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COCYCLE EQUATIONS VIA COCHAINS AND HYPERSTABILITY OF RELATED FUNCTIONAL EQUATIONS

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Abstract. This paper presents properties of the cocycle equations via cochains on a semigroup. And then we offer hyperstability results of related functional equations using the properties of cocycle equations via cochains. These results generalize hyperstability results of a class of linear functional equation by Maksa and Páles. The obtained results can be applied to obtain hyperstability of various functional equations such as Euler-Lagrange type quadratic equations.

1. Introduction

In 2001, Maksa and Páles [10] proved a new type of stability of a class of linear functional equation

$$f(s) + f(t) = \frac{1}{n}\sum_{i=1}^{n} f(s\psi_i(t)), \qquad (s, t \in S), \tag{1.1}$$

where f is a functional on a semigroup $S := (S, \cdot)$ and $\psi_1, \cdots, \psi_n : S \to S$ pairwise distinct automorphisms of S such that the set $\{\psi_1, \cdots, \psi_n\}$ is a group with the operation of composition of mappings. More precisely, they proved that if the error bound for the difference of two sides of (1.1) satisfies a certain asymptotic property, then in fact, the two sides have to be equal. Such a phenomenon is called the *hyperstability* of the functional equation on S. Since then numerous papers on this subject have been published [1]-[9].

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In 2015, Sirouni and Kabbaj [12] investigated of the hyperstability of an Euler-Lagrange type quadratic functional equation:

$$f(x+y)+\frac{f(x-y)+f(y-x)}{2}=2f(x)+2f(y) \tag{1.2}$$

in class of functions from an abelian group into a Banach space. The general solution and stability of this equation is established by Rassias [11].

Let $(G,+)$ denote a semigroup and X be a real normed space. Note that the function F satisfied with the equation:

$$F(x,y)+F(x+y,z)=F(x,y+z)+F(y,z), \quad (x,y,z\in G) \tag{1.3}$$

is called a *cocycle* on $G\times G$ into X and the equation is called the *cocycle equation*. If F is the Cauchy difference, that is $F(x,y)=f(x)+f(y)-f(x+y)$, then F satisfies the equation (1.3). It is well known that the cocycle equation plays an important role in the hyperstability. Note that every quadratic functional equation on G:

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

has the cocycle equation-type identity

$$F(x,y)+\frac{F(x+y,z)+F(x-y,z)}{2}=F(y,z)+\frac{F(x,y+z)+F(x,y-z)}{2},$$

where $F(x,y)=f(x)+f(y)-\frac{f(x+y)}{2}-\frac{f(x-y)}{2}$.

In this paper, we introduce the concept of cocycles via a cochain and present that cocycles via a cochain play an important role in the hyperstability. As results, we obtain that if F is a cocycle via a cochain $\{\varphi_i\}$ on a semigroup then there is a generating function f such that F is a coboundary of f. That is, if f is a generating function such that

$$F(x_1,\cdots,x_m)=f(x_1)+\cdots+f(x_m)-\frac{1}{n}\sum_{i=1}^{n}f(\varphi_i(x_1,\cdots,x_m)), \tag{1.4}$$

where $\{\varphi_i\}$ is a cochain, then F satisfies the cocycle equation via the cochain:

$$\begin{aligned}&F(x_1,\cdots,x_m)+\frac{1}{n}\sum_{i=1}^{n}F(\varphi_i(x_1,\cdots,x_m),y_1,\cdots,y_{m-1}) \qquad (1.5)\\ &=\frac{1}{n}\sum_{i=1}^{n}F(x_1,\cdots,x_{m-1},\varphi_i(x_m,y_1,\cdots,y_{m-1}))+F(x_m,y_1,\cdots,y_{m-1}).\end{aligned}$$

Also we show that the hyperstability of the functional equation

$$f(x_1) + \cdots + f(x_m) = \frac{1}{n} \sum_{i=1}^{n} f(\varphi_i(x_1, \cdots, x_m)). \tag{1.6}$$

This generalizes the results of a class of linear functional equation by Maksa and Páles [10]. Using the properties of cocycle equations via cochains we offer hyperstability results of related functional equations.

2. Cocycle equations by cochains

Throughout this paper, let $(G, +)$ denote a semigroup and X be a real normed space. Also let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of natural numbers, real numbers and complex numbers, respectively, and let $\mathbb{R}_+ = \{x \in \mathbb{R} \,|\, x > 0\}$ and $S_n = \{1, 2, \cdots, n\}$.

Definition 2.1. Let $\varphi_i : G^m \longrightarrow G$ be a function for each $i \in S_n$. $\{\varphi_i | i \in S_n\}$ is called a cochain in m-variables if there exists a bijective function $\lambda : S_n \times S_n \longrightarrow S_n \times S_n$ such that if $\lambda(i, j) = (a_{ij}, b_{ij})$ for any $i, j, a_{ij}, b_{ij} \in S_n$, then

$$\begin{aligned} &\varphi_i(\varphi_j(x_1, \cdots, x_m), y_1, \cdots, y_{m-1}) \\ &\quad = \varphi_{a_{ij}}(x_1, \cdots, x_{m-1}, \varphi_{b_{ij}}(x_m, y_1, \cdots, y_{m-1}) \end{aligned} \tag{2.1}$$

for all $x_1, \cdots x_m, y_1, \cdots, y_{m-1} \in G$.

Example 2.2. Consider the case $m = 2$ of the above definition. Let $\varphi_i : G \times G \longrightarrow G$ be a function for each $i \in S_4$ defined by

$$\begin{aligned} &\varphi_1(x, y) = x + y, \quad \varphi_2(x, y) = x - y, \\ &\varphi_3(x, y) = -x + y, \quad \varphi_4(x, y) = -x - y \end{aligned}$$

for all $x, y \in G$. Also we define a bijectve function $\lambda : S_4 \times S_4 \to S_4 \times S_4$ by

$$\begin{aligned} &\lambda(1,1) = (1,1), \quad \lambda(1,2) = (1,3), \quad \lambda(1,3) = (3,1), \quad \lambda(1,4) = (4,2), \\ &\lambda(2,1) = (1,2), \quad \lambda(2,2) = (1,4), \quad \lambda(2,3) = (3,2), \quad \lambda(2,4) = (4,1), \\ &\lambda(3,1) = (3,3), \quad \lambda(3,2) = (4,4), \quad \lambda(3,3) = (2,2), \quad \lambda(3,4) = (2,4), \\ &\lambda(4,1) = (3,4), \quad \lambda(4,2) = (4,3), \quad \lambda(4,3) = (2,1), \quad \lambda(4,4) = (2,3). \end{aligned}$$

If $\lambda(i, j) = (a_{ij}, b_{ij})$ for any $i, j, a_{ij}, b_{ij} \in S_4$, then we have the equation

$$\varphi_i(\varphi_j(x, y), z) = \varphi_{a_{ij}}(x, \varphi_{b_{ij}}(y, z))$$

for all $x, y, z \in G$. Thus $\{\varphi_i | i \in S_4\}$ is a cochain in 2-variables.

Example 2.3. Consider the case $m = 3$ of the above definition. Let $\varphi_i : G^3 \longrightarrow G$ be a function for each $i \in S_8$ defined by

$$\begin{aligned} &\varphi_1(x,y,z) = x+y+z, \quad \varphi_2(x,y,z) = x+y-z, \\ &\varphi_3(x,y,z) = x-y+z, \quad \varphi_4(x,y,z) = -x+y+z, \\ &\varphi_5(x,y,z) = -x-y-z, \quad \varphi_6(x,y,z) = -x-y+z, \\ &\varphi_7(x,y,z) = -x+y-z, \quad \varphi_8(x,y,z) = x-y-z \end{aligned}$$

for all $x, y \in G$. Then we can easily find a bijectve function $\lambda : S_8 \times S_8 \to S_8 \times S_8$ such that if $\lambda(i,j) = (a_{ij}, b_{ij})$ for any $i, j, a_{ij}, b_{ij} \in S_8$ then

$$\varphi_i(\varphi_j(x,y,z), v, w) = \varphi_{a_{ij}}(x, y, \varphi_{b_{ij}}(z, v, w))$$

for all $x, y, z, v, w \in G$. Thus $\{\varphi_i | i \in S_8\}$ is a cochain in 3-variales.

Definition 2.4. Let $\{\varphi_i | \varphi_i : G^m \longrightarrow G$ is a function for each $i \in S_n\}$ be a cochain in m-variables. Then the function $F : G^m \longrightarrow X$ satisfied with the functional equation (1.5) is called a *cocycle via the cochain* $\{\varphi_i\}$ and the equation is called the a *cocycle equation via the cochain* $\{\varphi_i\}$.

Example 2.5. Consider the case $m = 2$ of the above definition. Let $\{\varphi_i | \varphi_i : G \times G \longrightarrow G$ is a function for each $i \in S_n\}$ be a cochain in 2-variables. Then any solution $F : G \times G \longrightarrow X$ of the functional equation

$$F(x,y) + \frac{1}{n}\sum_{i=1}^{n} F(\varphi_i(x,y), z) = \frac{1}{n}\sum_{i=1}^{n} F(x, \varphi_i(y,z)) + F(y,z), \ (x,y,z \in G)$$

is a *cocycle via the cochain* $\{\varphi_i\}$ and the equation is the a *cocycle equation via the cochain* $\{\varphi_i\}$.

Example 2.6. Consider the case $m = 3$ of the above definition. Let $\{\varphi_i | \varphi_i : G^3 \longrightarrow G$ is a function for each $i \in S_n\}$ be a cochain in 3-variables. Then any solution $F : G^3 \longrightarrow X$ of the functional equation

$$\begin{aligned} &F(x,y,z) + \frac{1}{n}\sum_{i=1}^{n} F(\varphi_i(x,y,z), v, w) \\ &\quad = \frac{1}{n}\sum_{i=1}^{n} F(x, y, \varphi_i(z,v,w)) + F(z,v,w), \ (x,y,z,v,w \in G) \end{aligned}$$

is a *cocycle via the cochain* $\{\varphi_i\}$ and the equation is the a *cocycle equation via the cochain* $\{\varphi_i\}$.

Theorem 2.7. *Let $\{\varphi_i\}$ be a cochain in m-variables where $\varphi_i : G^m \longrightarrow G$ is a function for each $i \in S_n$. If there exists a function $f : G \longrightarrow X$ such that f generate a function $F : G^m \longrightarrow X$ by*

$$F(x_1, \cdots, y_m) := f(x_1) + \cdots + f(x_m) - \frac{1}{n}\sum_{i=1}^{n} f(\varphi_i(x_1, \cdots, x_m)) \tag{2.2}$$

for all $x_1, \cdots, x_m \in G$, then F satisfies the cocycle equation via the cochain $\{\varphi_i\}$.

Proof. Suppose that F is generated by f. Since $\{\varphi_i | i \in S_n\}$ is a cochain, we have

$$\begin{aligned}
&\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} f(\varphi_j(\varphi_i(x_1, \cdots, x_m), y_1, \cdots, y_{m-1})\\
&\quad = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} f(\varphi_j(x_1, \cdots, x_{m-1}, \varphi_i(x_m, y_1, \cdots, y_{m-1}))
\end{aligned}$$

for all $x_1, \cdots x_m, y_1, \cdots, y_{m-1} \in G$. Then we have

$$\begin{aligned}
&F(x_1, \cdots, x_m) + \frac{1}{n}\sum_{i=1}^{n} F(\varphi_i(x_1, \cdots, x_m), y_1, \cdots, y_{m-1})\\
&= f(x_1) + \cdots + f(x_m) - \frac{1}{n}\sum_{i=1}^{n} f(\varphi_i(x_1, \cdots, x_m))\\
&\quad + \frac{1}{n}\sum_{i=1}^{n}\Big[f(\varphi_i(x_1, \cdots, x_m)) + f(y_1) + \cdots + f(y_{m-1})\\
&\quad - \frac{1}{n}\sum_{j=1}^{n} f(\varphi_j(\varphi_i(x_1, \cdots, x_m), y_1, \cdots, y_{m-1}))\Big]\\
&= f(x_1) + \cdots + f(x_m) + f(y_1) + \cdots + f(y_{m-1})\\
&\quad - \frac{1}{n^2}\sum_{j=1}^{n}\sum_{i=1}^{n} f(\varphi_j(\varphi_i(x_1, \cdots, x_m), y_1, \cdots, y_{m-1})\\
&= \frac{1}{n}\sum_{i=1}^{n} F(x_1, \cdots, x_{m-1}, \varphi_i(x_m, y_1, \cdots, y_{m-1})) + F(x_m, y_1, \cdots, y_{m-1}).
\end{aligned}$$

for all $x_1, \cdots x_m, y_1, \cdots, y_{m-1} \in G$. Thus F satisfies the cocycle equation via the cochain. □

Corollary 2.8. *Let $\{\varphi_i\}$ be a cochain in 2-variables where $\varphi_i : G \times G \longrightarrow G$ is a function for each $i \in S_n$. If there exists a function $f : G \longrightarrow X$ such that*

f generates a function $F : G \times G \longrightarrow X$ by

$$F(x,y) := f(x) + f(y) - \frac{1}{n}\sum_{i=1}^{n} f(\varphi_i(x,y)) \tag{2.3}$$

for all $x, y \in G$, then F satisfies the cocycle equation via the cochain $\{\varphi_i\}$.

Proof. Suppose that F is generated by f. Since $\{\varphi_i | i \in S_n\}$ is a cochain, we have

$$\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} f(\varphi_j(\varphi_i(x,y),z)) = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} f(\varphi_j(x,\varphi_i(y,z))$$

for all $x, y, z \in G$. Then we have

$$\begin{aligned} &F(x,y) + \frac{1}{n}\sum_{i=1}^{n} F(\varphi_i(x,y),z)) \\ &\quad = f(x) + f(y) - \frac{1}{n}\sum_{i=1}^{n} f(\varphi_i(x,y)) \\ &\qquad + \frac{1}{n}\sum_{i=1}^{n}\Big[f(\varphi_i(x,y)) + f(z) - \frac{1}{n}\sum_{j=1}^{n} f(\varphi_j(\varphi_i(x,y),z))\Big] \\ &\quad = f(x) + f(y) + f(z) - \frac{1}{n^2}\sum_{j=1}^{n}\sum_{j=1}^{n} f(\varphi_j(\varphi_i(x,y),z)) \\ &\quad = F(y,z) + \frac{1}{n}\sum_{i=1}^{n} F(x,\varphi_i(y,z)) \end{aligned}$$

for all $x, y, z \in G$. Thus F satisfies the cocycle equation via the cochain. □

Theorem 2.9. *Let $\{\varphi_i\}$ be a cochain, where $\varphi_i : G \times G \longrightarrow G$ is a function for each $i \in S_n$. Also let $\varepsilon : G \times G \to X$ be a function for which that there exists a sequence $(s_k)_{k\in\mathbb{N}}$ of elements of G satisfying the following condition:*

$$\lim_{k\to\infty} \varepsilon(\varphi_i(s_k,x),y) = 0, \qquad (x,y \in G, i \in S_n). \tag{2.4}$$

Assume that a function $f : G \to X$ satisfies the inequality

$$\Big|\Big|f(x) + f(y) - \frac{1}{n}\sum_{i=1}^{n} f(\varphi_i(x,y))\Big|\Big| \le \varepsilon(x,y) \tag{2.5}$$

for all $x, y \in G$. Then f is a solution of the equation (2.3).

Proof. For any $x, y \in G$, let

$$F(x,y) := f(x) + f(y) - \frac{1}{n}\sum_{i=1}^{n} f(\varphi_i(x,y)).$$

Note that $||F(x,y)|| \le \varepsilon(x,y)$ for all $x, y \in G$. By Theorem 2.7, we have

$$F(x,y) + \frac{1}{n}\sum_{i=1}^{n} F(\varphi_i(x,y),z) = F(y,z) + \frac{1}{n}\sum_{i=1}^{n} F(x,\varphi_i(y,z))$$

for any $x, y, z \in G$, and thus

$$\Big|\Big|F(y,z)\Big|\Big| \le \Big|\Big|F(x,y)\Big|\Big| + \frac{1}{n}\sum_{i=1}^{n}\Big|\Big|F(\varphi_i(x,y),z))\Big|\Big| + \frac{1}{n}\sum_{i=1}^{n}\Big|\Big|F(x,\varphi_i(y,z))\Big|\Big|.$$

Since $\{\varphi_i\}$ is a cochain, there is a bijective function λ on $S_n \times S_n$ such that $\lambda(i,j) = (a_{ij}, b_{ij})$ and

$$\varphi_i(\varphi_j(s_k,x),y) = \varphi_{a_{ij}}(s_k, \varphi_{b_{ij}}(x,y))$$

for each $i, j, a_{ij}, b_{ij} \in S_n$. Letting x by $\varphi_j(s_k, x)$ for some $j \in S_n$,

$$\begin{aligned}\Big|\Big|F(y,z)\Big|\Big| &\le \Big|\Big|F(\varphi_j(s_k,x),y)\Big|\Big| + \frac{1}{n}\sum_{i=1}^{n}\Big|\Big|F(\varphi_i(\varphi_j(s_k,x),y),z))\Big|\Big| \\ &\quad + \frac{1}{n}\sum_{i=1}^{n}\Big|\Big|F(\varphi_j(s_k,x),\varphi_i(y,z))\Big|\Big| \\ &\le \varepsilon(\varphi_j(s_k,x),y) + \frac{1}{n}\sum_{i=1}^{n}\varepsilon(\varphi_{a_{ij}}(s_k, b_{ij}(x,y)),z) \\ &\quad + \frac{1}{n}\sum_{i=1}^{n}\varepsilon(\varphi_j(s_k,x),\varphi_i(y,z)) \\ &= 0\end{aligned}$$

as $k \to \infty$. □

Corollary 2.10. *Let $\varphi_i : G \times G \longrightarrow G$ be a function for each $i \in S_4$ defined by*

$$\varphi_1(x,y) = x + y, \quad \varphi_2(x,y) = x - y,$$
$$\varphi_3(x,y) = -x + y, \quad \varphi_4(x,y) = -x - y$$

for all $x, y \in G$. Also let $\varepsilon : G \times G \to \mathbb{R}$ be a a function for which that there exists a sequence $(s_k)_{k\in\mathbb{N}}$ of elements of G satisfying the following condition:

$$\lim_{k\to\infty} \varepsilon(\varphi_i(s_k,x),y) = 0, \quad (x, y \in G, i \in S_4).$$

Assume that a function $f : G \to X$ satisfies the stability inequality

$$\left|\left|f(x) + f(y) - \frac{1}{4}\sum_{i=1}^{4} f(\varphi_i(x,y))\right|\right| \leq \varepsilon(x,y)$$

for all $x, y \in G$. Then

$$f(x) + f(y) = \frac{1}{4}\sum_{i=1}^{4} f(\varphi_i(x,y))$$

for all $x, y \in G$, and in the case of even function f, f is a solution of the equation (1.2).

Proof. Define a function $F : G \times G \to X$ by

$$F(x,y) = f(x) + f(y) - \frac{1}{4}\sum_{i=1}^{4} f(\varphi_i(x,y)) \quad (x, y, z \in G).$$

Also we define a bijective function $\lambda : S_4 \times S_4 \to S_4 \times S_4$ by

$$\begin{array}{llll}
\lambda(1,1) = (1,1), & \lambda(1,2) = (1,3), & \lambda(1,3) = (3,1), & \lambda(1,4) = (4,2), \\
\lambda(2,1) = (1,2), & \lambda(2,2) = (1,4), & \lambda(2,3) = (3,2), & \lambda(2,4) = (4,1), \\
\lambda(3,1) = (3,3), & \lambda(3,2) = (4,4), & \lambda(3,3) = (2,2), & \lambda(3,4) = (2,4), \\
\lambda(4,1) = (3,4), & \lambda(4,2) = (4,3), & \lambda(4,3) = (2,1), & \lambda(4,4) = (2,3).
\end{array}$$

Then, for the case of $\lambda(i,j) = (k,l)$

$$\varphi_i(\varphi_j(x,y), z) = \varphi_k(x, \varphi_l(y,z))$$

for all $x, y, z \in G$. Thus $\{\varphi_i | i \in S_4\}$ is a cochain. By Theorem 2.9, the result holds. □

Lemma 2.11. *Let $(S, \cdot)$ be a semigroup and for each $i \in S_n$ let $\psi_i : S \longrightarrow S$ be pairwise distinct automorphisms of S such that the set $\{\psi_i | i \in S_n\}$ is a group with respect to the composition as a group operation. If $\varphi_i : S \times S \longrightarrow S$ is a function defined by $\varphi_i(x,y) = x\psi_i(y)$ for all $x, y \in S$ and for each $i \in S_n$, then $\{\varphi_i | i \in S_n\}$ is a cochain.*

Proof. Note that for any $x, y, z \in S$ and $i, j, k \in S_n$ we have

$$\varphi_i(\varphi_j(x,y), z) = x\psi_j(y)\psi_i(z),$$
$$\varphi_j(x, \varphi_k(y,z)) = x\psi_j(y)\psi_j\psi_k(z).$$

Since ψ_j is an automorphism for each $j \in S_n$ and $\{\psi_j\psi_i | i \in S_n\}$ is a permutation of $\{\psi_i | i \in S_n\}$, there is a unique k such that $\psi_j\psi_k = \psi_i$. Thus we can define a bijective function λ by $\lambda(i,j) = (j,k)$ for $i,j,k \in S_n$ such that for any $x,y,z \in S$

$$\varphi_i(\varphi_j(x,y),z) = \varphi_j(x,\varphi_k(y,z)).$$

$\square$

Theorem 2.12. *Let $(S,\cdot)$ be a semigroup and for each $i \in S_n$, let $\psi_i : S \longrightarrow S$ be pairwise distinct automorphosms of S such that the set $\{\psi_i | i \in S_n\}$ is a group with respect to the composition as a group operation. Also let $\varepsilon : S \times S \to \mathbb{R}$ be a a function for which that there exists a sequence $(s_k)_{k\in\mathbb{N}}$ of elements of G satisfying the following condition:*

$$\lim_{k\to\infty} \varepsilon(s_k x, y) = 0, \quad (x,y \in S).$$

Assume that a function $f : S \to X$ satisfies the inequality

$$\left|\left| f(x) + f(y) - \frac{1}{n}\sum_{i=1}^{n} f(x\psi_i(y)) \right|\right| \le \varepsilon(x,y)$$

for all $x,y \in S$. Then f is a solution of the equation (1.1).

Proof. Let $\varphi_i : S \times S \longrightarrow S$ be a function defined by $\varphi_i(x,y) = x\psi_i(y)$ for all $x,y \in S$ and for each $i \in S_n$. By Lemma 2.11, $\{\varphi_i | i \in S_n\}$ is a cochain. By Theorem 2.9, the result holds. $\square$

Remark 2.13. In Theorem 2.9, the inequality condition (2.2) can be replace by

$$\lim_{k\to\infty} \varepsilon(x,\varphi_i(s_k,y)) = 0 \quad \text{or} \quad \varepsilon(\varphi_i(s,x),y) \le q\varepsilon(x,y) \quad (0 \le q < 1, i \in S_n).$$

Let us go through the same procedure as in Theorem 2.9. If we define F by (2.1) then $\lim_{k\to\infty} F(x,\varphi_i(s_k,y)) = 0$ and eventually $F(x,y) = 0$ for all $x,y \in G$. If $\varepsilon(\varphi_i(s,x),y) \le q\varepsilon(x,y) \quad (0 \le q < 1, i \in S_n)$, then

$$\varepsilon(\varphi_i(\varphi_j(s,s),x),y) = \varepsilon(\varphi_{a_{ij}}(s,\varphi_{b_{ij}}(s,x)),y) \le q\varepsilon(\varphi_{b_{ij}}(s,x),y) \le q^2\varepsilon(x,y)$$

for all $i,j,a_{ij},b_{ij} \in S_n$. Letting

$$\varphi_j(s,s) = s_1, \quad \varphi_j(\varphi_j(s,s),s) = s_2,$$
$$\underbrace{\varphi_j(\varphi_j(\cdots\varphi_j}_{k}\underbrace{(s,s),\cdots),s)}_{k+1} = s_k,$$

we have

$$\lim_{k\to\infty} \varepsilon(\varphi_i(s_k,x),y) = 0, \quad (x,y \in G, i \in S_n).$$

Therefore (2.2) is satisfied and the statement follows from Theorem 2.9. Now we extend to n-variables by using a permutation. For all $P = (p_1, p_2, \ldots, p_n)$, $Q = (q_1, q_2, \ldots, q_n) \in G^n$, let $\sigma_i : G^n \to G^n$ be a permutation given by

$$\begin{aligned}
&\sigma_1(p_1, p_2, \ldots, p_n) = (p_1, p_2, \ldots, p_n),\\
&\sigma_i(p_1, p_2, \ldots, p_n) = (p_i, \ldots p_n, p_1, p_2, \ldots, p_{i-1}),\\
&\sigma_{n+1}(P) = \sigma_1 P,\\
&\sigma_{n+i}(P) = \sigma_i P
\end{aligned}$$

for each $i \in S_n$ and $P + Q = (p_1 + q_1, p_2 + q_2, \ldots, p_n + q_n)$.

Lemma 2.14. *Let $i \in S_n$ and $\varphi_i : G^n \times G^n \to G^n$ be functions defined by*

$$\begin{aligned}
\varphi_i(P, Q) &= P + \sigma_i(Q)\\
&= (p_1 + q_i, p_2 + q_{i+1}, \cdots, p_{n-i+1} + q_n, \cdots, p_n + q_{i-1})
\end{aligned}$$

for all $P, Q \in G^n$. Then $\{\varphi_i | i \in S_n\}$ is a cochain.

Proof. It can be easily checked that for all $P, Q, W \in G^n, i, j \in S_n$,

(a) $\varphi_i(\varphi_j(P, Q), W) = P + \sigma_j(Q) + \sigma_i(W)$,

(b) $\varphi_j(P, \varphi_i(Q, W)) = P + \sigma_j(Q) + \sigma_j\sigma_i(W) = P + \sigma_j(Q) + \sigma_{i+j-1}(W)$.

Define a bijective function $\lambda : S_n \times S_n \longrightarrow S_n \times S_n$ such that if $\lambda(i, j) = (a_{ij}, b_{ij})$ for any $i, j, a_{ij}, b_{ij} \in S_n$, where

$$\begin{aligned}
&a_{ij} = j - i + 1 \quad (\text{if } j - i + 1 \text{ is negative, } a_{ij} = n + j - i + 1),\\
&b_{ij} = i.
\end{aligned}$$

Then

$$\varphi_i(\varphi_j(P, Q), W) = \varphi_{a_{ij}}(P, \varphi_{b_{ij}}(Q, W))$$

for all $P, Q, W \in G^n$. □

Lemma 2.15. *Let $f : G^n \to X$ be an arbitrary function and $\varphi_i : G \times G \to G$ a function defined by Lemma 2.14 for each $i \in S_n$. Then the function $F : G^n \to X$ defined by*

$$F(P, Q) = f(P) + f(Q) - \frac{1}{n}\sum_{i=1}^{n} f(\varphi_i(P, Q))), \quad (P, Q \in G^n)$$

satisfies the following functional equation

$$F(P, Q) + \frac{1}{n}\sum_{i=1}^{n} F(\varphi_i(P, Q), W) = F(Q, W) + \frac{1}{n}\sum_{i=1}^{n} F(P, \varphi_i(Q, W))$$

for all $P, Q, W \in G^n$.

Proof. By Lemma 2.14, $\{\varphi_i | i \in S_n\}$ is a cochain. By the same procedure of Theorem 2.9, we complete the proof. □

By Lemma 2.14 and 2.15, we have the following theorem.

Theorem 2.16. *Let $\varphi_i : G \times G \to G$ be a function defined by Lemma 2.14 for each $i \in S_n$ and $\varepsilon : G^n \times G^n \to \mathbb{R}$ be function for which there exists a sequence $\{W_k\}_{k\in\mathbb{N}}$ of elements of G^n satisfying the following condition:*

$$\lim_{k\to\infty} \varepsilon(\varphi_i(W_k, P), Q) = 0, \qquad (P, Q \in G^n, i \in S_n).$$

Assume also that a function $f : G^n \to X$ satisfies the inequality

$$\left|\left| f(P) + f(Q) - \frac{1}{n}\sum_{i=1}^{n} f(\varphi_i(W_k, P), Q) \right|\right| \le \varepsilon(P, Q)$$

for all $P, Q \in G^n$. Then f is a solution of the functional equation (1.5). That is, for all $P, Q \in G^n$

$$f(P) + f(Q) = \frac{1}{n}\sum_{i=1}^{n} f(\varphi_i(W_k, P), Q).$$

Example 2.17. Assume that a function $f : \mathbb{R}^3_+ \to X$ satisfies the inequality

$$\begin{aligned} &\Big|\Big| f(p_1+q_1, p_2+q_2, p_3+q_3) + f(p_1+q_2, p_2+q_3, p_3+q_1) \\ &\qquad + f(p_1+q_3, p_2+q_1, p_3+q_2) - 3f(p_1,p_2,p_3) - 3f(q_1,q_2,q_3)\Big|\Big| \\ &\quad \le \frac{g_1q_2q_3}{p_1p_2p_3} \end{aligned}$$

for all $P = (p_1, p_2, p_3), Q = (q_1, q_2, q_3) \in \mathbb{R}^3_+$ and let $\varepsilon(P, Q) = \frac{g_1q_2q_3}{p_1p_2p_3}$ and $W_k = (k, k, k)$ for any $k \in \mathbb{N}$. Then by Theorem 2.16, we have

$$\begin{aligned} &f(p_1+q_1, p_2+q_2, p_3+q_3) + f(p_1+q_2, p_2+q_3, p_3+q_1) \\ &\qquad + f(p_1+q_3, p_2+q_1, p_3+q_2) \\ &\quad = 3f(p_1,p_2,p_3) + 3f(q_1,q_2,q_3) \end{aligned}$$

for all $P = (p_1, p_2, p_3), Q = (q_1, q_2, q_3) \in \mathbb{R}^3_+$.

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