappa)| = \|g\|_\infty \\ &= \frac{2}{\sqrt{\pi}}\left(\frac{4}{\Gamma(6)} + 1\right) + \frac{1}{\Gamma(7)} + \frac{1}{\Gamma(8)} \\ &= \frac{31}{15\sqrt{\pi}} + \frac{1}{630}, \end{aligned}$$

$$\begin{aligned} \chi_1^* &= \frac{1}{4}\sup_{\varkappa\in\hbar}\int_0^{\varkappa}|1+\varkappa-\xi|\,d\xi = \frac{1}{8}, \\ \chi_2^* &= \frac{5}{18}\sup_{\varkappa\in\hbar}\int_0^{\varkappa}\left|e^{\xi-\varkappa}\right|d\xi = \frac{7}{20}\sup_{\varkappa\in\hbar} e^{-\varkappa}\int_0^{\varkappa}\left|e^{\xi}\right|d\xi \\ &= \frac{5}{18}(1-\frac{1}{e}). \end{aligned}$$

Hence,

$$\Lambda_1 := \left(\ell_y + \frac{\sum_{j=1}^{2}\ell_{\mathcal{N}_j}\chi_j^*}{\Gamma(\nu_1+1)}\right) \approx 0.581\,63 < 1.$$

As consequence of Theorem 3.3, the problem (5.1)-(5.2) has a unique solution on $[0, 1]$.

Applying the operator $\left(\mathcal{I}_{0^+}^{\frac{1}{4};\varsigma},\ \varsigma(\varkappa) = \varkappa\right)$ to both sides of equation (5.1), we get

$$\begin{aligned}\omega(\varkappa) &= \frac{1}{4}\omega(\frac{1}{3}) + \mathcal{I}_{0^+}^{\frac{1}{4};\varkappa}\left(\frac{2}{\sqrt{\pi}}\left(\frac{4\varkappa^{\frac{3}{2}}}{\Gamma(6)} + \varkappa^{\frac{1}{2}}\right) + \frac{\varkappa^3}{\Gamma(7)} + \frac{\varkappa}{\Gamma(8)}\right)\\ &\quad + \mathcal{I}_{0^+}^{\frac{1}{4};\varkappa}\left(\frac{1}{4}\int_0^{\varkappa}(1+\varkappa-\eta)\omega(\eta)d\eta\right) + \mathcal{I}_{0^+}^{\frac{1}{4};\varkappa}\left(\frac{5}{18}\int_0^1 e^{\eta-\varkappa}\omega^2(\eta)d\eta\right).\end{aligned}$$

Suppose

$$\begin{aligned}R(\varkappa) &= \mathcal{I}_{0^+}^{\frac{1}{4};\varkappa}\left(\frac{2}{\sqrt{\pi}}\left(\frac{4\varkappa^{\frac{3}{2}}}{\Gamma(6)} + \varkappa^{\frac{1}{2}}\right) + \frac{\varkappa^3}{\Gamma(7)} + \frac{\varkappa}{\Gamma(8)}\right)\\ &= \frac{2}{\sqrt{\pi}}\frac{4}{\Gamma(6)}\left(\mathcal{I}_{0^+}^{\frac{1}{4};\varkappa}\eta^{\frac{3}{2}}\right)(\varkappa) + \frac{2}{\sqrt{\pi}}\left(\mathcal{I}_{0^+}^{\frac{1}{4};\varkappa}\eta^{\frac{1}{2}}\right)(\varkappa)\\ &\quad + \frac{1}{\Gamma(7)}\left(\mathcal{I}_{0^+}^{\frac{1}{4};\varkappa}\eta^{3}\right)(\varkappa) + \frac{1}{\Gamma(8)}\left(\mathcal{I}_{0^+}^{\frac{1}{4};\varkappa}\eta\right)(\varkappa)\\ &= \frac{2}{\sqrt{\pi}}\frac{4}{\Gamma(6)}\left(\frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{5}{2}+\frac{1}{4})}\varkappa^{\frac{5}{2}}\right) + \frac{2}{\sqrt{\pi}}\left(\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{4}+\frac{1}{2})}\varkappa^{-\frac{1}{4}}\right)\\ &\quad + \frac{1}{\Gamma(7)}\left(\frac{\Gamma(4)}{\Gamma(\frac{1}{4}+4)}\varkappa^{\frac{13}{4}}\right) + \frac{1}{\Gamma(8)}\left(\frac{\Gamma(2)}{\Gamma(\frac{3}{4})}\varkappa^{\frac{5}{4}}\right).\end{aligned}$$

Now, we apply the modified Adomian decomposition method,

$$R(\varkappa) = R_1(\varkappa) + R_2(\varkappa),$$

where

$$R_1(\varkappa) = \frac{8}{\sqrt{\pi}\Gamma(6)}\frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{11}{4})}\varkappa^{\frac{5}{2}}$$

and

$$R_2(\varkappa) = \frac{2}{\sqrt{\pi}}\left(\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4})}\varkappa^{-\frac{1}{4}}\right) + \frac{1}{\Gamma(7)}\left(\frac{\Gamma(4)}{\Gamma(\frac{17}{4})}\varkappa^{\frac{13}{4}}\right) + \frac{1}{\Gamma(8)}\left(\frac{\Gamma(2)}{\Gamma(\frac{3}{4})}\varkappa^{\frac{5}{4}}\right).$$

The modified recursive relation

$$\omega_0(\varkappa) = R_1(\varkappa) = \frac{8}{\sqrt{\pi}\Gamma(6)}\frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{11}{4})}\varkappa^{\frac{5}{2}},$$

$$\begin{aligned}
\omega_1(\varkappa) &= R_2(\varkappa) + \mathcal{I}_{0^+}^{\frac{1}{4};\varkappa}\left(\frac{1}{4}\int_0^{\varkappa}(1+\varkappa-\eta)A_0(\eta)d\eta\right) \\
&\quad + \mathcal{I}_{0^+}^{\frac{1}{4};\varkappa}\left(\frac{5}{18}\int_0^{1} e^{\eta-\varkappa}B_0(\eta)d\eta\right) + D_0(\varkappa) \\
&= \frac{2}{\sqrt{\pi}}\left(\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4})}\varkappa^{-\frac{1}{4}}\right) + + \frac{1}{\Gamma(7)}\left(\frac{\Gamma(4)}{\Gamma(\frac{17}{4})}\varkappa^{\frac{13}{4}}\right) + \frac{1}{\Gamma(8)}\left(\frac{\Gamma(2)}{\Gamma(\frac{3}{4})}\varkappa^{\frac{5}{4}}\right) \\
&\quad + \mathcal{I}_{0^+}^{\frac{1}{4};\varkappa}\left(\frac{1}{4}\int_0^{\varkappa}(1+\varkappa-\eta)\omega_0(\eta)d\eta\right) \\
&\quad + \mathcal{I}_{0^+}^{\frac{1}{4};\varkappa}\left(\frac{5}{18}\int_0^{1} e^{\eta-\varkappa}\omega_0(\eta)d\eta\right) + \frac{1}{5}\omega_0(\frac{1}{4}) \\
&= \frac{2}{\sqrt{\pi}}\left(\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4})}\varkappa^{-\frac{1}{4}}\right) + + \frac{1}{\Gamma(7)}\left(\frac{\Gamma(4)}{\Gamma(\frac{17}{4})}\varkappa^{\frac{13}{4}}\right) + \frac{1}{\Gamma(8)}\left(\frac{\Gamma(2)}{\Gamma(\frac{3}{4})}\varkappa^{\frac{5}{4}}\right) \\
&\quad + \mathcal{I}_{0^+}^{\frac{1}{4};\varkappa}\left(\frac{1}{4}\int_0^{\varkappa}(1+\varkappa-\eta)\left(\frac{15}{\sqrt{\pi}}\frac{\Gamma(\frac{11}{2})}{(\frac{1}{3})^{\frac{1}{2}}\,\Gamma(7)\Gamma(6)}\eta^{\frac{5}{3}}\right)d\eta\right) \\
&\quad + \mathcal{I}_{0^+}^{\frac{1}{4};\varkappa}\left(\frac{5}{18}\int_0^{1} e^{\eta-\varkappa}\left(\frac{8}{\sqrt{\pi}\Gamma(6)}\frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{11}{4})}\eta^{\frac{5}{2}}\right)d\eta\right) \\
&\quad + \frac{1}{4}\left(\frac{8}{\sqrt{\pi}\Gamma(6)}\frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{11}{4})}\right)\left(\frac{1}{3}\right)^{\frac{5}{2}} \\
&= 0,
\end{aligned}$$

$$\omega_2(\varkappa) = 0,$$

$$\vdots$$

$$\omega_n(\varkappa) = 0.$$

Therefore, the obtained solution is

$$\omega(\varkappa) = \sum_{j=0}^{\infty}\omega_j(\varkappa) = \frac{8}{\sqrt{\pi}\Gamma(6)}\frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{11}{4})}\varkappa^{\frac{5}{2}}.$$

6. Conclusion

In this work, we have already studied the fractional Volterra Fredholm integrodifferential equation involving $\varsigma-$Hilfer fractional derivative. Also, we have derived the solution representation of the problem (1.1). Moreover, the convergence of approximated solutions and existence solutions has been obtained. The theoretical analysis is based on some classic fixed point theories such as Banach and Krasnoselskii and the fractional Adomian decomposition technique. An example was provided as relevant to the results.

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