Nonlinear Functional Analysis and Applications
Vol. 28, No. 4 (2023), pp. 877-886
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2023.28.04.02
http://nfaa.kyungnam.ac.kr/journal-nfaa

FIXED POINT THEOREMS IN b-METRIC AND EXTENDED b-METRIC SPACES

P. Swapna[1], T. Phaneendra[2] and M. N. Rajashekhar[3]

[1]Research Scholar, Department of Mathematics, JNTUH, Hyderabad - 500 085
(MVSR Engineering College, Rangareddy, Hyderabad-501510, Telangana State, India)
e-mail: swapna.pothuguntla@gmail.com

[2]Department of Mathematics, School of Advanced Sciences,
Vellore Institute of Technology, Vellore-632014, Tamil Nadu, India
e-mail: drtp.indra@gmail.com

[3]Department of Mathematics, JNTUH, Hyderabad - 500 085, Telangana, India
e-mail: mnrs@jntuh.ac.in

Abstract. The first result of this paper is to give a revised proof of Sanatammappa et al.'s recent result in a b-metric space, under appropriate choice of constants without using the continuity of the b-metric. The second is to prove a fixed point theorem under a contraction type condition in an extended b-metric space.

1. Introduction

Let X be a nonempty set and $\rho : X \times X \to \mathbb{R}$ be such that

(m1) $\rho(x, y) \geq 0$ for all $x, y \in X$,
(m2) $\rho(x, y) = 0$ if and only if $x = y$,
(m3) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$,
(m4) $\rho(x, y) \leq \rho(x, z) + \rho(y, z)$ for all $x, y, z \in X$.

Then the pair (X, ρ) denotes a metric space with metric ρ. Let $X = \mathbb{R}$. Then the metric $\rho(x, y) = |x - y|$ for all $x, y \in X$ is called the *usual metric* and it

[0]Received January 19, 2023. Revised May 25, 2023. Accepted May 29, 2023.
[0]2020 Mathematics Subject Classification: 54H25.
[0]Keywords: b-metric space, extended b-metric space, convergent and Cauchy-sequences, fixed point.
[0]Corresponding author: T. Phaneendra(drtp.indra@gmail.com).

gives the *distance* between the points x and y on the number line $\mathbb{R}^1$. Let $X = \mathbb{R} \times \mathbb{R}$ and $\rho(x, y) = |x - y|$ for all $x, y \in X$. Condition (m4) says that the length of one side in a triangle with vertices x, y and z never exceeds the sum of the lengths of other sides in it. Hence it is referred to as the *triangle inequality* of the metric ρ. The notion of metric space was due to Frechet in 1906.

In the last few decades, fixed point theorems were developed in a metric space, normed linear space, topological space etc., while the conditions imposed on the underlying mappings are usually metrical or compact type conditions. Further, new ambient algebraic structures were formulated to improve the results. One such was a b-metric, introduced by Bakhtin [1], by generalizing the triangle inequality (m4). For all the definitions of this section, one can refer to [2, 7, 8]:

Definition 1.1. ([2]) Let $s \geq 1$, X be a nonempty set and $\rho_s : X \times X \to [0, \infty)$ be such that

($b1$) $\rho_s(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$,
($b2$) $\rho_s(x, y) = \rho_s(y, x)$ for all $x, y \in X$,
($b3$) $\rho_s(x, y) \leq s[\rho_s(x, z) + \rho_s(y, z)]$ for all $x, y, z \in X$.

Then ρ_s is a b-metric on X, and (X, ρ_s) denotes a b-metric space.

A b-metric space (X, ρ_s) reduces to a metric space (X, ρ), if $s = 1$. However, a b-metric space is not necessarily a metric space. For instance, consider the pair (X, ρ_s), where $X = \mathbb{R}$ and $\rho_s(x, y) = |x - y|^2$ for all $x, y \in \mathbb{R}$. Then the conditions ($b1$) and ($b2$) are obvious. Further,

$$\begin{aligned}\rho_s(x, y) = |x - y|^2 &= |x - z + z - y|^2 \\ &\leq 2\left(|x - z|^2 + |z - y|^2\right) \\ &= 2[\rho_s(x, z) + \rho_s(y, z)]\end{aligned}$$

for all $x, y \in X$. Thus $(X = \mathbb{R}, \rho_s)$ is a b-metric space with $s = 2$. Since $\rho_s(1, 3) + \rho_s(1, 0) = 4 + 1 = 5$ and $\rho_s(0, 3) = 9$, (m3) fails to hold good, showing that ρ_s is not a metric. Thus a b-metric space is not a metric space.

In view of the convexity of $f(x) = x^p$, where $x > 0$ and $1 < p < \infty$, it follows that $(\mathbb{R}, |x - y|^p)$ is a b-metric space, which is not a metric space. In other words, the class of b-metric spaces contains that of metric spaces.

Definition 1.2. ([2]) A b-ball in a b-metric space (X, ρ_s) is defined by

$$B_{\rho_s}(x, r) = \{y \in X : \rho_s(x, y) < r\}.$$

The family of all b-balls forms a base topology, called the *b-metric topology* $\tau(\rho_s)$ on X.

Definition 1.3. ([2]) Let (X, ρ_s) be a b-metric space with parameter s. A sequence $\{x_n\}_{n=1}^{\infty}$ in X is said to be

(a) b-convergent, with limit $p \in X$, if it converges to p in the b-metric topology $\tau(\rho_s)$,
(b) b-Cauchy, if $\lim_{n,m\to\infty} \rho_s(x_n, x_m) = 0$.

Like in a metric space, every b-convergent sequence has a unique limit, and is necessarily b-Cauchy.

Definition 1.4. ([2]) A b-metric space (X, ρ_s) is said to be b-complete, if every b-Cauchy sequence in X is b-convergent in it.

Remark 1.5. A b-metric is not jointly continuous in general in its coordinate variables x and y, though a metric d is known to be continuous (See Example 2.13, [9]).

Lemma 1.6. ([4]) *Let (X, ρ_s) be a b-metric space with parameter s. Suppose that $\{x_n\}_{n=1}^{\infty}$ is b-convergent with limit x and $\{y_n\}_{n=1}^{\infty}$ is b-convergent with limit y in X. Then*

$$\frac{1}{s^2}\rho_s(x, y) \le \liminf_{n\to\infty} \rho_s(x_n, y_n) \le \limsup_{n\to\infty} \rho_s(x_n, y_n) \le s^2\rho_s(x, y). \tag{1.1}$$

In particular, $x = y$, then $\lim_{n\to\infty} \rho_s(x_n, y_n) = 0$. Further, for each $z \in X$, we have

$$\frac{1}{s}\rho_s(x, z) \le \liminf_{n\to\infty} \rho_s(x_n, z) \le \limsup_{n\to\infty} \rho_s(x_n, z) \le s\rho_s(x, z). \tag{1.2}$$

2. A modified proof of Sanatammappa et al.'s result

Sanatammappa et al. [6] recently proved the following result:

Theorem 2.1. *Let $s \ge 1$ and (X, ρ_s) be a complete b-metric space. If a self-map f on X is such that*

$$\begin{aligned} \rho_s(fx, fy) \le{}& a_1\rho_s(x, y) + a_2\rho_s(x, fx) + a_3\rho_s(y, fy) + a_4\rho_s(x, fy) \\ &+ a_5\rho_s(y, fx) + a_6[\rho_s(y, fx) + \rho_s(x, fy)], \quad \forall x, y \in X, \end{aligned} \tag{2.1}$$

where a_j, $1 \le j \le 6$ are non-negative real numbers, not all zero, with

$$a_1 + a_2 + a_3 + 2a_4 + a_5 + 2a_6 < 1. \tag{2.2}$$

Then the sequence $\{x_n\}_{n=1}^{\infty}$ defined by

$$x_n = fx_{n-1} = f^n x_0, \; n \ge 1, \tag{2.3}$$

converges to a point $p \in X$, which is a unique fixed point of f.

In the proof of Theorem 2.1, the authors obtained that for each $m > n$,

$$\rho_s(x_n, x_m) \leq \omega^n \rho_s(x_0, fx_0), \quad \forall n, \tag{2.4}$$

where

$$\omega = \frac{a_1 + a_2 + sa_4 + sa_6}{1 - a_3 - sa_4 - sa_6}. \tag{2.5}$$

Employing the limit as $n \to \infty$ in (2.4), they concluded that $\rho_s(x_n, x_m) \to 0$.

We assert that $\rho_s(f^n x_0, f^m x_0) \to 0$ as $n \to \infty$ holds good only if $\omega < 1$. But the choice (2.2) does not guarantee to give $\omega < 1$. Also, the authors used the continuity of the b-metric ρ_s in Theorem 2.1, without mentioning it. However, in view of Remark 1.5, ρ_s is not continuous. Therefore, we restate Theorem 2.1 as follows:

Theorem 2.2. *Let $s \geq 1$, (X, ρ_s) be a complete b-metric space, and $f : X \to X$ satisfy the inequality* (2.1)*, where a_j, $1 \leq j \leq 6$ are nonnegative real numbers, not all zero, such that*

$$a_1 + a_2 + a_3 + 2sa_4 + a_5 + 2sa_6 < 1. \tag{2.6}$$

Then the sequence $\{x_n\}_{n=1}^{\infty}$ defined by (2.3) *converges to a point $p \in X$, which is a unique fixed point of f.*

Proof. Given $x_0 \in X$, consider the sequence $\{x_n\}_{n=1}^{\infty}$ with the choice (2.3). Writing $x = x_{n-1}$ and $y = x_n$ in (2.1) and then using (2.3) and (b3),

$$\begin{aligned}
\rho_s(x_n, x_{n+1}) &= \rho_s(fx_{n-1}, fx_n) \\
&\leq a_1\rho_s(x_{n-1}, x_n) + a_2\rho_s(x_{n-1}, fx_{n-1}) + a_3\rho_s(x_n, fx_n) \\
&\quad + a_4\rho_s(x_{n-1}, fx_n) + a_5\rho_s(x_n, fx_{n-1}) \\
&\quad + a_6[\rho_s(x_n, fx_{n-1}) + \rho_s(x_{n-1}, fx_n)] \\
&= a_1\rho_s(x_{n-1}, x_n) + a_2\rho_s(x_{n-1}, x_n) + a_3\rho_s(x_n, x_{n+1}) \\
&\quad + a_4\rho_s(x_{n-1}, x_{n+1}) + a_5\rho_s(x_n, x_n) \\
&\quad + a_6[\rho_s(x_n, x_n) + \rho_s(x_{n-1}, x_{n+1})] \\
&\leq (a_1 + a_2)\rho_s(x_{n-1}, x_n) + a_3\rho_s(x_n, x_{n+1}) \\
&\quad + a_4.s[\rho_s(x_{n-1}, x_n) + \rho_s(x_n, x_{n+1})] + a_5.0 \\
&\quad + a_6.s[\rho_s(x_{n-1}, x_n) + \rho_s(x_n, x_{n+1})].
\end{aligned}$$

Rearranging the terms and simplifying, this gives

$$(1 - a_3 - sa_4 - sa_6)\,\rho_s(x_n, x_{n+1}) \leq (a_1 + a_2 + sa_4 + sa_6)\,\rho_s(x_{n-1}, x_n)$$

or

$$\rho_s(x_n, x_{n+1}) \leq \omega\rho_s(x_{n-1}, x_n), \quad \forall n,$$

where ω is given by (2.5). By induction,

$$\rho_s(x_n, x_{n+1}) \leq \omega^n \rho_s(x_0, x_1) \quad \text{for all } n \geq 1. \tag{2.7}$$

Therefore, for all $m > n$, repeatedly using ($b3$) and (2.7), we have

$$\begin{aligned}
\rho_s(x_n, x_m) &\leq s\left[\rho_s(x_n, x_{n+1}) + \rho_s(x_{n+1}, x_m)\right] \\
&\leq s\rho_s(x_n, x_{n+1}) + s^2\left[\rho_s(x_{n+1}, x_{n+2}) + \rho_s(x_{n+2}, x_m)\right] \\
&\vdots \\
&\leq \underbrace{s\rho_s(x_n, x_{n+1}) + s^2\rho_s(x_{n+1}, x_{n+2}) + \cdots + s^{m-n}\rho_s(x_{m-1}, x_m)}_{m-n \text{ terms}} \\
&\leq \left[s\omega^n + s^2\omega^{n+1} + \cdots + s^{m-n}\omega^{m-1}\right]\rho_s(x_0, x_1) \\
&= s\omega^n\left(1 + s\omega + \cdots + s^{m-n-1}\omega^{m-n-1}\right)\rho_s(x_0, x_1) \\
&\leq \frac{s\omega^n}{1 - s\omega} \cdot \rho_s(x_0, x_1) \quad \text{for all } n.
\end{aligned}$$

As $m, n \to \infty$, this implies that $\rho_s(x_m, x_n) \to 0$. Thus $\{x_n\}_{n=1}^{\infty}$ is a b-Cauchy sequence in X. Since X is b-complete, there exists a point $z \in X$ such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} f^n x_0 = z. \tag{2.8}$$

Now we show that z is a fixed point of f. In fact, writing $x = x_n$ and $y = z$ in (2.1), the we have

$$\begin{aligned}
\rho_s(fx_n, fz) \leq\ & a_1\rho_s(x_n, z) + a_2\rho_s(x_n, fx_n) + a_3\rho_s(z, fz) \\
&+ a_4\rho_s(x_n, fz) + a_5\rho_s(z, fx_n) \\
&+ a_6[\rho_s(z, fx_n) + \rho_s(x_n, fz)]
\end{aligned}$$

or

$$\begin{aligned}
\rho_s(x_{n+1}, fz) \leq\ & a_1\rho_s(x_n, z) + a_2\rho_s(x_n, x_{n+1}) + a_3\rho_s(z, fz) \\
&+ a_4\rho_s(x_n, fz) + a_5\rho_s(z, x_{n+1}) \\
&+ a_6[\rho_s(z, x_{n+1}) + \rho_s(x_n, fz)] \quad \text{for all } n.
\end{aligned}$$

Employing the limit superior as $n \to \infty$ in this and using Lemma 1.6, this gives

$$\begin{aligned}
\frac{1}{s^2}\rho_s(z, fz) &\leq \limsup_{n\to\infty} \rho_s(x_{n+1}, fz) \\
&\leq a_3\rho_s(z, fz) + (a_4 + a_6)\limsup_{n\to\infty} \rho_s(x_{n+1}, fz) \\
&\leq a_3\rho_s(z, fz) + (a_4 + a_6)s\rho_s(z, fz).
\end{aligned}$$

If $\rho_s(z, fz) > 0$, this would give a contradiction that

$$0 < \frac{1}{s^2}\rho_s(z, fz) \leq (a_3 + sa_4 + sa_6)\rho_s(z, fz) < \rho_s(z, fz).$$

Therefore, $\rho_s(z, fz) = 0$. The uniqueness of the fixed point is easily established. □

Writing $a_2 = a_3 = a_6 = 0$ in Theorem 2.2, we have:

Corollary 2.3. *Let $s \geq 1$, (X, ρ_s) be a complete b-metric space, and $f : X \to X$ satisfy the inequality*

$$\rho_s(fx, fy) \leq \alpha\rho_s(x, y) + \beta\rho_s(x, fy) + \gamma\rho_s(y, fx) \text{ for all } x, y \in X, \tag{2.9}$$

where $0 < \alpha + \beta s + \gamma < 1$. Then f has a unique fixed point.

3. Result in extended b-metric space

Kamran et al. ([3]) generalized a b-metric space as an extended b-metric space as follows:

Definition 3.1. Let $s \geq 1$, X be a nonempty set and $\theta : X \times X \to [1, \infty)$. Consider $\rho_\theta : X \times X \to [0, \infty)$ such that

(eb1) $\rho_\theta(x, y) = 0$ for all $x, y \in X$,
(eb2) $\rho_\theta(x, y) = 0$ implies that $x = y$ for all $x, y \in X$,
(eb3) $\rho_\theta(x, y) = \rho_\theta(y, x)$ for all $x, y \in X$,
(eb4) $\rho_\theta(x, y) \leq \theta(x, y)[\rho_\theta(x, z) + \rho_\theta(z, y)]$ for all $x, y, z \in X$.

Then ρ_θ is called an extended b-metric on X, and (X, ρ_θ) is an extended b-metric space.

If $\theta(x, y) = s \geq 1$ for all $x, y \in X$, then ρ_θ reduces to a b-metric ρ_s. In this paper, we denote an extended b-metric space by (X, ρ_θ).

The notions of convergence and completeness in an extended b-metric space are similar to that in a b-metric space.

Definition 3.2. A sequence $\{x_n\}_{n=1}^{\infty} \subset X$ is said to be convergent to $z \in X$, written as $\lim_{n\to\infty} x_n = z$, if for every $\epsilon > 0$ there exists a natural number N such that $\rho_\theta(x_n, z) < \epsilon$ for all $n \geq N$.

If ρ_θ is continuous, then every convergent sequence in X has a unique limit in it.

Definition 3.3. A sequence $\{x_n\}_{n=1}^{\infty} \subset X$ is said to be Cauchy, if for every $\epsilon > 0$ there exists a natural number N such that $\rho_\theta(x_n, x_m) < \epsilon$ for all $m, n \geq N$.

A Cauchy sequence in X need not be convergent in it. But, if Cauchy sequence in X is convergent in it, then we say that X is complete. Banach contraction mapping theorem in an extended b-metric space was proved in [3]. We establish fixed point theorems for some contraction types other than Banach's, in an extended b-metric space. In this sequel, we employ the following notion:

Definition 3.4. Let f be a self-map on an extended b-metric space (X, ρ_θ) and $x_0 \in X$. Then the orbit $O_f(x_0)$ at x_0 is the sequence of f-iterates $x_0, fx_0, ..., f^n x_0, ...$.

We need the following lemmas:

Lemma 3.5. ([5, Theorem 3.22, p. 59]) The infinite series $\sum_{n=1}^{\infty} u_n$ of positive terms converges if and only if, given $\epsilon > 0$, there is a natural number n_0 such that $\sum_{j=n}^{m} u_n \leq \epsilon$ for all $m \geq n \geq n_0$.

Lemma 3.6. ([5, Theorem 3.34, p. 66]) The infinite series $\sum_{n=1}^{\infty} u_n$ of positive terms converges, provided $\limsup_{n\to\infty} (u_{n+1}/u_n) < 1$.

Our main result in extended b-metric space (X, ρ_θ), where ρ_θ is continuous, is the following:

Theorem 3.7. *Let (X, ρ_θ) be a complete extended b-metric space, where ρ_θ is continuous. Suppose that $f : X \to X$ satisfies the condition*

$$\rho_\theta(fx, fy) \leq \alpha\rho_\theta(x, y) + \beta\rho_\theta(x, fy) + \gamma\rho_\theta(y, fx) \text{ for all } x, y \in X, \tag{3.1}$$

where $0 < \alpha + 2\beta + \gamma < 1$ is such that for each $x_0 \in X$,

$$\lim_{n,m\to\infty} \left\{ \frac{\alpha + \beta\theta(f^n x_0, f^{n+2}x_0)}{1 - \beta\theta(f^n x_0, f^{n+2}x_0))} \right\} \theta(f^{n+1}x_0, f^m x_0) < 1. \tag{3.2}$$

Then f has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary. Define $\{x_n\}_{n=1}^{\infty} \subset X$ by

$$x_n = fx_{n-1} \quad \text{for } n \geq 1. \tag{3.3}$$

By induction, (3.3) gives

$$x_1 = fx_0, x_2 = f^2x_0, ..., x_n = f^n x_0 \quad \text{for } n \geq 1. \tag{3.4}$$

Now writing $x = x_{n-1}$ and $y = x_n$ in (3.1) and using (3.4) and ($eb4$), we find that

$$\begin{aligned}
\rho_\theta(x_n, x_{n+1}) &= \rho_\theta(fx_{n-1}, fx_n) \\
&\leq \alpha\rho_\theta(x_{n-1}, x_n) + \beta\rho_\theta(x_{n-1}, fx_n) + \gamma\rho_\theta(x_n, fx_{n-1}) \\
&= \alpha\rho_\theta(x_{n-1}, x_n) + \beta\rho_\theta(x_{n-1}, x_{n+1}) \\
&\leq \alpha\rho_\theta(x_{n-1}, x_n) + \beta\theta(x_{n-1}, x_{n+1})[\rho_\theta(x_{n-1}, x_n) + \rho_\theta(x_n, x_{n+1})].
\end{aligned}$$

Rearranging and simplifying, this gives

$$\rho_\theta(x_n, x_{n+1}) \leq \frac{\alpha + \beta\theta(x_{n-1}, x_{n+1})}{1 - \beta\theta(x_{n-1}, x_{n+1})} \cdot \rho_\theta(x_{n-1}, x_n)\cdot$$

By induction, it follows that

$$\rho_\theta(x_n, x_{n+1}) \leq \psi_n \cdot \rho_\theta(x_0, x_1),\ n = 1,\, 2\, ..., \tag{3.5}$$

where

$$\psi_n = \Pi_{j=1}^n \left\{ \frac{\alpha + \beta\theta(x_{j-1}, x_{j+1})}{1 - \beta\theta(x_{j-1}, x_{j+1})} \right\} \text{ for all } n. \tag{3.6}$$

Now for $m > n$, by using $(eb4)$ repeatedly and (3.6), we obtain

$$\begin{aligned} \rho_\theta(x_n, x_m) &\leq \theta(x_n, x_m)[\rho_\theta(x_n, x_{n+1}) + \rho_\theta(x_{n+1}, x_m)] \\ &\leq \theta(x_n, x_m)\left[\psi_n \rho_\theta(x_0, x_1) + \rho_\theta(x_{n+1}, x_m)\right] \\ &\vdots \\ &\leq \rho_\theta(x_0, x_1)\theta(x_n, x_m)\big[\psi_n + \psi_{n+1} \cdot \theta(x_{n+1}, x_m) \\ &\quad + \cdots + \psi_{m-1} \cdot \theta(x_{n+1}, x_m)\theta(x_{n+2}, x_m) \cdots \theta(x_{m-1}, x_m)\big]. \end{aligned}$$

Since, $\theta(x, y) \geq 1$ for all x and y, this can be written as

$$\begin{aligned} \rho_\theta(x_n, x_m) \leq \rho_\theta(x_0, x_1)\big[&\psi_n \cdot \theta(x_1, x_m)\theta(x_2, x_m) \cdots \theta(x_n, x_m) \\ &+ \psi_{n+1} \cdot \theta(x_1, x_m)\theta(x_2, x_m) \cdots \theta(x_n, x_m)\theta(x_{n+1}, x_m) \\ &+ \cdots + \psi_{m-1} \cdot \theta(x_1, x_m)\theta(x_2, x_m) \cdots \theta(x_n, x_m) \\ &\times \theta(x_{n+1}, x_m)\theta(x_{n+2}, x_m) \cdots \theta(x_{m-1}, x_m)\big]. \end{aligned} \tag{3.7}$$

Consider the series $P = \sum_{n=1}^\infty \psi_n \cdot \Pi_{i=1}^n \theta(x_i, x_m)$ for each $m \geq 1$. Write $v_n = \psi_n \cdot \Pi_{i=1}^n \theta(x_i, x_m)$ for each m and $n \geq 1$. Then

$$\begin{aligned} \lim_{n\to\infty} \frac{v_{n+1}}{v_n} &= \lim_{n\to\infty} \frac{\psi_{n+1} \cdot \Pi_{i=1}^{n+1} \theta(x_i, x_m)}{\psi_n \cdot \Pi_{i=1}^n \theta(x_i, x_m)} \\ &= \lim_{n\to\infty} \left\{ \frac{\alpha + \beta\theta(x_n, x_{n+2})}{1 - \beta\theta(x_n, x_{n+2})} \right\} \theta(x_{n+1}, x_m) \text{ for each } m. \end{aligned} \tag{3.8}$$

Now, from (3.2), we find that $\lim_{n\to\infty} \frac{v_{n+1}}{v_n} < 1$, and hence in view of Lemma 3.6, the series P converges. Also, the partial sums of P, given by

$$P_n = \sum_{j=1}^n \psi_j \cdot \Pi_{i=1}^j \theta(x_i, x_m),\ n = 1, 2, \cdots,\ \text{ for each } m \tag{3.9}$$

are bounded. Using (3.9) in (3.7), it follows that

$$\rho_\theta(x_n, x_m) \leq \rho_\theta(x_0, x_1)\big(P_{m-1} - P_n\big) \text{ for } m > n. \tag{3.10}$$

Given $\epsilon > 0$, using the convergence of P and Lemma 3.5, (3.10) implies that

$$\rho_\theta(x_n, x_m) \le \rho_\theta(x_0, x_1)\epsilon \quad \text{for } m > n \ge n_0, \tag{3.11}$$

for some natural number n_0. Thus $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Since X is complete, we can find a point $z \in X$ such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} fx_{n-1} = z. \tag{3.12}$$

Now we establish that z is a fixed point of f. In fact, writing $x = x_{n-1}$ and $y = z$, the inequality (3.1) gives

$$\rho_\theta(fx_{n-1}, fz) \le \alpha\rho_\theta(x_{n-1}, z) + \beta\rho_\theta(x_{n-1}, fz) + \gamma\rho_\theta(z, fx_{n-1})$$

or

$$\rho_\theta(x_n, fz) \le \alpha\rho_\theta(x_{n-1}, z) + +\beta\rho_\theta(x_{n-1}, fz) + \gamma\rho_\theta(z, x_n).$$

Applying the limit as $n \to \infty$ and using (3.12) and the continuity of ρ_θ, we obtain

$$\rho_\theta(z, fz) \le \beta\, \rho_\theta(z, fz)$$

or $\rho_\theta(z, fz) = 0$. That is, $fz = z$.

To establish the uniqueness of the fixed point, let $q \neq z$ be also a fixed point of f. Then with $x = z$ and $y = z$ in (3.1),

$$\begin{aligned} 0 < \rho_\theta(z, q) = \rho_\theta(fz, fq) &\le \alpha\rho_\theta(z, q) + \beta\rho_\theta(z, fq) + \gamma\rho_\theta(q, fz) \\ &= (\alpha + \beta + \gamma)\rho_\theta(q, z) < \rho_\theta(z, fq), \end{aligned}$$

which is a contradiction. Hence $z = q$, and the fixed point is unique. □

Writing $\beta = 0$ and $\gamma = 0$ in Theorem 3.7, we get the following version of Banach's contraction mapping theorem in extended b-metric space, proved in [3]:

Corollary 3.8. *Let (X, ρ_θ) be a complete extended b-metric space, where ρ_θ is continuous. Suppose that $f : X \to X$ satisfies the condition*

$$\rho_\theta(fx, fy) \le \alpha\rho_\theta(x, y) \text{ for all } x, y \in X, \tag{3.13}$$

where $0 < \alpha < 1$ is such that for each $x_0 \in X$,

$$\lim_{n,m\to\infty} \alpha\theta(f^{n+1}x_0, f^m x_0) < 1. \tag{3.14}$$

Then f has a unique fixed point.

Now, writing $\theta(x, y) = s \ge 1$ in Theorem 3.7, we obtain Corollary 2.3. It may be noted from the proof of Theorem 2.2 that the continuity of $\rho_s(x, y)$ is not needed to obtain a unique fixed point.

Conclusion

In the first result of this paper, a revised proof has been given to a recent theorem of Sanatammappa et al. in a b-metric space, under appropriate choice of constants, without using the continuity of the b-metric. As a second result, a general fixed point theorem has been proved in an extended b-metric space under a contraction type condition.

Acknowledgements: The authors express sincere thanks to the referees for their useful suggestions in improving the paper.

References

[1] I.A. Bakhtin, *The contraction mapping principle in quasi-metric spaces*, Funct. Anal. Unianowsk Gos. Ped. Inst., **30** (1989), 26–37.

[2] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostraviensis. **1** (1993), 511.

[3] T. Kamran, M. Samreen and Q.Ul. Ain, *A generalization of b-metric space and some fixed point theorems*, Mathematics, **5**(19) (2017), DOI: 10.3390/math5020019.

[4] J.R. Roshan, N. Shobkolaei, S. Sedghi and M. Abbas, *Common fixed point of four maps in b-metric spaces*, Hacettepe J. Math. and Statistics, **43**(4) (2014), 613–624.

[5] W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill Int. Edn. Third Edition, 1976.

[6] N.P. Sanatammappa, R. Krishnakumar and K. Dinesh, *Some results in b-metric spaces*, Malaya J. Matematik, **9**(1) (2021), 539–541.

[7] Sonam, C. S. Chauhan, Ramakant Bharadwaj and Satyendra Narayan, *Fixed point results in soft rectangular b-metric space*, Nonlinear Funct. Anal. Appl., **28**(3) (2023), 753-774.

[8] T. Stephen, Y. Rohen, M. K. Singh and K. S. Devi, *Some rational F-contractions in b-metric spaces and fixed points,* Nonlinear Funct. Anal. Appl., **27**(2) (2022), 309-322

[9] B. Wu, F. He and T. Xu, *Common fixed point theorems for Ciric type mappings in b-metric spaces without any completeness assumption*, J. Nonlinear Sci. Appl., **10** (2017), 3180–3190.