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ALMOST QUADRATIC LIE $*$-DERIVATIONS ON CONVEX MODULAR $*$-ALGEBRAS

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Abstract. In this article, we investigate an approximate quadratic Lie $*$-derivation of a quadratic functional equation

$$f(ax+by)+abf(x-y)=(a+b)(af(x)+bf(y)),$$

where $ab\neq 0,\ a,b\in\mathbb{N}$, associated with the identity $f([x,y])=[f(x),y^2]+[x^2,f(y)]$ on a ρ-complete convex modular $*$-algebra χ_ρ by using Δ_2-condition via convex modular ρ.

1. Introduction

Let us recall that the problem of stability of functional equations has been inspired by a question of Ulam concerning the stability of homomorphisms on groups. In 1940, Ulam [31] at the Mathematics Club of the University of Wisconsin has presented the question concerning the stability of group homomorphisms: when a solution of an equation of group homomorphism, differing slightly from a given one, must be near to the exact solution of the given equation. Hyers [12] has solved the problem of Ulam for the case of additive mappings in 1941. The result was generalized by Aoki [1] in 1950, by

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Rassias [28] in 1978, by Rassias [26] in 1992, and by Găvruta [9] in 1994. Over the last few decades, many mathematicians have investigated the stability problems of several different types of functional equations between various linear spaces together with functionals [3, 5, 7, 8, 13, 14, 20, 29, 33]. In particular, Rassias [27] investigated the Hyers–Ulam stability of the Euler–Lagrange quadratic equation

$$f(ax+by)+f(bx-ay)=(a^2+b^2)(f(x)+f(y)), \tag{1.1}$$

which is a generalized form of the classical quadratic equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.2}$$

between vector spaces. Concerning stability problems of functional equations, the stability theorems of various functional equations in modular spaces have been intensively established by many authors (see, e.g., [10, 15, 23, 24, 25]).

In the present paper, we first investigate generalized Hyers–Ulam stability of the following modified Euler–Lagrange quadratic functional equation

$$f(ax+by)+abf(x-y)=(a+b)(af(x)+bf(y)), \tag{1.3}$$

associated with quadratic Lie $*$-derivations, where a, b are any nonzero fixed natural numbers in $\mathbb{N}$, without using both Fatou property and Δ_2-condition, and then alternatively present generalized Hyers–Ulam stability of the equation (1.3) associated with quadratic Lie $*$-derivations using necessarily Δ_2-condition but not using the Fatou property in ρ-complete convex modular $*$-algebras.

2. Definitions and preliminaries

First of all, the concept of modular spaces has been introduced by Nakano [22], and then by Musielak and Orlicz [21]. Now, we recall some basic definitions and remarks of modular spaces with modular functions, which are primitive notions corresponding to norm or metric, as in the followings [16, 19, 25, 30, 32].

Definition 2.1. Let χ be a linear space and $\mathbb{C}$ be a set of complex numbers.

(a) A function $\rho : \chi \to [0, +\infty]$ is called a modular, (convex modular, resp.) if for arbitrary $x, y \in \chi$,
   (1) $\rho(x) = 0$ if and only if $x = 0$,
   (2) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$,
   (3) $\rho(\alpha x + \beta y) \le \rho(x) + \rho(y)$, ($\rho(\alpha x + \beta y) \le \alpha\rho(x) + \beta\rho(y)$, resp.) for every scalars α, β, where $\alpha + \beta = 1$ and $\alpha, \beta \ge 0$, acting on the real linear space χ,

(b) alternatively, if (3) is replaced by

(3)' $\rho(\alpha x + \beta y) \leq |\alpha|\rho(x) + |\beta|\rho(y)$ for any scalars $\alpha, \beta \in \mathbb{C}$ with $|\alpha| + |\beta| = 1$, acting on the complex linear space χ, then we say that ρ is a convex modular on the complex linear space χ [17, 18].

Now, we observe that a modular ρ defines a corresponding modular space, that is, the linear space χ_ρ given by

$$\chi_\rho = \{x \in \chi : \rho(\lambda x) \to 0 \quad \text{as} \quad \lambda \to 0\}.$$

It is well known that for a convex modular ρ on χ, $\rho(tx)$ is an increasing function in $t \geq 0$ for each fixed $x \in \chi$, that is, $\rho(\alpha x) \leq \rho(\beta x)$ whenever $0 \leq |\alpha| < |\beta|$. Moreover, we see that $\rho(\alpha x) \leq \alpha\rho(x)$ for all $x \in \chi$ and for all α with $0 \leq \alpha \leq 1$, and that $\rho(\alpha x) \leq |\alpha|\rho(x)$ for all $x \in \chi$ and all $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$.

Remark 2.2. (a) In general, we note that $\rho\big(\sum_{i=1}^n \alpha_i x_i\big) \leq \sum_{i=1}^n \alpha_i \rho(x_i)$ for all $x_i \in \chi$ and $\alpha_i \geq 0$ $(i = 1, \cdots, n)$ if $0 < \sum_{i=1}^n \alpha_i := \alpha \leq 1$ [16].

(b) Consequently, we lead to $\rho\big(\sum_{i=1}^n \alpha_i x_i\big) \leq \sum_{i=1}^n |\alpha_i|\rho(x_i)$ for all $x_i \in \chi$ and all $\alpha_i \in \mathbb{C}$ if $0 < \sum_{i=1}^n |\alpha_i| := \alpha \leq 1$.

Definition 2.3. Let χ_ρ be a modular space and let $\{x_n\}$ be a sequence in χ_ρ. Then,

(a) $\{x_n\}$ is ρ-convergent to $x \in \chi_\rho$ and write $x_n \overset{\rho}{\to} x$ if $\rho(x_n - x) \to 0$ as $n \to +\infty$.
(b) $\{x_n\}$ is called ρ-Cauchy in χ_ρ if $\rho(x_n - x_m) \to 0$ as $n, m \to +\infty$.
(c) A subset K of χ_ρ is called ρ-complete if any ρ-Cauchy sequence is ρ-convergent to an element in K.

Now, we say that χ_ρ is called a convex modular algebra if the fundamental space X is an algebra with convex modular ρ subject to $\rho(ab) \leq \rho(a)\rho(b)$ for all $a, b \in X$. A subset K of a convex modular algebra χ_ρ is called ρ-complete if any ρ-Cauchy sequence in K is ρ-convergent to an element in K.

It is said that the modular ρ has the Fatou property if

$$\rho(x) \leq \liminf_{n\to\infty} \rho(x_n),$$

whenever the sequence $\{x_n\}$ is ρ-convergent to x. For a given natural number $n > 1$, a modular function ρ is said to satisfy the Δ_n-condition if there exists a constant $\kappa_n > 0$ such that $\rho(nx) \leq \kappa_n \rho(x)$ for all vectors $x \in \chi_\rho$. Then, it is noted that if a convex modular ρ satisfies the Δ_2-condition, then $\kappa_2 \geq 2$ for nontrivial convex modular ρ [16], and ρ also satisfies the Δ_n-condition for any natural number $n > 2$ because there exists a natural number $l \in \mathbb{N}$ such that $\frac{n}{2^l} \leq 1$, and thus $\rho(nx) = \rho(\frac{n}{2^l} 2^l x) \leq \kappa_n \rho(x)$ for all $x \in \chi_\rho$, where $\kappa_n := \frac{n}{2^l}\kappa_2^l$.

Remark 2.4. A convex modular function ρ satisfies the Δ_2-condition if and only if the modular ρ satisfies the Δ_n-condition.

Now, it is said that χ_ρ is called a convex modular $*$-algebra if the basic space χ is a $*$-algebra with convex modular ρ subject to $\rho(ab) \leq \rho(a)\rho(b)$ and $\rho(c^*) = \rho(c)$ for all $a, b, c \in \chi_\rho$. A subset K of a convex modular $*$-algebra χ_ρ is called ρ-complete if and only if any ρ-Cauchy sequence in K is ρ-convergent to an element in K. It is said that a linear mapping f is called a Lie $*$-derivation if

$$f([x,y]) = [f(x), y] + [x, f(y)] \quad \text{and } f(z^*) = f(z)^*$$

for all vectors x, y, z, where $[x,y] = xy - yx$. In a similar way, they say that a quadratic mapping f is quadratic homogeneous if $f(\lambda x) = \lambda^2 f(x)$ for all vectors x and all scalars λ, and in addition a quadratic homogeneous mapping f is called a quadratic Lie $*$-derivation if

$$f([x,y]) = [f(x), y^2] + [x^2, f(y)] \quad \text{and } f(z^*) = f(z)^*$$

for all vectors $x, y, z \in \chi_\rho$ [17, 18].

Throughout the paper, χ_ρ will denote a ρ-complete convex modular $*$-algebra with nontrivial convex modular ρ unless we give any specific reference.

3. Approximate quadratic Lie $*$-derivations

First of all, we remark that the equation (1.3) is equivalent to the original quadratic functional equation (1.2), and so every solution of equation (1.3) is a quadratic mapping [4]. For notational convenience, we set $\mathbb{T}_{n_0} := \left\{e^{i\theta} \in \mathbb{C} : 0 \leq \theta \leq \frac{2\pi}{n_0}\right\}$ for a given $n_0 \in \mathbb{N}$, and we denote the quadratic difference operator QE_f^λ and quadratic Lie $*$-derivation QD_f associated with quadratic equation (1.3) as follows, respectively:

$$\begin{aligned} QE_f^\lambda(x,y) &:= f(\lambda ax + \lambda by) + \lambda^2 abf(x-y) \\ &\qquad - \lambda^2(a+b)[af(x) + bf(y), \\ QD_f(x,y) &:= f([x,y]) - [f(x), y^2] - [x^2, f(y)] \end{aligned}$$

for all x, y in χ_ρ and $\lambda \in \mathbb{T}_{n_0}$, which act as perturbing terms for given approximate quadratic Lie $*$-derivations $f : \chi_\rho \to \chi_\rho$. In the following, we present a generalized Hyers–Ulam stability of the equation (1.3) via direct method associated with approximate quadratic Lie $*$-derivations in ρ-complete modular $*$-algebras without using both Fatou property and Δ_2-condition.

Theorem 3.1. *Suppose that a mapping $f : \chi_\rho \to \chi_\rho$ with $f(0) = 0$ satisfies*

$$\rho(QE_f^\lambda(x,y) + f(z^*) - f(z)^*) \leq \phi_1(x,y,z), \tag{3.1}$$
$$\rho(QD_f(x,y)) \leq \phi_2(x,y)$$

and $\phi_1, \phi_2 : \chi_\rho^2 \to [0, +\infty)$ *are mappings such that*

$$\Phi_1(x,y,z) := \sum_{j=0}^{\infty} \frac{\phi_1((a+b)^j x, (a+b)^j y, (a+b)^j z)}{(a+b)^{2(j+1)}} < +\infty, \tag{3.2}$$

$$\lim_{n\to\infty} \frac{\phi_2((a+b)^n x, (a+b)^n y)}{(a+b)^{4n}} = 0$$

for all $x, y \in \chi_\rho$ *and* $\lambda \in \mathbb{T}_{n_0}$. *If for each* $x \in \chi_\rho$ *the mapping* $t \to f(tx)$ *from* $\mathbb{R}$ *to* χ_ρ *is continuous, then there exists a unique quadratic Lie ∗-derivation* $F_1 : \chi_\rho \to \chi_\rho$, *defined as*

$$F_1(x) = \rho - \lim_{n\to\infty} \frac{f((a+b)^n x)}{(a+b)^{2n}},$$

which satisfies the equation (1.3) *and the approximation*

$$\rho(f(x) - F_1(x)) \leq \Phi(x,x,0) \tag{3.3}$$

near f *for all* $x \in \chi_\rho$.

Proof. Interchanging (x,y,z) with $(x,x,0)$ in (3.1), we obtain

$$\rho(QE_f^1(x,x)) = \rho\big(f((a+b)x) - (a+b)^2 f(x)\big) \leq \phi_1(x,x,0), \tag{3.4}$$

which yields

$$\begin{aligned}\rho\Big(f(x) - \frac{f((a+b)x)}{(a+b)^2}\Big) &\leq \frac{1}{(a+b)^2}\rho\big(f((a+b)x) - (a+b)^2 f(x)\big)\\ &\leq \frac{1}{(a+b)^2}\phi_1(x,x,0)\end{aligned}$$

for all $x \in \chi_\rho$. Since $\sum_{j=n}^{m-1} \frac{1}{(a+b)^{2(j+1)}} \leq 1$, it follows from (3.4) and the property of convex modular ρ that

$$\begin{aligned}&\rho\Big(\frac{1}{(a+b)^{2m}} f((a+b)^m x) - \frac{1}{(a+b)^{2n}} f((a+b)^n x)\Big) \qquad (3.5)\\ &= \rho\Big(\sum_{i=0}^{m-n-1} \frac{1}{(a+b)^{2(n+i+1)}}\Big(f((a+b)^{n+i+1}x) - (a+b)^2 f((a+b)^{n+i}x)\Big)\Big)\\ &\leq \sum_{i=0}^{m-n-1} \frac{1}{(a+b)^{2(n+i+1)}}\phi_1((a+b)^{n+i}x, (a+b)^{n+i}x, 0)\\ &= \sum_{j=n}^{m-1} \frac{1}{(a+b)^{2(j+1)}}\phi_1((a+b)^j x, (a+b)^j x, 0)\end{aligned}$$

for all $x \in \chi_\rho$ and for any integers m, n with $m > n \geq 0$. Since the right hand side of (3.5) tends to zero as $n \to \infty$, the sequence $\left\{ \frac{f((a+b)^n x)}{(a+b)^{2n}} \right\}$ is ρ-Cauchy in χ_ρ, and thus it converges for all $x \in \chi_\rho$. Therefore, one may define a mapping $F_1 : \chi_\rho \to \chi_\rho$ as

$$F_1(x) := \rho - \lim_{n\to\infty} \frac{f((a+b)^n x)}{(a+b)^{2n}} \Longleftrightarrow \lim_{n\to\infty} \rho\Big(\frac{f((a+b)^n x)}{(a+b)^{2n}} - F_1(x)\Big) = 0,$$

which leads to a unique quadratic mapping satisfying the approximation (3.3), as desired, using the direct method ([2, 7, 11]).

In fact, if we put $(x, y, z) := ((a+b)^n x, (a+b)^n y, 0)$ in (3.1), and then divide the resulting inequality by $(a+b)^{2n}$, one obtains

$$\begin{aligned} \rho\Big(\frac{QE_f^\lambda((a+b)^n x, (a+b)^n y)}{(a+b)^{2n}}\Big) &\leq \frac{\rho(QE_f^\lambda((a+b)^n x, (a+b)^n y))}{(a+b)^{2n}} \\ &\leq \frac{\phi_1((a+b)^n x, (a+b)^n y, 0)}{(a+b)^{2n}}, \end{aligned}$$

which tends to zero as $n \to +\infty$ for all $x, y \in \chi_\rho$ and all $\lambda \in \mathbb{T}_{n_0}$. Thus, choosing a natural number L with $\frac{a^2+3ab+b^2+2}{L} \leq 1$ we figure out

$$\begin{aligned} &\rho(\frac{1}{L} QE_{F_1}^\lambda(x, y)) \\ &= \rho\Big(\frac{1}{L} QE_{F_1}^\lambda(x, y) - \frac{QE_f^\lambda((a+b)^n x, (a+b)^n y)}{L \cdot (a+b)^{2n}} + \frac{QE_f^\lambda((a+b)^n x, (a+b)^n y)}{L \cdot (a+b)^{2n}}\Big) \\ &\leq \frac{1}{L}\rho\Big(F_1\big(\lambda a x + \lambda b y\big) - \frac{f\big((a+b)^n(\lambda a x + \lambda b y)\big)}{(a+b)^{2n}}\Big) \\ &\quad + \frac{|\lambda|^2 ab}{L}\rho\Big(F_1\big(x - y\big) - \frac{f\big((a+b)^n(x - y)\big)}{(a+b)^{2n}}\Big) \\ &\quad + \frac{|\lambda|^2 (a+b)a}{L}\rho\Big(F_1(x) - \frac{f((a+b)^n x)}{(a+b)^{2n}}\Big) \\ &\quad + \frac{|\lambda|^2 (a+b)b}{L}\rho\Big(F_1(y) - \frac{f((a+b)^n y)}{(a+b)^{2n}}\Big) + \frac{1}{L}\rho\Big(\frac{QE_f^\lambda\big((a+b)^n x, (a+b)^n y\big)}{(a+b)^{2n}}\Big) \end{aligned}$$

for all $x, y \in \chi_\rho$ and all positive integers n by Remark 2.2. Taking the limit as $n \to +\infty$ in the last inequality, we arrive at the desired functional identity $\rho(\frac{1}{L} QE_{F_1}^\lambda(x, y)) = 0$, and so

$$QE_{F_1}^\lambda(x, y) = 0 \tag{3.6}$$

for all $x, y \in \chi_\rho$ and all $\lambda \in \mathbb{T}_{n_0}$. Hence F_1 satisfies the equation (1.3) and so it is quadratic. It follows from (3.6) that

$$QE^{\lambda}_{F_1}(x,x) = 0 \iff F_1((a+b)\lambda x) = (a+b)^2\lambda^2 F_1(x)$$

for all $x \in \chi_\rho$ and all $\lambda \in \mathbb{T}_{n_0}$, which yields $F_1(\lambda x) = \lambda^2 F_1(x)$ for all $x \in \chi_\rho$ and all unit scalars $\lambda \in \mathbb{T}_1$. From the assumption that for each $x \in \chi_\rho$ the mapping $t \to f(tx)$ from $\mathbb{R}$ to χ_ρ is continuous, it follows that $F_1(tx) = t^2 F_1(x)$ for all $x \in \chi_\rho$ and all $t \in \mathbb{R}$ by the same argument as in the paper [28]. Thus, for any nonzero $\lambda \in \mathbb{C}$

$$\begin{aligned} F_1(\lambda x) &= F_1\left((a+b)\frac{\lambda}{|\lambda|}\frac{|\lambda|}{(a+b)}x\right) = (a+b)^2\left(\frac{\lambda}{|\lambda|}\right)^2 F_1\left(\frac{|\lambda|}{(a+b)}x\right) \\ &= (a+b)^2\left(\frac{\lambda}{|\lambda|}\right)^2\left(\frac{|\lambda|}{(a+b)}\right)^2 F_1(x) = \lambda^2 F_1(x), \end{aligned}$$

which concludes that F_1 is quadratic homogeneous over $\mathbb{C}$.

On the other hand, since

$$\sum_{i=0}^{n} \frac{1}{(a+b)^{2(i+1)}} + \frac{1}{(a+b)^2} \leq 1$$

for all $n \in \mathbb{N}$, it follows from (3.4) and Remark 2.2 that

$$\begin{aligned} \rho(f(x) - F_1(x)) &= \rho\Big(\sum_{i=0}^{n} \frac{1}{(a+b)^{2(i+1)}}\Big((a+b)^2 f((a+b)^i x) - f((a+b)^{i+1}x)\Big) \\ &\quad + \frac{f((a+b)^{n+1}x)}{(a+b)^{2(n+1)}} - \frac{F_1((a+b)x)}{(a+b)^2}\Big) \\ &\leq \sum_{i=0}^{n} \frac{1}{(a+b)^{2(i+1)}}\phi_1((a+b)^i x, (a+b)^i x, 0) \\ &\quad + \frac{1}{(a+b)^2}\rho\Big(\frac{f((a+b)^n \cdot (a+b)x)}{(a+b)^{2n}} - F_1((a+b)x)\Big), \end{aligned}$$

without applying the Fatou property of ρ for all $x \in \chi_\rho$ and all $n \in \mathbb{N}$, from which we obtain the approximation (3.3) near f by taking $n \to +\infty$ in the last inequality.

In the last part, we claim that F_1 is a quadratic Lie $*$-derivation. In view of the inequality in (3.1) and the second condition in (3.2), we arrive at

$$\begin{aligned}\rho(\frac{1}{4}QD_{F_1}(x,y)) &= \rho\Big(\frac{1}{4}QD_{F_1}(x,y) - \frac{QD_f(a+b)^n(x,y)}{4\cdot(a+b)^{4n}} + \frac{QD_f(a+b)^n(x,y)}{4\cdot(a+b)^{4n}}\Big)\\ &\le \frac{1}{4}\rho\Big(F_1\big([x,y]\big) - \frac{f\big((a+b)^{2n}[x,y]\big)}{(a+b)^{4n}}\Big)\\ &\quad + \frac{1}{4}\rho\Big(\frac{[x^2, f((a+b)^n y)]}{(a+b)^{2n}} - [x^2, F_1(y)]\Big)\\ &\quad + \frac{1}{4}\rho\Big(\frac{[f((a+b)^n x), y^2]}{(a+b)^{2n}} - [F_1(x), y^2]\Big) + \frac{\phi_2(a+b)^n(x,y)}{4\cdot(a+b)^{4n}}\end{aligned}$$

for all $x, y \in \chi_\rho$, which tends to zero as n tends to $+\infty$. Therefore, F_1 is a quadratic Lie derivation. In addition, we get the following inequality

$$\begin{aligned}\rho\Big(\frac{1}{3}\Big(F_1(z^*) - F_1(z)^*\Big)\Big) &\le \frac{1}{3}\rho\Big(F_1(z^*) - \frac{f((a+b)^n z^*)}{(a+b)^{2n}}\Big)\\ &\quad + \frac{1}{3}\rho\Big(\frac{f((a+b)^n z)^*}{(a+b)^{2n}} - F_1(z)^*\Big)\\ &\quad + \frac{1}{3}\rho\Big(\frac{f((a+b)^n z^*)}{(a+b)^{2n}} - \frac{f((a+b)^n z)^*}{(a+b)^{2n}}\Big)\\ &\le \frac{1}{3}\rho\Big(F_1(z^*) - \frac{f((a+b)^n z^*)}{(a+b)^{2n}}\Big)\\ &\quad + \frac{1}{3}\rho\Big(\frac{f((a+b)^n z)^*}{(a+b)^{2n}} - F_1(z)^*\Big)\\ &\quad + \frac{\phi_1(0,0,(a+b)^n z)}{3\cdot(a+b)^{2n}}\end{aligned}$$

for all vector $z \in \chi_\rho$. Taking $n \to +\infty$, one concludes F_1 is a quadratic Lie $*$-derivation. Therefore, the mapping F_1 is a unique quadratic Lie $*$-derivation near f satisfying the approximation (3.3) in the ρ-complete convex modular $*$-algebra χ_ρ. □

As a corollary, we obtain a stability result under strictly quadratical contractive conditions over control functions of perturbing terms QE_f^λ and QD_f.

Corollary 3.2. *Suppose there exist two functions* $\phi_1 : \chi_\rho^3 \to [0, +\infty)$ *and* $\phi_2 : \chi_\rho^2 \to [0, +\infty)$ *and two constant* l_i *with* $0 < l_i < 1\,(i = 1, 2)$ *for which a* *mapping* $f : \chi_\rho \to \chi_\rho$ *with* $f(0) = 0$ *satisfies*

$$\rho(QE_f^\lambda(x,y) + f(z^*) - f(z)^*) \leq \phi_1(x,y,z),$$
$$\phi_1(a+b)(x,y,z) \leq (a+b)^2 l_1 \phi_1(x,y,z),$$
$$\rho(QD_f(x,y)) \leq \phi_2(x,y),$$
$$\phi_2(a+b)(x,y) \leq (a+b)^4 l_2 \phi_2(x,y)$$

for all $x, y, z \in \chi_\rho$ *and all* $\lambda \in \mathbb{T}_{n_0}$. *If for each* $x \in \chi_\rho$ *the mapping* $t \to f(tx)$ *from* $\mathbb{R}$ *to* χ_ρ *is continuous, then there exists a unique quadratic Lie* $*$*-derivation* $F_1 : \chi_\rho \to \chi_\rho$ *which satisfies the equation* (1.3) *and*

$$\rho(f(x) - F_1(x)) \leq \frac{1}{(a+b)^2(1-l_1)}\phi_1(x,x,0)$$

for all $x \in \chi_\rho$.

In the following, we are going to investigate alternatively generalized Hyers–Ulam stability of the equation (1.3) associated with approximate quadratic Lie $*$-derivations via direct method using necessarily Δ_2-condition but not using the Fatou property in ρ-complete convex modular $*$-algebras.

Theorem 3.3. *Let* χ_ρ *be a* ρ*-complete convex modular* $*$*-algebra with* Δ_2*-condition. Suppose there exist two functions* $\varphi_1 : \chi_\rho^3 \to [0,+\infty)$ *and* $\varphi_2 : \chi_\rho^2 \to [0,+\infty)$ *for which a mapping* $f : \chi_\rho \to \chi_\rho$ *satisfies*

$$\rho(QE_f^\lambda(x,y) + f(z^*) - f(z)^*) \leq \varphi_1(x,y,z), \tag{3.7}$$
$$\sum_{j=1}^{\infty} \frac{\kappa^{3j}}{(a+b)^j}\varphi_1\Big(\frac{x,y,z}{(a+b)^j}\Big) := \Psi(x,y,z) < \infty,$$
$$\rho(QD_f(x,y)) \leq \varphi_2(x,y), \tag{3.8}$$
$$\lim_{n\to\infty} \kappa^{4n}\varphi_2((a+b)^{-n}(x,y)) = 0$$

for all $x, y, z \in \chi_\rho$ *and all* $\lambda \in \mathbb{T}_{n_0}$, *where* κ *is the smallest positive real number such that* $\rho((a+b)x) \leq \kappa\rho(x)$, $(a+b) \leq \kappa$ *derived from the* Δ_2*-condition, for any* $x \in \chi_\rho$. *If in addition for each* $x \in \chi_\rho$ *the mapping* $t \to f(tx)$ *from* $\mathbb{R}$ *to* χ_ρ *is continuous, then there exists a unique quadratic Lie* $*$*-derivation* $F_2 : \chi_\rho \to \chi_\rho$ *satisfies the equation* (1.3) *and the approximation*

$$\rho(f(x) - F_2(x)) \leq \frac{1}{(a+b)\kappa}\Psi(x,x,0) \tag{3.9}$$

for all $x \in \chi_\rho$.

Proof. First, we remark that since $\sum_{j=1}^{\infty}\frac{\kappa^{3j}}{(a+b)^j}\varphi_1(0,0,0) = \Psi(0,0,0) < +\infty$ and $\rho(QE_f^1(0,0)) \leq \varphi_1(0,0,0)$, we lead to $\varphi_1(0,0,0) = 0$, $QE_f^1(0,0) = 0$ and so $f(0) = 0$. Thus, it follows from (3.4) that

$$\rho\Big(f(x) - (a+b)^2 f\Big(\frac{x}{(a+b)}\Big)\Big) \le \varphi_1\Big(\frac{x,x,0}{(a+b)}\Big) \le \frac{\kappa}{(a+b)}\varphi_1\Big(\frac{x,x,0}{(a+b)}\Big)$$

for all $x \in \chi_\rho$. Thus, one obtains the following inequality by the convexity of the modular ρ and Δ_2-condition

$$\begin{aligned}
&\rho\Big(f(x) - (a+b)^4 f\Big(\frac{x}{(a+b)^2}\Big)\Big)\\
&\quad\le \frac{1}{(a+b)}\rho\Big((a+b)f(x) - (a+b)^3 f\Big(\frac{x}{(a+b)}\Big)\Big)\\
&\qquad + \frac{1}{(a+b)^2}\rho\Big((a+b)^4 f\Big(\frac{x}{(a+b)}\Big) - (a+b)^6 f\Big(\frac{x}{(a+b)^2}\Big)\Big)\\
&\quad\le \frac{\kappa}{(a+b)}\varphi_1\Big(\frac{x,x,0}{(a+b)}\Big) + \frac{\kappa^4}{(a+b)^2}\varphi_1\Big(\frac{x,x,0}{(a+b)^2}\Big)
\end{aligned}$$

for all $x \in \chi_\rho$. Then using the inductive process for any $n \ge 2$, we prove the following functional inequality

$$\rho\Big(f(x) - 4^n f\Big(\frac{x}{2^n}\Big)\Big) \le \frac{1}{\kappa^2}\sum_{j=1}^{n}\frac{\kappa^{3j}}{(a+b)^j}\varphi_1\Big(\frac{x,x,0}{(a+b)^j}\Big) \tag{3.10}$$

for all $x \in \chi_\rho$. In fact, it is true for $j = 1, 2$. Assume that the inequality (3.10) holds true for n. Thus, using the convexity of the modular ρ, we deduce

$$\begin{aligned}
&\rho\Big(f(x) - (a+b)^{2(n+1)} f\Big(\frac{x}{(a+b)^{n+1}}\Big)\Big)\\
&\quad= \rho\Big(\frac{1}{(a+b)}\Big\{(a+b)f(x) - (a+b)^3 f\Big(\frac{x}{(a+b)}\Big)\Big\}\\
&\qquad + \frac{1}{(a+b)}\Big\{(a+b)^3 f(\frac{x}{(a+b)}) - (a+b)^{2n+3} f(\frac{x}{(a+b)^{n+1}})\Big\}\Big)\\
&\quad\le \frac{\kappa}{(a+b)}\varphi_1\Big(\frac{x,x,0}{(a+b)}\Big) + \frac{\kappa^3}{(a+b)}\cdot\frac{1}{\kappa^2}\sum_{j=1}^{n}\frac{\kappa^{3j}}{(a+b)^j}\varphi_1\Big(\frac{x,x,0}{(a+b)^{j+1}}\Big)\\
&\quad= \frac{\kappa}{(a+b)}\varphi_1\Big(\frac{x,x,0}{(a+b)}\Big) + \frac{1}{\kappa^2}\sum_{j=1}^{n}\frac{\kappa^{3(j+1)}}{(a+b)^{j+1}}\varphi_1\Big(\frac{x,x,0}{(a+b)^{j+1}}\Big)\\
&\quad= \frac{1}{\kappa^2}\sum_{j=1}^{n+1}\frac{\kappa^{3j}}{(a+b)^j}\varphi_1\Big(\frac{x,x,0}{(a+b)^j}\Big),
\end{aligned}$$

which proves (3.10) for $n+1$. Now, replacing x by $(a+b)^{-m}x$ in (3.10), we have

$$\begin{aligned}
&\rho\Big((a+b)^{2m}f\Big(\frac{x}{(a+b)^m}\Big)-(a+b)^{2(m+n)}f\Big(\frac{x}{(a+b)^{m+n}}\Big)\Big)\\
&\quad\le \kappa^{2m}\rho\Big(f\Big(\frac{x}{(a+b)^m}\Big)-(a+b)^{2n}f\Big(\frac{x}{(a+b)^{m+n}}\Big)\Big)\\
&\quad\le \frac{\kappa^{2m}}{\kappa^2}\sum_{j=1}^{n}\frac{\kappa^{3j}}{(a+b)^j}\varphi_1\Big(\frac{x}{(a+b)^{j+m}},\frac{x}{(a+b)^{j+m}},0\Big)\\
&\quad\le \frac{\kappa^{2m}}{\kappa^2}\sum_{j=1}^{n}\frac{\kappa^{3j}}{(a+b)^j}\varphi_1\Big(\frac{x}{(a+b)^{j+m}},\frac{x}{(a+b)^{j+m}},0\Big)\cdot\frac{\kappa^m}{(a+b)^m}\\
&\quad= \frac{1}{\kappa^2}\sum_{j=1}^{n}\frac{\kappa^{3(j+m)}}{(a+b)^{j+m}}\varphi_1\Big(\frac{x}{(a+b)^{j+m}},\frac{x}{(a+b)^{j+m}},0\Big)\\
&\quad= \frac{1}{\kappa^2}\sum_{j=m+1}^{m+n}\frac{\kappa^{3j}}{(a+b)^j}\varphi_1\Big(\frac{x}{(a+b)^{j}},\frac{x}{(a+b)^{j}},0\Big),
\end{aligned}$$

which converges to zero as $m\to+\infty$ by the assumption (3.8). Thus, the sequence $\{(a+b)^{2n}f(\frac{x}{(a+b)^n})\}$ is ρ-Cauchy for all $x\in\chi_\rho$ and so it is ρ-convergent in χ_ρ since the space χ_ρ is ρ-complete. Thus, we may define a mapping $F_2:\chi_\rho\to\chi_\rho$ as

$$\begin{aligned}
F_2(x) &:= \rho-\lim_{n\to\infty}(a+b)^{2n}f\Big(\frac{x}{(a+b)^n}\Big)\\
&\Longleftrightarrow \lim_{n\to\infty}\rho\Big((a+b)^{2n}f\Big(\frac{x}{(a+b)^n}\Big)-F_2(x)\Big)=0
\end{aligned}$$

for all $x\in\chi_\rho$.

Now, we prove the mapping F_2 satisfies the equation (1.3). Setting $(x,y,z):=(a+b)^{-n}(x,y,0)$ in (3.7), and then multiplying the resulting inequality by $(a+b)^{2n}$, we get

$$\begin{aligned}
\rho((a+b)^{2n}QE_f^{\lambda}((a+b)^{-n}(x,y))) &\le \kappa^{2n}\varphi_1((a+b)^{-n}(x,y,0))\\
&\le \frac{\kappa^{3n}}{(a+b)^n}\varphi_1((a+b)^{-n}(x,y,0)),
\end{aligned}$$

which tends to zero as $n \to +\infty$ for all $x, y \in \chi_\rho$. Thus, it follows that

$$\begin{aligned}
\rho\Big(\frac{1}{L} QE^\lambda_{F_2}(x,y)\Big) \le{}& \frac{1}{L}\rho\Big(F_2(\lambda ax + \lambda by) - (a+b)^{2n} f\Big(\frac{\lambda ax + \lambda by}{(a+b)^n}\Big)\Big) \\
&+ \frac{|\lambda|^2 ab}{L}\rho\Big(F_2(x-y) - (a+b)^{2n} f\Big(\frac{x-y}{(a+b)^n}\Big)\Big) \\
&+ \frac{|\lambda|^2 (a+b)a}{L}\rho\Big(F_2(x) - (a+b)^{2n} f\Big(\frac{x}{(a+b)^n}\Big)\Big) \\
&+ \frac{|\lambda|^2 (a+b)b}{L}\rho\Big(F_2(y) - (a+b)^{2n} f\Big(\frac{y}{(a+b)^n}\Big)\Big) \\
&+ \frac{1}{L}\rho\Big((a+b)^{2n} QE^\lambda_f\Big((a+b)^{-n}(x,y)\Big)\Big)
\end{aligned}$$

for all $x, y \in \chi_\rho$ and all positive integers n, where L is the smallest natural number with $a^2 + 3ab + b^2 + 2 \le L$. Taking the limit as $n \to +\infty$ in the last inequality, we arrive at $QE^\lambda_{F_2}(x,y) = 0$ for all $x, y \in \chi_\rho$ and all $\lambda \in \mathbb{T}_{n_0}$. Hence F_2 satisfies the equation (1.3), and so it is quadratic homogeneous by the same reasoning as in Theorem 3.1.

Furthermore, without using the Fatou property one can see the following inequality

$$\begin{aligned}
\rho(f(x) - F_2(x)) ={}& \rho\Big(\frac{1}{(a+b)}\Big\{(a+b)f(x) - (a+b)^{2n+1} f\Big(\frac{x}{2^n}\Big)\Big\} \\
&+ \frac{1}{(a+b)}\Big\{(a+b)^{2n+1} f\Big(\frac{x}{2^n}\Big) - (a+b)F_2(x)\Big\}\Big) \\
\le{}& \frac{\kappa}{(a+b)} \cdot \frac{1}{\kappa^2} \sum_{j=1}^{n} \frac{\kappa^{3j}}{(a+b)^j} \varphi_1(\frac{x,x,0}{2^j}) \\
&+ \frac{\kappa}{(a+b)}\rho\Big((a+b)^{2n} f(\frac{x}{(a+b)^n}) - F_2(x)\Big) \\
\le{}& \frac{1}{(a+b)\kappa} \sum_{j=1}^{\infty} \frac{\kappa^{3j}}{(a+b)^j} \varphi_1\Big(\frac{x,x,0}{2^j}\Big) \\
={}& \frac{1}{2\kappa}\Psi(x,x,0),
\end{aligned}$$

which yields the approximation (3.9) by taking $n \to +\infty$.

To prove that F_2 is a quadratic Lie ∗-derivation, we observe by the inequality (3.8) that

$$\begin{aligned}\rho\Big(\frac{1}{4}QD_{F_2}(x,y)\Big) &= \rho\Big(\frac{1}{4}QD_{F_2}(x,y) - (a+b)^{4n}\frac{QD_f((a+b)^{-n}(x,y))}{4} \\ &\quad + (a+b)^{4n}\frac{QD_f((a+b)^{-n}(x,y))}{4}\Big) \\ &\le \frac{1}{4}\rho\big(F_2([x,y]) - (a+b)^{4n}f((a+b)^{-2n}[x,y])\big) \\ &\quad + \frac{1}{4}\rho\big([x^2,(a+b)^{2n}f((a+b)^{-n}y) - F_2(y)]\big) \\ &\quad + \frac{1}{4}\rho\big([(a+b)^{2n}f((a+b)^{-n}x) - F_2(x), y^2]\big) \\ &\quad + \frac{\kappa^{4n}}{4}\varphi_2\big(2^{-n}(x,y)\big)\end{aligned}$$

for all $x,y \in \chi_\rho$, from which $QD_{F_2}(x,y) = 0$ by taking $n \to +\infty$ and so F_2 is a quadratic Lie derivation. In addition, it follows from the definition of F_2 that the following inequality

$$\begin{aligned}\rho\Big(\frac{1}{3}\Big(F_2(z^*) - F_2(z)^*\Big)\Big) &\le \frac{1}{3}\rho\Big(F_2(z^*) - (a+b)^{2n}f\Big(\frac{z^*}{(a+b)^n}\Big)\Big) \\ &\quad + \frac{1}{3}\rho\Big((a+b)^{2n}f\Big(\frac{z}{(a+b)^n}\Big)^* - F_2(z)^*\Big) \\ &\quad + \frac{\kappa^{3n}}{3(a+b)^n}\varphi_1\Big(0,0,\frac{z}{(a+b)^n}\Big)\end{aligned}$$

holds for all vectors $z \in \chi_\rho$, which goes to zero as $n \to +\infty$. Hence, one concludes that F_2 is a quadratic Lie ∗-derivation. Hence, the mapping F_2 is a unique quadratic Lie ∗-derivation satisfying the estimation (3.9) near f. □

Corollary 3.4. *Let χ_ρ be a ρ-complete convex modular ∗-algebra with Δ_2-condition. Suppose there exist two functions $\varphi_1 : \chi_\rho^3 \to [0,+\infty)$ and $\varphi_2 : \chi_\rho^2 \to [0,+\infty)$ and two constant l_i with $0 < l_1 < \frac{(a+b)^3}{\kappa^3}$ and $0 < l_2 < \frac{(a+b)^4}{\kappa^4}$ for which a mapping $f : \chi_\rho \to \chi_\rho$ satisfies*

$$\rho(QE_f^\lambda(x,y) + f(z^*) - f(z)^*) \le \varphi_1(x,y,z), \varphi_1(\frac{x,y,z}{(a+b)}) \le \frac{l_1}{(a+b)^2}\varphi_1(x,y,z),$$

$$\rho(QD_f(x,y)) \le \varphi_2(x,y), \varphi_2(\frac{x,y}{(a+b)}) \le \frac{l_2}{(a+b)^4}\varphi_2(x,y)$$

for all $x,y,z \in \chi_\rho$ and all $\lambda \in \mathbb{T}_{n_0}$. If for each $x \in \chi_\rho$ the mapping $t \to f(tx)$ from $\mathbb{R}$ to χ_ρ is continuous, then there exists a unique quadratic Lie

$*$-derivation $F_2 : \chi_\rho \to \chi_\rho$ satisfies the equation (1.3) and

$$\rho(f(x) - F_2(x)) \le \frac{\kappa^2 l_1}{(a+b)((a+b)^3 - \kappa^3 l_1)} \varphi_1(x, x, 0)$$

for all $x \in \chi_\rho$.

Remark 3.5. In Theorem 3.3, if χ_ρ is a Banach $*$-algebra with norm $\|\cdot\| := \rho$, and so $\rho((a+b)x) = (a+b)\rho(x),\ \kappa := (a+b)$, then we see from (3.7) and (3.8) that there exists a unique quadratic Lie $*$-derivation $F_2 : \chi_\rho \to \chi_\rho$, defined as $F_2(x) = \lim_{n\to\infty}(a+b)^{2n} f(\frac{x}{(a+b)^n}),\ x \in \chi_\rho$, which satisfies the equation (1.3) and the estimation

$$\rho(f(x) - F_2(x)) \le \frac{1}{(a+b)^2} \sum_{j=1}^{\infty} (a+b)^{2j} \varphi_1\left(\frac{x}{(a+b)^j}, \frac{x}{(a+b)^j}, 0\right)$$

near f for all $x \in \chi_\rho$.

As a corollary of Theorem 3.1 and Theorem 3.3, we obtain the following stability result of approximate quadratic Lie $*$-derivations on complete normed $*$-algebras χ_ρ, which may be considered as χ_ρ equipped with norm $\|\cdot\| = \rho(\cdot)$.

Corollary 3.6. *Let χ_ρ be a complete normed $*$-algebra. For given nonnegative real numbers θ_i, ϑ_i together with $r_i \neq 2 (i = 1, 2, 3)$ and p_1, p_2 with $p_1 + p_2 \neq 2$, suppose that a mapping $f : \chi_\rho \to \chi_\rho$ with $f(0) = 0$ satisfies*

$$\|QE_f^\lambda(x, y) + f(z^*) - f(z)^*\| \le \theta_1\|x\|^{r_1} + \theta_2\|y\|^{r_2} + \theta_3(\|x\|^{p_1}\|y\|^{p_2} + \|z\|^{r_3}),$$
$$\|QD_f(x, y)\| \le \vartheta_1\|x\|^{2r_1} + \vartheta_2\|y\|^{2r_2} + \vartheta_3\|x\|^{2p_1}\|y\|^{2p_2}$$

for all $x, y, z \in \chi_\rho$ and all $\lambda \in \mathbb{T}_{n_0}$. If for each $x \in \chi_\rho$ the mapping $t \to f(tx)$ from $\mathbb{R}$ to χ_ρ is continuous, then there exists a unique quadratic Lie $*$-derivation $F : \chi_\rho \to \chi_\rho$ such that

$$\|f(x) - F(x)\| \le \frac{\theta_1\|x\|^{r_1}}{|(a+b)^2 - (a+b)^{r_1}|} + \frac{\theta_2\|x\|^{r_2}}{|(a+b)^2 - (a+b)^{r_2}|} + \frac{\theta_3\|x\|^{p_1+p_2}}{|(a+b)^2 - (a+b)^{p_1+p_2}|}$$

for all $x \in \chi_\rho$.

4. Conclusion

In the paper, we are devoted to proving stability results for an approximate quadratic Lie $*$-derivation associated with a quadratic functional equation in ρ-complete convex modular $*$-algebra by way of the direct method. As results, we have obtained stability results of approximate quadratic Lie $*$-derivations in Banach ∗-algebras, and these stability results could be applied to various ∗-algebras.

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References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, **2** (1950), 64–66.

[2] C. Borelli and G.L. Forti, *On a general Hyers–Ulam stability result*, Internat. J. Math. Math. Sci., **18** (1995), 229–236, doi:10.1155/S0161171295000287.

[3] N. Brillouët-Belluot, J. Brzdęk, K. Ciepliński, *On some recent developments in Ulam's type stability*, Abst. Appl. Anal., **2012** (2012), Article ID 716936, 41 pages.

[4] I.-S. Chang, H.-M. Kim, H.-W. Lee, *Approximation of almost quadratic mappings via modular functional*, J. Math. Comput. Sci., **29**(2) (2023), 106–117.

[5] S. Czerwik, *Functional equations and inequalities in several variables*, World Scientific, New Jersey, London, Singapore, Hong Kong, 2002.

[6] El-Sayed El-Hady, *On hyperstability of Cauchy functional equation in (2,γ)-Banach spaces*, J. Math. Comput. Sci., **23**(4) (2021), 354–363.

[7] G.L. Forti, *Comments on the core of the direct method for proving Hyers–Ulam stability of functional equations*, J. Math. Anal. Appl., **295** (2004), 127–133.

[8] O. Fouad, A. Radouane and M. Driss, *A fixed point theorem on some multi-valued maps in modular spaces*, Nonlinear Funct. Anal. Appl. **27**(3) (2022), 641–648.

[9] P. Găvruta, *A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., **184** (1994), 431–436, doi:10.1006/jmaa.1994.1211.

[10] M.B. Ghaemi, M. Choubin, G. Sadeghi, M.E. Gordji, *A fixed point approach to stability of quintic functional equations in modular spaces*, Kyungpook Math. J., **55** (2015), 313–326.

[11] A. Grabiec, *The generalized Hyers–Ulam stability of a class of functional equations*, Publ. Math. Debrecen, **48** (1996), 217–235.

[12] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224, doi:10.1073/pnas.27.4.222.

[13] D.H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.

[14] S.-M. Jung, *Hyers–Ulam–Rassias stability of functional equations in nonlinear analysis*, Springer Optimization and its applications, Vol. **48**, Springer, New York, 2011.

[15] S. Karthikeyan, C. Park, P. Palani, T.R.K. Kumar, *Stability of an additive–quartic functional equation in modular spaces*, J. Math. Comput. Sci., **26** (2022), 22–40.

[16] H.-M. Kim and Y. Hong, *Approximate quadratic mappings in modular spaces*, Int. J. Pure Appl. Math., **116** (2017), 31–43.

[17] H.-M. Kim and Y. Hong, *Additional stability results for quartic Lie ∗-derivations*, Nonlinear Funct. Anal. Appl., **24**(3) (2019), 583–593.

[18] H.-M. Kim, J.-S. Park and H.-Y. Shin, *Approximation of quadratic Lie ∗-derivations on ρ-complete convex modular algebras*, J. Math. Inequal., **14**(1) (2020), 121–134.

[19] Y.-H. Lee and S.-M. Jung, *Stability of some cubic-additive functional equations*, Nonlinear Funct. Anal. Appl., **25**(1) (2020), 35–54.

[20] Z. Moszner, *On the stability of functional equations*, Aequationes Math., **77** (2009), 33–88.

[21] J. Musielak and W. Orlicz, *On modular spaces*, Studia Math., **18** (1959), 49–65.

[22] H. Nakano, *Modulared Semi-Ordered Linear Spaces*, Tokyo Math. Book Ser., **1** Maruzen Co. Ltd. Tokyo, 1950.

[23] C. Park, A. Bodaghi, S.O. Kim, *A fixed point approach to stability of additive mappings in modular spaces without Δ_2-conditions*, J. Comput. Anal. Appl., **24** (2018), 1038–1048.

[24] C. Park, J.M. Rassias, A. Bodaghi, S.O. Kim, *Approximate homomorphisms from ternary semigroups to modular spaces*, Rev. Real Acad. Cienc. Exactas, Fis. Nat. Ser. A-Mat., **2019**, 113, 2175–2188.

[25] M. Ramdoss, D. Pachaiyappan, I. Hwang and C. Park, *Stability of an n-variable mixed type functional equation in probabilistic modular space*, AIMS Mathematics, **5**(6), (2020), 5903–5915.

[26] J.M. Rassias, *On the stability of the Euler–Lagrange functional equation*, Chinese J. Math., **20**(2) (1992), 185–190.

[27] J.M. Rassias, *On the stability of the general Euler–Lagrange functional equation*, Demonstratio Math., **XXIX** (1996), 755–766.

[28] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72** (1978), 297–300, doi:10.1090/S0002-9939-1978-0507327-1.

[29] Th.M. Rassias, *Functional equations, inequalities and applications*, Kluwer Acadmic Publishers, Dordrecht, Boston and London, 2003.

[30] G. Sadeghi, *A fixed point approach to stability of functional equations in modular spaces*, Bull. Malays. Math. Sci. Soc., **37**(2) (2014), 333–344.

[31] S.M. Ulam, *Problems in Modern Mathematics*, Chap. VI, Wiley, New York, 1960.

[32] K. Wongkum, P. Chaipunya and P. Kumam, *On the generalized Ulam–Hyers–Rassias stability of quadratic mappings in modular spaces without Δ_2-conditions*, J. Funct. Spaces, Vol. 2015, Article ID 461719, 6 pages, doi:10.1155/2015/461719.

[33] K. Wongkum and P. Kumam, *The stability of sextic functional equation in fuzzy modular spaces*, J. Nonlinear Sci. Appl., **9** (2016), 3555–3569.