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SEVEN-PARAMETER MITTAG-LEFFLER OPERATOR WITH SECOND-ORDER DIFFERENTIAL SUBORDINATION RESULTS

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Abstract. This paper constructs a new linear operator associated with a seven parameters Mittag-Leffler function using the convolution technique. In addition, it investigates some significant second-order differential subordination properties with considerable sandwich results concerning that operator.

1. Introduction

The special function, Mittag-Leffler function, arose in 1903 as an immediate generalization of the exponential function by the mathematician Gosta Mittag-Leffler as

$$E_\tau(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\tau n + 1)}, \tag{1.1}$$

where z $\in \mathbb{C}$ and $Re(\tau) > 0$ [12]. Then, Wiman proposed a new general form of Mittag-Leffler function as

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$$E_{\tau,\lambda}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\tau n+\lambda)}, \tag{1.2}$$

where $z\in\mathbb{C}, Re(\tau)>0$ and $Re(\lambda)>0$ [13]. Thereafter, many researchers interested in studying those functions, their various properties and applications, due to their significant in the solution of fractional order differential and integral equations [8, 9, 14, 16, 19, 27].

Recently, Rasheed and Majeed [18], introduced a new generalized Mittag-Leffler function that considering seven complex parameters by

$$E^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n\ n!}\frac{z^n}{\Gamma(\tau_1 n+\lambda_1)\Gamma(\tau_2 n+\lambda_2)}, \tag{1.3}$$

where $z,\tau_1,\tau_2\in\mathbb{C}$ and $\min\{Re(a),Re(b),Re(c),Re(\lambda_1),Re(\lambda_2)\}>0$. They also confirmed that this function is an entire function of finite order. Further, its noteworthy to mention that the function (1.3) generalizes the standard Mittag-Leffler function, Kummer function and Gaussian hypergeometric function.

The Mittag-Leffler function and their generalizations had attracted wide interest to involved it in the geometric function theory and its applications, such as operators defining and their consequent properties [1, 4, 7, 20, 26].

Let $\mathbb{H}$ be a class of holomorphic functions in the open unit disk $\mathbb{U}$, also let A be a subclass of $\mathbb{H}$ that containing the functions normalized by the form [10]

$$f(z)=z+\sum_{n=2}^{\infty}u_n z^n,\quad u_i\in\mathbb{C}\ \ (i=2,...,n). \tag{1.4}$$

The Hadamard product (or convolution) of two functions $g_1,g_2\in A$ is denoted by g_1*g_2 and defined as

$$(g_1*g_2)(z)=z+\sum_{n=2}^{\infty}\alpha_n\mu_n z^n, \tag{1.5}$$

where α_n,μ_n are the respective coefficient from the series representation of the functions g_1 and g_2, such that $\alpha_i,\mu_i\in\mathbb{C},(i=2,...,n)$. Noting that, the convolution of two functions in A is again a function in A [21]. That product technique basically appeared as significant tool for constructing operators, as well as, describing many differential and integral operators in terms of convolution.

Let f_1 and f_2 be members of the class $\mathbb{H}$, we say that the function f_1 subordinate to f_2, denoted $f_1\prec f_2$ if there exist a schwarz function w such that

$f_1(z) = f_2(w(z))$. If f_2 univalent, then $f_1 \prec f_2$ if and only if $f_1(0) = f_2(0)$ and $f_1(\mathbb{U}) \subset f_2(\mathbb{U})$. Note that, if f_1 subordinate to f_2, then f_2 superordinate to f_1 [17].

Let $\pi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ and h be univalent in $\mathbb{U}$. If θ is holomorphic in $\mathbb{U}$ and satisfies the second-order differential subordination

$$\pi\left(\theta(z), z\acute{\theta}(z), z^2\acute{\acute{\theta}}(z); z\right) \prec h(z), \tag{1.6}$$

then θ is called a solution of (1.6). The univalent function ν is called dominant of the solutions of (1.6), if $\theta \prec \nu$ for all θ satisfying (1.6). A dominant $\tilde{\nu}$ that satisfies $\tilde{\nu} \prec \nu$ for all dominant ν of (1.6) is said to be the best dominant [24].

Likewise, a corresponding concept to the second-order differential subordination had been committed, known as the second-order differential superordination. Let $\pi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ and h be univalent in $\mathbb{U}$. If θ is holomorphic in $\mathbb{U}$ and satisfies the second-order differential superordination

$$h(z) \prec \pi\left(\theta(z), z\acute{\theta}(z), z^2\acute{\acute{\theta}}(z); z\right), \tag{1.7}$$

then θ is called a solution of (1.7). The univalent function ν is called subordinant of the solutions of (1.7), if $\theta \prec \nu$ for all θ satisfying (1.7). A subordinant $\tilde{\nu}$ that satisfies $\tilde{\nu} \prec \nu$ for all subordinants ν of (1.7) is said to be a best subordinant [11].

Subsequently, Ali et al. [3] assumed certain sufficient conditions for the function $f \in A$ to satisfy

$$\nu_1(z) \prec \frac{z\acute{f}(z)}{f(z)} \prec \nu_2(z), \tag{1.8}$$

where ν_1 and ν_2 are univalent functions in $\mathbb{U}$ with $\nu_1(0) = 1$. Following that, Shanmugam et al. [22, 23] had established another conditions for the function $f \in A$ for the same conditions on ν that Ali et. al. set with $\nu_2(0) = 1$, to achieve the following implications

$$\nu_1(z) \prec \frac{f(z)}{z\acute{f}(z)} \prec \nu_2(z), \tag{1.9}$$

$$\nu_1(z) \prec \frac{z^2\acute{f}(z)}{(f(z))^2} \prec \nu_2(z). \tag{1.10}$$

Thereafter, numerous researchers investigate many various properties and applications concerning to differential subordination and superordinations in crucial fields of mathematics, such as kinetic equations, fractional calculus, and geometric theory of functions, see [2, 6, 15, 25, 28, 29, 30].

Throughout this paper, we apply the convolution method to define new linear operator linked to seven-parameter Mittag-Leffler function which given in (1.3). In order to construct that operator, we assume the following normalization

$$T^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z) = \frac{c\Gamma(\tau_1+\lambda_1)\Gamma(\tau_2+\lambda_2)}{ab}\left(E^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z) - \frac{1}{\Gamma(\lambda_1)\Gamma(\lambda_2)}\right), \tag{1.11}$$

where $z, \tau_1, \tau_2 \in \mathbb{C}$ and $\min\{Re(a), Re(b), Re(c), Re(\lambda_1), Re(\lambda_2)\} > 0$.

Let $f \in A$, we introduce a new linear operator $M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} : A \to A$ such that

$$\begin{aligned} M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z) &= T^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{(a)_n}{a}\frac{(b)_n}{b}\frac{c}{(c)_n}\frac{\Gamma(\tau_1+\lambda_1)\Gamma(\tau_2+\lambda_2)}{\Gamma(\tau_1 n+\lambda_1)\Gamma(\tau_2 n+\lambda_2)\, n!}\, u_n z^n. \end{aligned} \tag{1.12}$$

Observe that, $M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)$ involve some well-known functions as special cases:

(1) $M^{1,b,b}_{0,\lambda_1,0,\lambda_2} f(z) = f(z)$,
(2) $M^{1,b,b}_{1,0,0,\lambda_2} f(z) = z\acute{f}(z)$ (Alexander operator),
(3) $M^{2,b,b}_{1,1,0,1} f(z) = \frac{(zf(z))\acute{}}{2}$ (Livingstone operator),
(4) $M^{1,b,b}_{1,1,0,\lambda_2}(\frac{z}{1-z}) = e^z - 1$,
(5) $M^{1,b,b}_{2,1,0,\lambda_2}(\frac{z}{1-z}) = \cosh(\sqrt{z}) - 2$.

In addition, we achieved the following necessary relations in view of (1.12):

$$z\left(M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)\right)\acute{} = (a+1)M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z) - aM^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z). \tag{1.13}$$

The major idea of this paper, is to introduce a new operator linked to Mittag-Leffler function with seven complex parameters in terms of convolution method. Additionally, it illustrates certain interesting second-order differential subordination results for that operator. Besides, specific sandwich results have been established.

2. Preliminaries

We state some necessary definition and lemmas which are required to establish our basic results:

Definition 2.1. ([10]) Let Λ be a set of all functions $f(z)$ that are holomorphic and univalent on $\overline{U}/E(f)$, where $\overline{U} = \mathbb{U} \cup \partial\mathbb{U}$ and

$$E(f) = \left\{ s \in \partial\mathbb{U} : \lim_{z\to s} f(z) = \infty \right\}$$

such that $\acute{f}(z) \neq 0$ for $s \in \partial\mathbb{U} \backslash E(f)$.

Lemma 2.2. ([11]) *Let ν be a convex function in $\mathbb{U}$ and let $\kappa, \delta \in \mathbb{C}$ with $\delta \neq 0$ such that*

$$Re\left\{ \frac{z\acute{\acute{\nu}}(z)}{\acute{\nu}(z)} + 1 \right\} > max\left\{0; -Re\left(\frac{\kappa}{\delta}\right)\right\},\ z \in \mathbb{U}.$$

If ρ is holomorphic in $\mathbb{U}$ and $\kappa\,\theta(z) + \delta z\acute{\theta}(z) \prec \kappa\,\nu(z) + \delta z\,\acute{\nu}(z)$, then $\theta(z) \prec \nu(z)$ and ν is the best dominant.

Lemma 2.3. ([10]) *Let ν be a univalent function in $\mathbb{U}$ and let the functions F and G be holomorphic in a domain D containing $\nu(\mathbb{U})$ with $G(s) \neq 0$ when $s \in \nu(\mathbb{U})$. Put*

$$\nu(z) = z\,\acute{\nu}(z)\,G(z), \quad T(z) = F(\nu(z)) + \nu(z).$$

In addition, suppose that

(1) *ν is a starlike function in $\mathbb{U}$,*
(2) *$Re\left\{\frac{z\acute{T}(z)}{\nu(z)}\right\} > 0$ for $z \in \mathbb{U}$.*
If θ is holomorphic function in $\mathbb{U}$ with $\theta(0) = \nu(0)$, $\theta(\mathbb{U}) \subseteq D$ and

$$F[\,\theta(z)] + z\,\acute{\theta}(z)\,G[\theta(z)] \prec F[\nu(z)] + z\,\acute{\nu}(z), G[\nu(z)].$$

Then $\theta(z) \prec \nu(z)$ and $\nu(z)$ is the best dominant.

Lemma 2.4. ([5]) *Let ν be a convex univalent function in $\mathbb{U}$ and let F and G be holomorphic in a domain D containing $\nu(\mathbb{U})$. Suppose that*

(1) *$Re\left\{\frac{\acute{F}(\nu(z))}{G(\nu(z))}\right\} > 0, \quad z \in \mathbb{U}$.*
(2) *$z\,\acute{\nu}(z)\,G[\nu(z)]$ is starlike in $\mathbb{U}$.*

If $\theta \in \mathbb{H}[\nu(0), 1] \cap \Lambda$ with $\theta(\mathbb{U}) \subset D$ and $F[(\theta(z)] + z, \acute{\theta}(z), G[\theta(z)]$ is univalent in $\mathbb{U}$ such that

$$F[(\nu(z)] + z\,\acute{\nu}(z)\,G[\nu(z)] \prec F[(\theta(z)] + z\,\acute{\theta}(z)\,G[\theta(z)],$$

then $\nu(z) \prec \theta(z)$ and $\nu(z)$ is the best subordinant.

Lemma 2.5. ([11]) *Let ν be a convex function in $\mathbb{U}$ and let $\beta \in \mathbb{C}$ with $Re(\beta) > 0$. If $\theta \in \mathbb{H}[\nu(0), 1] \cap \Lambda$ and $\theta(z) + \beta z \acute{\theta}(z)$ univalent in $\mathbb{U}$, then*

$$\nu(z) + \beta z \acute{\nu}(z) \prec \theta(z) + \beta z \acute{\theta}(z),$$

implies $\nu(z) \prec \theta(z)$ and ν is the best subordinant.

Lemma 2.6. ([10]) *Let $\nu(z)$ be a univalent function in $\mathbb{U}$. Consider F and G be holomorphic functions in a domain D containing $\nu(\mathbb{U})$ with $G(w) \neq 0$ when $w \in \nu(z)$. Set*

$$\phi(z) = z \acute{\nu}(z)\, G[\nu(z)], \quad T(z) = F[\nu(z)] + \phi(z).$$

Suppose that either $T(z)$ is convex or $\phi(z)$ is starlike. In addition, assume that

$$Re\left(\frac{z\acute{T}(z)}{\phi(z)}\right) > 0.$$

If

$$F[\theta(z)] + z \acute{\theta}(z)\, G[\theta(z)] \prec F[\nu(z)] + z \acute{\nu}(z)\, G[\nu(z)] = T(z),$$

then $\theta(z) \prec \nu(z)$ and $\nu(z)$ is the best dominant.

3. Second-order differential subordinations involving $M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)$

Here, we confirm certain second-order differential subordination major results concerning the linear operator introduced in (1.12).

Theorem 3.1. *Let ν be a convex univalent function in U with $\nu(0) = 1, \rho > 0$ and $\xi \in \mathbb{C}\backslash\{0\}$. Assume*

$$Re\left\{\frac{z\acute{\acute{\nu}}(z)}{\acute{\nu}(z)} + 1\right\} > max\left\{0; -\rho Re\left(\frac{1}{\xi}\right)\right\}. \tag{3.1}$$

If $f \in A$ satisfies the following relation

$$\left(\frac{M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)}{z}\right)^{\rho} + \xi(a+1)\left(\frac{M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)}{z}\right)^{\rho}\left(\frac{M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)}{M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)} - 1\right)$$
$$\prec \nu(z) + \frac{\xi}{\rho} z\acute{\nu}(z), \tag{3.2}$$

then

$$\left(\frac{M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)}{z}\right)^{\rho} \prec \nu(z) \tag{3.3}$$

and $\nu(z)$ *is the best dominant of* (3.2).

Proof. Suppose that

$$\theta(z) = \left(\frac{M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)}{z}\right)^{\rho}. \tag{3.4}$$

Then it is obvious that the function $\theta(z)$ is holomorphic in $\mathbb{U}$ and $\theta(0) = 1$. Differentiate the function θ logarithmically with respect to z then use identity (1.13), yields

$$\frac{z\acute{\theta}(z)}{\theta(z)} = \rho(a+1)\left[\frac{M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)}{M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)} - 1\right].$$

Hence,

$$\frac{z\acute{\theta}(z)}{\rho} = (a+1)\left(\frac{M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)}{z}\right)^{\rho}\left[\frac{M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)}{M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)} - 1\right],$$

follows that expression (3.2) can be written as

$$\theta(z) + \frac{\xi}{\rho}z\acute{\theta}(z) \prec \nu(z) + \frac{\xi}{\rho}z\acute{\nu}(z).$$

Therefore, after applying Lemma 2.2 with $\kappa = 1$ and $\delta = \frac{\xi}{\rho}$, implies (3.3). □

The next theorems, discuss another subordination relation linked to the operator $M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)$ concurring polynomial of the dominant function $\nu(z)$.

Theorem 3.2. *Let* ν *be a convex univalent function in* U *with* $\nu(0) = 1$ *and* $\nu(z) \neq 0$. *Also, let* $\gamma, \beta_i \in \mathbb{C}$, $(i = 1, 2, 3)$, $\eta \in \mathbb{C}\backslash\{0\}$ *and* $\rho > 0$ *such that*

$$Re\left\{1 + \frac{\beta_1}{\eta}\nu(z) + \frac{2\beta_2}{\eta}\nu^2(z) + \frac{3\beta_3}{\eta}\nu^3(z) + \frac{z\acute{\acute{\nu}}(z)}{\acute{\nu}(z)} - \frac{z\acute{\nu}(z)}{\nu(z)}\right\} > 0, \tag{3.5}$$

where $z \in \mathbb{U}$. *Assume that* $\frac{z\acute{\nu}(z)}{\nu(z)}$ *is a starlike univalent function in* $\mathbb{U}$. *If* $f \in A$ *satisfies the following subordination relation*

$$\begin{aligned}\psi(\gamma,\beta_1,\beta_2,\beta_3,\eta,\rho,a,b,c,\tau_1,\tau_2,\lambda_1,\lambda_2;z) \prec\, &\gamma + \beta_1\nu(z) + \beta_2\nu^2(z) \\ &+ \beta_3\nu^3(z) + \eta\frac{z\acute{\nu}(z)}{\nu(z)},\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}
&\psi(\gamma,\beta_1,\beta_2,\beta_3,\eta,\rho,a,b,c,\tau_1,\tau_2,\lambda_1,\lambda_2;z)\\
&=\gamma+\beta_1\left(\frac{M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)}{M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)}\right)^{\rho}+\beta_2\left(\frac{M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)}{M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)}\right)^{2\rho}\\
&\quad+\beta_3\left(\frac{M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)}{M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)}\right)^{3\rho}+\eta\rho(a+1)\left[\frac{M^{a+2,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)}{M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)}-\frac{M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)}{M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)}\right],
\end{aligned} \tag{3.7}$$

then

$$\left(\frac{M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)}{M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)}\right)^{\rho}\prec\nu(z)$$

and $\nu(z)$ *is the best dominant of* (3.6).

Proof. Define the function θ as

$$\theta(z)=\left(\frac{M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)}{M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)}\right)^{\rho},\quad z\in\mathbb{U}. \tag{3.8}$$

Obviously, the function θ is a holomorphic in $\mathbb{U}$ with $\theta(0)=1$. Also after some computations and by virtue of (1.13) we see that

$$\begin{aligned}
&\psi(\gamma,\beta_1,\beta_2,\beta_3,\eta,\rho,a,b,c,\tau_1,\tau_2,\lambda_1,\lambda_2;z)\\
&=\gamma+\beta_1\theta(z)+\beta_2\theta^2(z)+\beta_3\theta^3(z)+\eta\frac{z\acute{\theta}(z)}{\theta(z)}.
\end{aligned}$$

Hence, from (3.6) implies

$$\gamma+\beta_1\theta(z)+\beta_2\theta^2(z)+\beta_3\theta^3(z)+\eta\frac{z\acute{\theta}(z)}{\theta(z)}\prec\gamma+\beta_1\nu(z)+\beta_2\nu^2(z)+\beta_3\nu^3(z)+\eta\frac{z\acute{\nu}(z)}{\nu(z)}.$$

Now, set

$$F(w)=\gamma+\beta_1w+\beta_2w^2+\beta_3w^3\ \text{ and }\ G(w)=\frac{\eta}{w},\ w\neq0,$$

we can easily notice that F is holomorphic in $\mathbb{C}$, and G is holomorphic in $\mathbb{C}\backslash\{0\}$ with $G(w)\neq0, w\in\mathbb{C}\backslash\{0\}$. Additionally

$$\nu(z)=z\acute{\nu}(z)G[\nu(z)]=\eta\frac{z\acute{\nu}(z)}{\nu(z)}$$

and

$$T(z)=F[\nu(z)]+\nu(z)=\gamma+\beta_1\nu(z)+\beta_2\nu^2(z)+\beta_3\nu^3(z)+\eta\frac{z\acute{\nu}(z)}{\nu(z)}.$$

In addition, $\nu(z)$ is clearly starlike in $\mathbb{U}$

$$Re\left\{\frac{z\acute{T}(z)}{\nu(z)}\right\} = Re\left\{1+\frac{\beta_1}{\eta}\nu(z)+\frac{2\beta_2}{\eta}\nu^2(z)+\frac{3\beta_3}{\eta}\nu^3(z)+\frac{z\acute{\acute{\nu}}(z)}{\acute{\nu}(z)}-\frac{z\acute{\nu}(z)}{\nu(z)}\right\}$$
$$> 0.$$

Hence, by Lemma 2.3, we conclude that $\theta(z) \prec \nu(z)$ and from expression (3.8), we obtain the acquired result. □

Theorem 3.3. *Let ν be a convex univalent function in U with $\nu(0) = 1$ and $\nu(z) \neq 0$. Also, let $\gamma, \beta_i \in \mathbb{C}$ $(i = 1, 2, 3)$, $\eta \in \mathbb{C}\backslash\{0\}$ and $\rho > 0$ such that ν satisfies* (3.5)*. Assume that $\frac{z\acute{\nu}(z)}{\nu(z)}$ is starlike univalent function in $\mathbb{U}$. If $f \in A$ satisfies the subordination relation*

$$\psi(\gamma,\beta_1,\beta_2,\beta_3,\eta,\rho,a,b,c,\tau_1,\tau_2,\lambda_1,\lambda_2;z) \prec \gamma+\beta_1\nu(z)+\beta_2\nu^2(z)+\beta_3\nu^3(z) + \eta\frac{z\acute{\nu}(z)}{\nu(z)}, \tag{3.9}$$

where

$$\begin{aligned}
&\psi(\gamma,\beta_1,\beta_2,\beta_3,\eta,\rho,a,b,c,\tau_1,\tau_2,\lambda_1,\lambda_2;z)\\
&= \gamma+\beta_1\frac{zM^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)}{\left(M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)\right)^{\rho}}+\beta_2\frac{z^2\left(M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)\right)^2}{\left(M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)\right)^{2\rho}}\\
&\quad+\beta_3\frac{z^3\left(M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)\right)^3}{\left(M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)\right)^{3\rho}}\\
&\quad+\eta(a+1)\left[1+\frac{M^{a+2,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)}{M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)}-\rho\frac{M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)}{M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)}\right],
\end{aligned}\tag{3.10}$$

then

$$\frac{zM^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)}{\left(M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)\right)^{\rho}} \prec \nu(z)$$

and $\nu(z)$ is the best dominant of (3.9)*.*

Proof. Define the function θ as

$$\theta(z) = \frac{zM^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)}{\left(M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)\right)^{\rho}}, \quad z \in \mathbb{U}. \tag{3.11}$$

Note that, the function θ is a holomorphic in $\mathbb{U}$ with $\theta(0) = 1$. The rest of the proof is similar to the proof of Theorem 3.2, so one can easily confirm it. $\square$

Remark 3.4. The superordination results for the operator $M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)$, which are dual to the subordination features of the previous theorems, can be obtained analogously in view of Lemma 2.4 and Lemma 2.5.

The following result, discuss a subordination property for the operator $M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)$, concerning its derivative.

Theorem 3.5. *Let* $\frac{M^{a+i,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)}{z} \neq 0$ $(i = 0, 1)$ *and* $\nu(z)$ *be univalent in* $\mathbb{U}$ *with* $\nu(0) = 1$ *which satisfies the following subordination relation*

$$\frac{z(\left(M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)\right)'}{M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)} \prec \nu(z) + \frac{z\acute{\nu}(z)}{\nu(z) + a} \tag{3.12}$$

such that

$$Re\{\nu(z) + a\} > 0 \;\; and \;\; Re\left\{1 + \frac{z\acute{\acute{\nu}}(z)}{\acute{\nu}(z)} - \frac{z\acute{\nu}(z)}{\nu(z) + a}\right\} > 0.$$

Then

$$\frac{z(\left(M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)\right)'}{M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)} \prec \nu(z) \tag{3.13}$$

and $\nu(z)$ *is the best dominant of* (3.13).

Proof. Define a function θ as

$$\theta(z) = \frac{z\left(M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)\right)'}{M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)}. \tag{3.14}$$

Note that, $\theta(z)$ is holomorphic in $\mathbb{U}$ with $\theta(0) = 1$. In view of (1.13) we have

$$(\theta(z) + a)\, M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z) = (a + 1) M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z), \tag{3.15}$$

which implies

$$\frac{z(\left(M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)\right)'}{M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)} = \theta(z) + \frac{z\acute{\theta}(z)}{\theta(z) + a}. \tag{3.16}$$

Hence, (3.12) becomes

$$\theta(z) + \frac{z\acute{\theta}(z)}{\theta(z) + a} \prec \nu(z) + \frac{z\acute{\nu}(z)}{\nu(z) + a}. \tag{3.17}$$

Now, set $F(w) = w$ and $G(w) = \frac{1}{w+a}$. It is obvious that the function $F(w)$ is a entire function, hence both $F(w)$ and $G(w)$ are holomorphic in $D = \mathbb{C}\backslash\{-a\}$ that contain $\nu(\mathbb{U})$ with $G(w) \neq 0$ when $w \in \nu(\mathbb{U})$. In addition, we define

$$\phi(z) = z\acute{\nu}(z)G[\nu(z)].$$

See that,

$$T(z) = F[\nu(z)] + \phi(z) = \nu(z) + \frac{z\acute{\nu}(z)}{\nu(z)+a},$$

moreover,

$$\frac{z\acute{\phi}(z)}{\phi(z)} = 1 + \frac{z\acute{\acute{\nu}}(z)}{\acute{\nu}(z)} - \frac{z\acute{\nu}(z)}{\nu(z)+a},$$

that makes $\phi(z)$ is a starlike function in $\mathbb{U}$. Furthermore,

$$Re\left\{\frac{z\acute{T}(z)}{\phi(z)}\right\} = Re\left\{\nu(z) + a + \frac{z\acute{\phi}(z)}{\phi(z)}\right\} > 0.$$

Since $\{-a\} \notin \theta(\mathbb{U})$, $\theta(\mathbb{U}) \subset D$. Apply Lemma 2.6, we obtain that $\theta(z) \prec \nu(z)$ and $\nu(z)$ is the best dominant. $\square$

4. Sandwich theorems

This section, concludes some sandwich theorems linked to the linear operator $M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)$, from combining subordination and superordination results associated to Theorem 3.1, Theorem 3.2 and Theorem 3.3.

Theorem 4.1. *Let ν_1 and ν_2 be convex univalent in $\mathbb{U}$ with $\nu_1(0) = \nu_2(0) = 1$. Assume that ν_2 satisfies* (3.1) *such that $\rho > 0$ and $Re(\xi) > 0$. Let $f \in A$ satisfies*

$$\left(\frac{M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)}{z}\right)^{\rho} \in \mathbb{H}[1,1] \cap \Lambda$$

and

$$\left(\frac{M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)}{z}\right)^{\rho} + (a+1)\left(\frac{M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)}{z}\right)^{\rho}\left(\frac{M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)}{M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}f(z)} - 1\right)$$

be univalent in $\mathbb{U}$*. If*

$$\begin{aligned}\nu_1(z) + \frac{\xi}{\rho} z\, \acute{\nu}_1(z) &\prec \left(\frac{M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)}{z}\right)^{\rho} \\ &\qquad + (a+1)\left(\frac{M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)}{z}\right)^{\rho}\left(\frac{M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)}{M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)} - 1\right) \\ &\prec \nu_2(z) + \frac{\xi}{\rho} z \acute{\nu}_2(z),\end{aligned}$$

then

$$\nu_1(z) \prec \left(\frac{M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)}{z}\right)^{\rho} \prec \nu_2(z)$$

such that ν_1 *and* ν_2 *are respectively the best subordinate and the best dominant.*

Theorem 4.2. *Let* ν_1 *and* ν_2 *be convex univalent in* $\mathbb{U}$ *with* $\nu_1(0) = \nu_2(0) = 1$*. Assume that* ν_1 *satisfies*

$$Re\left\{\frac{\beta_1}{\eta}\nu_1(z) + \frac{2\beta_2}{\eta}\nu_1^2(z) + \frac{3\beta_3}{\eta}\nu_1^3(z)\right\} > 0 \tag{4.1}$$

and ν_2 *satisfies* (3.5)*. Also, let* $f \in A$ *satisfies*

$$\left(\frac{M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)}{M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)}\right)^{\rho} \in \mathbb{H}[1,1] \cap \Lambda$$

and $\psi(\gamma, \beta_1, \beta_2, \beta_3, \eta, \rho, a, b, c, \tau_1, \tau_2, \lambda_1, \lambda_2; z)$ *is univalent in* $\mathbb{U}$*, where* ψ *is given in* (3.7)*. If*

$$\begin{aligned}\gamma + \beta_1\nu_1(z) + \beta_2\nu_1^2(z) + \beta_3\nu_1^3(z) + \eta\frac{z\acute{\nu}_1(z)}{\nu_1(z)} \\ \prec \psi(\gamma, \beta_1, \beta_2, \beta_3, \eta, \rho, a, b, c, \tau_1, \tau_2, \lambda_1, \lambda_2; z) \\ \prec \gamma + \beta_1\nu_2(z) + \beta_2\nu_2^2(z) + \beta_3\nu_2^3(z) + \eta\frac{z\acute{\nu}_2(z)}{\nu_2(z)},\end{aligned} \tag{4.2}$$

then

$$\nu_1(z) \prec \left(\frac{M^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)}{M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)}\right)^{\rho} \prec \nu_2(z)$$

such that ν_1 *and* ν_2 *are respectively the best subordinate and the best dominant.*

Theorem 4.3. *Let ν_1 and ν_2 be convex univalent in $\mathbb{U}$ with $\nu_1(0) = \nu_2(0) = 1$. Assume that ν_1 satisfies* (4.1) *and ν_2 satisfies* (3.5)*. Also, let $f \in A$ satisfies*

$$\frac{zM^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)}{\left(M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)\right)^{\rho}} \in \mathbb{H}[1,1] \cap \Lambda$$

and $\psi(\gamma, \beta_1, \beta_2, \beta_3, \eta, \rho, a, b, c, \tau_1, \tau_2, \lambda_1, \lambda_2; z)$ is univalent in $\mathbb{U}$, where ψ is given in (3.10)*. If $\psi(\gamma, \beta_1, \beta_2, \beta_3, \eta, \rho, a, b, c, \tau_1, \tau_2, \lambda_1, \lambda_2; z)$ satisfies* (4.2)*, then*

$$\nu_1(z) \prec \frac{zM^{a+1,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)}{\left(M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)\right)^{\rho}} \prec \nu_2(z)$$

such that ν_1 and ν_2 are respectively the best subordinate and the best dominant.

5. Conclusion and discussion

Involving the seven-parameter Mittag-Leffler function, we obtained a linear operator $M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z)$ by using the Hadamard product method, then we discuss the special case's well-known operators. Further, we employed this new operator to achieve some second-order differential subordination results in the open unit disk $\mathbb{U}$, in order to find a best dominant to some consequences of that operator and its derivative. Moreover, we could conclude some sandwich type theorems respecting the subordination results that associated to its dual superordination. It is noteworthy to mention that all the results concurring that operator in this paper holds for $M^{a,b,c}_{\tau_1,\lambda_1+1,\tau_2,\lambda_2} f(z)$ and $M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2+1} f(z)$ by the following relations that we established

$$\tau_1 z \left(M^{a,b,c}_{\tau_1,\lambda_1+1,\tau_2,\lambda_2} f(z)\right)' = (\tau_1 + \lambda_1) M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z) - \lambda_1 M^{a,b,c}_{\tau_1,\lambda_1+1,\tau_2,\lambda_2} f(z),$$
$$\tau_2 z \left(M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2+1} f(z)\right)' = (\tau_2 + \lambda_2) M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2} f(z) - \lambda_2 M^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2+1} f(z),$$

as well as, the corresponding superordinations results and sandwich theorems.

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