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ON THE CHARACTERIZATION OF F_0-SPACES

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Abstract. Let X be a simply connected rationally elliptic space such that $H^2(X;\mathbb{Q}) \neq 0$. In this paper, we show that if $H^{2n}(X^{[2n-2]};\mathbb{Q}) = 0$ or if $\pi_{2n}(X^{2n}) \otimes \mathbb{Q} = 0$ for all n, then X is an F_0-space.

1. Introduction

In this paper, by a space we mean a simply connected CW-complex X of finite type, i.e., $\dim H^n(X;\mathbb{Q}) < \infty$ for all n. A space X is called rationally elliptic if both the graded vector spaces $H^*(X;\mathbb{Q})$ and $\pi_*(X) \otimes \mathbb{Q}$ are finite dimensional. Furthermore, if $H^{\text{odd}}(X;\mathbb{Q}) = 0$, then X is called an F_0-space. For instance, products of even spheres, complex Grassmannian manifolds and homogeneous spaces G/H such that rank G = rank H are F_0-spaces.

For any positive integer n, let $X^{[n]}$ denote the n-th Postnikov section of X and X^n its n-skeleton. The aim of this paper is to characterize an F_0-spaces X in terms of the homotopy groups of the spaces X^n and the rational cohomology of the spaces $X^{[n]}$. In other words, by exploiting the well-known properties of a rationally elliptic space X as well as the virtue of the Whitehead exact sequences associated, respectively, with the Sullivan model and the Quillen model of X, we prove the following.

Theorem 1. *Let X be a rationally elliptic space such that $H^2(X;\mathbb{Q}) \neq 0$. We have the following.*

(1) *If $\pi_{2n}(X^{2n}) = 0$ for all n, then X is an F_0-space.*
(2) *If $H^{2n}(X^{[2n-2]};\mathbb{Q}) = 0$ for all n, then X is an F_0-space.*

We show our results using standard tools of rational homotopy theory by working algebraically on the models of Quillen and Sullivan of X. We refer to [8] for a general introduction to these techniques. Recall that every simply connected space of finite type has a corresponding differential commutative cochain algebra called the Sullivan model of X, unique up to isomorphism, that encodes the rational homotopy of X. Dually, every simply connected space X

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has a differential graded Lie algebra (DGL for short), called the Quillen model of X and unique up to isomorphism, which determines completely the rational homotopy type of X.

2. Whitehead exact sequences in rational homotopy theory

2.1. Whitehead exact sequence associated with a DGL

Let $W = (W_{\geq 1})$ be a finite dimensional graded vector space over $\mathbb{Q}$ and let $(\mathbb{L}(W), \delta)$ be a DGL. For any positive integer n, we define the linear maps $j_n : H_n(\mathbb{L}(W_{\leq n})) \to W_n$ and $b_n : W_n \to H_{n-1}(\mathbb{L}(W_{\leq n-1}))$ by setting

$$j_n([w + y]) = w,$$

and

$$b_n(w) = [\delta(w)], \tag{1}$$

respectively, where $[\delta(w)]$ denotes the homology class of $\delta(w)$ in the sub-Lie algebra $\mathbb{L}_{n-1}(W_{\leq n-1})$. Recall that if $x \in H_n(\mathbb{L}(W_{\leq n}))$, then $x = [w + y]$, where $w \in W_n$, $y \in \mathbb{L}_n(W_{\leq n-1})$ and $\delta(w + y) = 0$.

To every DGL $(\mathbb{L}(W), \delta)$ we can assign (see [3, 5, 7] for more details) the following long exact sequence

$$\cdots \to W_{n+1} \xrightarrow{b_{n+1}} \Gamma_n \to H_n(\mathbb{L}(W)) \xrightarrow{h_n} W_n \xrightarrow{b_n} \cdots \tag{2}$$

called the Whitehead exact sequence of $(\mathbb{L}(W), \delta)$, where

$$\Gamma_n = \ker(j_n : H_n(\mathbb{L}(W_{\leq n})) \to W_n) \tag{3}$$

for all n.

Remark 2.1. It worth mentioning that Γ_n is the sub vector space of $H_n(\mathbb{L}(W_{\leq n}))$ formed by the homology classes $\{z\}$ with $z \in \mathbb{L}_n(W_{\leq n-1})$, i.e., z is a decomposable element.

Remark 2.2. If X is a given space and $(\mathbb{L}(W), \delta)$ is its Quillen model, then the virtue of properties of this model implies the following identifications valid for all n:

$$\begin{aligned} \pi_n(X) \otimes \mathbb{Q} &\cong H_{n-1}(\mathbb{L}(W)), \\ H_n(X; \mathbb{Q}) &\cong W_{n-1}, \\ \Gamma_n(X) &\cong \Gamma_{n-1}, \end{aligned} \tag{4}$$

where $\Gamma_n(X) = \ker(\pi_n(X^n) \otimes \mathbb{Q} \longrightarrow \pi_n(X^n; X^{n-1}) \otimes \mathbb{Q})$.

Moreover, the linear map h_n can be identified with the dual of the rational Hurewicz homomorphism $h^X_{n+1} : \pi_{n+1}(X) \otimes \mathbb{Q} \to H_{n+1}(X, \mathbb{Q})$ for every n.

2.2. Whitehead exact sequence of Sullivan algebra

Likewise, let $(\Lambda V, \partial)$ be a Sullivan algebra. In [1, 2, 4, 6], it is shown that with $(\Lambda V, \partial)$, we can associate the following long exact sequence

$$\cdots \to V^n \xrightarrow{b^n} H^{n+1}(\Lambda V^{\leq n-1}) \xrightarrow{i^{n+1}} H^{n+1}(\Lambda V) \xrightarrow{h^{n+1}} V^{n+1} \xrightarrow{b^{n+1}} \cdots \tag{5}$$

called the Whitehead exact sequence of $(\Lambda V, \partial)$. Recall that the linear map $b^n : V^n \to H^{n+1}(\Lambda(V^{\leq n}))$ is defined by setting

$$b^n(v) = [\partial(v)]. \tag{6}$$

Here $[\partial(v)]$ denotes the cohomology class of $\partial(v) \in (\Lambda V^{\leq n})^{n+1}$.

Remark 2.3. If X is a given space and $(\Lambda V, \partial)$ is its Sullivan minimal model, then by virtue of the properties of this model we obtain the following identifications valid for every n:

$$\begin{aligned} H^n(X; \mathbb{Q}) &\cong H^n(\Lambda V), \\ \operatorname{Hom}(\pi_n(X), \mathbb{Q}) &\cong V^n, \\ H^{n+1}(X^{[n-1]}; \mathbb{Q}) &\cong H^n(\Lambda V). \end{aligned} \tag{7}$$

Moreover, the linear map h^n can be identified with the dual of the rational Hurewicz homomorphism $h_n^X : \pi_n(X) \otimes \mathbb{Q} \to H_n(X, \mathbb{Q})$ for every n. Hence, from the exactness of the sequence (5) we deduce that

$$\ker b^{\text{odd}} = 0 \Longleftrightarrow \ker h^{\text{odd}} = 0 \Longleftrightarrow h_*^X = 0. \tag{8}$$

Theorem 2.4. *If* $(\Lambda V, \partial)$ *is the Sullivan model of a given space* X *and* $(\mathbb{L}(W), \delta)$ *is its Quillen model, then for all* n *we have the following*:

$$\Gamma_n = H^{n+2}(\Lambda V^{\leq n-2}), \tag{9}$$

where Γ_n *is defined as in* (3).

Proof. Applying the exact functor $\operatorname{Hom}(\cdot, \mathbb{Q})$ to the exact sequence (2) we obtain

$$\begin{aligned} \cdots \leftarrow \operatorname{Hom}(W_{n+1}, \mathbb{Q}) \leftarrow \operatorname{Hom}(\Gamma_n, \mathbb{Q}) \leftarrow \operatorname{Hom}(H_n(\mathbb{L}(W)), \mathbb{Q}) \\ \leftarrow \operatorname{Hom}(W_n, \mathbb{Q}) \xleftarrow{b_n} \cdots . \end{aligned} \tag{10}$$

Taking into account that by virtues of the Quillen and Sullivan models we have the following.

- $\operatorname{Hom}(W_n, \mathbb{Q}) \cong H^{n+1}(\Lambda V)$ for all n.
- $\operatorname{Hom}(H_n(\mathbb{L}(W)), \mathbb{Q}) \cong V^{n+1}$ for all n.
- All groups involved are vector spaces of finite dimensions.
- The two maps $H^{n+1}(\Lambda V) \to V^{n+1}$ and $\operatorname{Hom}(W_n, \mathbb{Q}) \to \operatorname{Hom}(H_n(\mathbb{L}(W)), \mathbb{Q})$ appearing in (5) and (10) are the same morphism.

By comparing the sequences (5) and (10), we get (9). □

Corollary 2.5. *If X is a given space, then*

$$\Gamma_{n+1}(X) \cong H^{n+2}(X^{[n]};\mathbb{Q}) \tag{11}$$

as vector spaces for all n.

Proof. It suffices to apply the identifications (4), (7) and Theorem 2.4 to the Sullivan model and the Quillen model of the space X. □

3. The main result

As it is stated in the introduction, a rationally elliptic space X is called an F_0-space, if $H^{\mathrm{odd}}(X;\mathbb{Q}) = 0$ or equivalently if $H^{\mathrm{odd}}(\Lambda V) = 0$, where $(\Lambda V, \partial)$ is the Sullivan model of X (see [8], §32 for more details).

Let us consider the Whitehead exact sequence of $(\Lambda V, \partial)$ given in (5)

$$\cdots \to V^n \xrightarrow{b^n} H^{n+1}(\Lambda V^{\leq n-1}) \to H^{n+1}(\Lambda V) \to V^{n+1} \xrightarrow{b^{n+1}} \cdots .$$

In order to simplify let us put

$$L^n = H^n(\Lambda V^{\leq n-2}) \tag{12}$$

for all n.

Proposition 3.1. *Let $(\Lambda V, \partial)$ be an elliptic Sullivan algebra such that $V^2 \neq 0$. If b^{odd} is not injective, then $L^{odd} \neq 0$.*

Proof. Assume that b^{2n-1} is not injective for some integer $n \geq 1$. Then there exists $v \in V^{2n-1}$ such that $b^{2n-1}(v) = 0$ and from the relation (6) there exists $q \in \Lambda V$ such that $\partial(v+q) = 0$. Therefore, if $v_2 \in V^2$, then $v_2(v+q)$ is decomposable cocycle of degree $2n+1$ providing a non-zero element in L^{2n+1} according to (12). □

Corollary 3.2. *Let X be a rationally elliptic space such that $H^2(X;\mathbb{Q}) \neq 0$. If $H^{2n+1}(X^{[2n-1]};\mathbb{Q}) = 0$ for all n, then the rational Hurewicz homomorphism h^X_{odd} is trivial.*

Proof. Let $(\Lambda V, \partial)$ be the Sullivan model of X. Note that because $H^2(X;\mathbb{Q}) \neq 0$ it follows that $V^2 \neq 0$. Using the identifications (7), the relation (12) and the hypothesis we get

$$L^{2n+1} = H^{2n+1}(X^{[2n-1]};\mathbb{Q}) = 0 \tag{13}$$

for all n. Next, the relation (8) and Proposition 3.1 imply that h^X_{odd} is trivial. □

Proposition 3.3. *Let $(\Lambda V, \partial)$ be an elliptic Sullivan algebra such that $V^2 \neq 0$. If $L^{\mathrm{odd}} = 0$, then $H^{\mathrm{odd}}(\Lambda V) = 0$.*

Proof. Proposition 3.1 implies that b^{odd} is injective. Let us consider the Whitehead exact sequence of $(\Lambda V, \partial)$ given in (5)

$$\cdots \to V^{2n} \xrightarrow{b^{2n}} L^{2n+1} \to H^{2n+1}(\Lambda V) \to V^{2n+1} \xrightarrow{b^{2n+1}} \cdots .$$

Since b^{2n+1} is injective and $L^{2n+1}=0$, it follows that $H^{2n+1}(\Lambda V)=0$ for all n. □

Theorem 3.4. *Let X be a rationally elliptic space such that $H^2(X;\mathbb{Q})\neq 0$. If $H^{2n+1}(X^{[2n-1]};\mathbb{Q})=0$ for all n, then X is an F_0-space.*

Proof. Let $(\Lambda V,\partial)$ be the Sullivan model of X. Note that because $H^2(X;\mathbb{Q})\neq 0$ it follows that $V^2\neq 0$. Now, if $H^{2n+1}(X^{[2n-1]};\mathbb{Q})=0$ for all n, then from (13) we deduce that $L^{2n+1}=0$ for all n. Applying Proposition 3.3 we derive that $H^{\text{odd}}(\Lambda V)=0$ ensuring that X is an F_0-space. □

Theorem 3.5. *Let X be a rationally elliptic space such that $H^2(X;\mathbb{Q})\neq 0$. If $\Gamma_{2n}(X)=0$ for all n, then X is an F_0-space.*

Proof. First, if $\Gamma_{2n}(X)=0$, then apply Corollary 2.5 to get

$$\Gamma_{2n}(X)\cong H^{2n+1}(X^{[2n-1]};\mathbb{Q})=0$$

for all n. Next, Theorem 3.4 ensures that X is an F_0-space. □

It worth noting that $\Gamma_{2n}(X)\subset \pi_{2n}(X^{2n})\otimes\mathbb{Q}$ for all n (see (4)). Therefore, Theorem 3.5 implies the following result.

Corollary 3.6. *Let X be a rationally elliptic space such that $H^2(X;\mathbb{Q})\neq 0$. If $\pi_{2n}(X^{2n})\otimes\mathbb{Q}=0$ for all n, then X is an F_0-space.*

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References

[1] M. Benkhalifa, *Postnikov decomposition and the group of self-equivalences of a rationalized space*, Homology Homotopy Appl. **19** (2017), no. 1, 209–224. `https://doi.org/10.4310/HHA.2017.v19.n1.a11`

[2] M. Benkhalifa, *On the group of self-homotopy equivalences of an elliptic space*, Proc. Amer. Math. Soc. **148** (2020), no. 6, 2695–2706. `https://doi.org/10.1090/proc/14900`

[3] M. Benkhalifa, *On the Euler-Poincaré characteristics of a simply connected rationally elliptic CW-complex*, J. Homotopy Relat. Struct. **17** (2022), no. 2, 163–174. `https://doi.org/10.1007/s40062-022-00301-2`

[4] M. Benkhalifa, *On the group of self-homotopy equivalences of an almost formal space*, Quaestiones Mathematicae (2022). `https://doi.org/10.2989/16073606.2022.2044405`

[5] M. Benkhalifa, *The effect of cell-attachment on the group of self-equivalences of an elliptic space*, Michigan Math. J. **71** (2022), no. 2, 611–617.

[6] M. Benkhalifa, *The group of self-homotopy equivalences of a rational space cannot be a free abelian group*, J. Math. Soc. Japan. `https://doi.org/10.2969/jmsj/87158715`

[7] M. Benkhalifa and S. B. Smith, *The effect of cell-attachment on the group of self-equivalences of an R-localized space*, J. Homotopy Relat. Struct. **10** (2015), no. 3, 549–564. `https://doi.org/10.1007/s40062-014-0076-5`

[8] Y. Félix, S. Halperin, and J.-C. Thomas, *Rational homotopy theory*, Graduate Texts in Mathematics, 205, Springer-Verlag, New York, 2001. `https://doi.org/10.1007/978-1-4613-0105-9`

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