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C_{12}-SPACE FORMS

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Abstract. The aim of this paper is two-fold. First, we study the Chinea-Gonzalez class C_{12} of almost contact metric manifolds and we discuss some fundamental properties. We show there is a one-to-one correspondence between C_{12} and Kählerian structures.

Secondly, we give some basic results for Riemannian curvature tensor of C_{12}-manifolds and then establish equivalent relations among φ-sectional curvature. Concrete examples are given.

1. Introduction

The warped product provides a way to construct new pseudo-Riemannian manifolds from the given ones. In 1960's and 1970's, the notion of an almost contact structure has been initiated by Boothby and Wang [4], these manifolds were studied as an odd dimensional counterpart of almost complex manifolds, the warped product was used to make examples of almost contact manifolds. There are different classifications of almost contact structures which one of the most significant classes is trans-sasakian manifolds. The trans-sasakian manifolds are divided into three groups Sasakian, Kenmotsu and cosymplectic manifolds. In this classification, Kenmotsu manifolds are generated locally by warped product of a Kähler manifold and an interval of the real line $\mathbb{R}$. Also, In 1985, using the warped product, Oubiña showed that there is a one-to-one correspondence between Sasakian and Kählerian structures [12].

On the other hand, in the classification of Chinea and Gonzalez [8] of almost contact metric manifolds there is a class C_{12}-manifolds which can be integrable but never normal. Recently, in [5], the authors have study some properties of three dimensional C_{12}-manifolds and construct some relations between class C_{12} and other classes as C_6 and $C_2 \oplus C_9$ or $|C|$.

Here, by generalizing our work on 3-dimensional C_{12}-manifolds [5], we show there is a one-to-one correspondence between C_{12} and Kählerian structures and we introduce a new concept, namely, generalized C_{12}-manifold. Finally, we give a study on the φ-holomorphic sectional curvature of C_{12}-manifolds.

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First of all, we will start by introducing the basic concepts that we need in this research.

1.1. Kählerian manifolds

An almost complex manifold with a Hermitian metric is called an almost Hermitian manifold. For an almost Hermitian manifold (N, J, h) we thus have

$$J^2 = -1, \quad h(JX, JY) = h(X, Y) \tag{1.1}$$

for all X and Y vector fields on N. An almost complex structure J is integrable, and hence the manifold is a complex manifold, if and only if its Nijenhuis tensor N_J vanishes, with

$$N_J(X, Y) = J^2[X, Y] + [JX, JY] - J[X, JY] - J[JX, Y]. \tag{1.2}$$

For an almost Hermitian manifold (N, J, h), we define the fundamental Kähler form Ω as:

$$\Omega(X, Y) = h(X, JY). \tag{1.3}$$

(N, J, h) is then called almost Kähler if Ω is closed, i.e., $\mathrm{d}\Omega = 0$, where d denotes the exterior derivative. It can be shown that this condition for (N, J, h) to be almost Kähler is equivalent to

$$h((\nabla_X J)Y, Z) + h((\nabla_Y J)Z, X) + h((\nabla_Z J)X, Y) = 0.$$

An almost Kähler manifold with integrable J is called a Kähler manifold, and thus is characterized by the conditions: $\mathrm{d}\Omega = 0$ and $N_J = 0$. One can prove that these both conditions combined are equivalent with the single condition

$$\nabla J = 0. \tag{1.4}$$

Definition 1.1 ([11])**.** A Hermitian manifold (M, J, g) is called a locally conformal Kähler (conformally Kähler) manifold if there exists a closed (exact) one-form θ (called the Lee form) such that:

$$\mathrm{d}\Omega = \theta \wedge \Omega.$$

For more background on almost complex structure manifolds, we recommend the reference [13].

1.2. Almost contact metric manifolds

An odd-dimensional Riemannian manifold (M^{2n+1}, g) is said to be an almost contact metric manifold if there exist on M a $(1,1)$-tensor field φ, a vector field ξ (called the structure vector field) and a 1-form η such that

$$\begin{cases} \eta(\xi) = 1, \\ \varphi^2(X) = -X + \eta(X)\xi, \\ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \end{cases} \tag{1.5}$$

for any vector fields X, Y on M.

In particular, in an almost contact metric manifold we also have

$$\varphi\xi = 0 \quad \text{and} \quad \eta \circ \varphi = 0.$$

The fundamental 2-form ϕ is defined by

$$\phi(X,Y) = g(X, \varphi Y).$$

It is known that the almost contact structure (φ, ξ, η) is said to be normal if and only if

$$N^{(1)}(X,Y) = N_\varphi(X,Y) + 2d\eta(X,Y)\xi = 0 \tag{1.6}$$

for any X, Y on M, where N_φ denotes the Nijenhuis torsion of φ, given by

$$N_\varphi(X,Y) = \varphi^2[X,Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]. \tag{1.7}$$

Given an almost contact structure, one can associate in a natural manner an almost CR-structure $(\mathcal{D}, \varphi|_{\mathcal{D}})$, where $\mathcal{D} := Ker(\eta) = Im(\varphi)$ is the distribution of rank $2n$ transversal to the characteristic vector field ξ. If this almost CR-structure is integrable (i.e., $N_\varphi = 0$), the manifold M^{2n+1} is said to be CR-integrable. It is known that normal almost contact manifolds are CR-manifolds.

For more background on almost contact metric manifolds, we recommend the references [3, 6, 13].

2. C_{12}-manifolds

In the classification of Chinea and Gonzalez [8] of almost contact metric manifolds there is a class C_{12}-manifolds which can be integrable but never normal. In this classification, C_{12}-manifolds are defined by

$$(\nabla_X \phi)(Y,Z) = \eta(X)\eta(Z)(\nabla_\xi \eta)\varphi Y - \eta(X)\eta(Y)(\nabla_\xi \eta)\varphi Z. \tag{2.1}$$

In [5] and [7], the $(2n+1)$-dimensional C_{12}-manifolds is characterized by:

$$(\nabla_X \varphi)Y = \eta(X)\big(\omega(\varphi Y)\xi + \eta(Y)\varphi\psi\big) \tag{2.2}$$

for all X and Y vector fields on M, where $\omega = -\nabla_\xi \eta$ is a closed 1-form and $\psi = -\nabla_\xi \xi$ is a vector field such that $\omega(X) = g(X, \psi)$. Firstly, notice that the vector field ψ is perpendicular to ξ because $g(\psi, \xi) = -g(\nabla_\xi \xi, \xi) = 0$.

Therefore, we present here the following definition:

Definition 2.1 ([5]). Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost contact manifold. M is called an almost C_{12}-manifold if there exists a one-form ω which satisfies

$$\mathrm{d}\eta = \omega \wedge \eta \quad \text{and} \quad \mathrm{d}\phi = 0.$$

In addition, if $N_\varphi = 0$ we say that M is a C_{12}-manifold and we denote it by $(M, \varphi, \xi, \psi, \eta, \omega, g)$.

In (2.2), putting $Y = \xi$ one can get

$$\nabla_X \xi = -\eta(X)\psi \quad \text{and} \quad (\nabla_X \eta)Y = -\eta(X)\omega(Y). \tag{2.3}$$

We give here examples of C_{12}-manifold. The first type of examples are closely related to the warped product space defined by Bishop-O'Neill [2]. Let (N_1, g_1)

and (N_2, g_2) be Riemannian manifolds and ρ a differentiable function on N_1. Consider the product manifold $N_1 \times N_2$ with its projections $\pi_1 : N_1 \times N_2 \to N_1$ and $\pi_2 : N_1 \times N_2 \to N_2$. The warped product $M = N_1 \times_\rho N_2$ is the manifold $N_1 \times N_2$ furnished with the Riemannian metric g such that

$$g = \pi_1^* g_1 + \mathrm{e}^{2\rho} \pi_2^* g_2.$$

Theorem 2.2. *Let (N, J, h) be a Kählerian manifold and ρ a non-zero function on N. Then the warped product space $M = N \times_\rho L$ have a C_{12}-structure.*

Proof. Under the above assumptions, we define a Riemannian metric tensor g, a vector field ξ, a 1-form η and a $(1,1)$-tensor field φ on M as follows:

$$g = h + \mathrm{e}^{2\rho} dt^2, \quad \xi = \mathrm{e}^{-\rho} \partial_t, \quad \eta = \mathrm{e}^{\rho} dt, \quad \varphi X = JX \quad \text{and} \quad \varphi \partial_t = 0 \tag{2.4}$$

for any vector field X on N, where ∂_t denote the unit tangent field to L.

By a direct calculation using (1.5), one can check that (φ, ξ, η, g) is an almost contact metric structure.

In addition, we have $\mathrm{d}\eta = \mathrm{d}\rho \wedge \eta$ which implies $\omega = \mathrm{d}\rho$.

Also, the fundamental 2-form ϕ of (φ, ξ, η, g) is

$$\phi\big((X\ , a\partial t), \big(Y, b\partial t)\big) = g\big((X, \partial t), \varphi(Y, \partial t)\big),$$

we can check that is very simply as follows:

$$\phi = \Omega, \tag{2.5}$$

since (N, J, h) is a Kählerian manifold then we have $\mathrm{d}\Omega = 0$ and $N_J = 0$ which gives $\mathrm{d}\phi = 0$. Given the definition of φ, we can show that

$$N_\varphi\big((X\ , a\partial t), \big(Y, b\partial t)\big) = N_J(X, Y) = 0.$$

Therefore, $(M, \varphi, \xi, \psi, \eta, \omega, g)$ is a 1-parameter family of C_{12}-manifold. □

On the other hand, let $(M, \varphi, \xi, \psi, \eta, \omega, g)$ be a C_{12}-manifold with $\omega = \mathrm{d}\tau$ where τ is a non-zero function on M.

On $\tilde{M} = \mathbb{R} \times M$, where $\mathbb{R}$ be the real line with natural coordinate system ∂t, we define a metric $\tilde{g}$ as follows:

$$\tilde{g}(X, Y) = dt^2 + \mathrm{e}^{-2\tau} g(X, Y), \tag{2.6}$$

where X and Y are vector fields on M.

Since the dimension of $\tilde{M}$ is $2n+2$, one can suspect that it is equipped with an almost complex structure. We can put

$$JX = \varphi X + \mathrm{e}^{-\tau} \eta(X) \partial t \quad \text{and} \quad J\partial t = -\mathrm{e}^{\tau} \xi. \tag{2.7}$$

The following is well known.

Proposition 2.3. *The triplet $(\tilde{M}, J, \tilde{g})$ constructed as above is an almost Hermitian manifold.*

Proof. It is obvious. □

Looking to obtain the classification of this structure, one needs the fundamental form Ω and N_J. The manifold $(\tilde{M}, J, \tilde{g})$ possesses a fundamental 2-form, Ω, the Kähler form, defined by

$$\begin{aligned}\Omega\big((X, a\partial t), (Y, b\partial t)\big) &= \tilde{g}\big((X, a\partial t), J(Y, b\partial t)\big)\\ &= \tilde{g}\big((X, a\partial t), (\varphi Y - b\mathrm{e}^{\tau}\xi, \mathrm{e}^{-\tau}\eta(Y)\partial t)\big),\end{aligned}$$

we can check that is very simply as follows:

$$\Omega = \mathrm{e}^{-2\tau}\phi - 2\mathrm{e}^{-\tau}\eta \wedge \mathrm{d}t, \tag{2.8}$$

where ϕ denotes the fundamental 2-form of (φ, ξ, η). We have immediately that,

$$\mathrm{d}\Omega = 2\mathrm{e}^{-2\tau}\mathrm{d}\tau \wedge \phi + \mathrm{e}^{-2\tau}\mathrm{d}\phi + 2\mathrm{e}^{-\tau}(\mathrm{d}\tau \wedge \eta - \mathrm{e}^{-\tau}\mathrm{d}\eta) \wedge \mathrm{d}t.$$

Since M is a C_{12}-manifold. That is, $\mathrm{d}\phi = 0$ and $\mathrm{d}\eta = \omega \wedge \eta$ with $\omega = \mathrm{d}\tau$, then

$$\mathrm{d}\Omega = 2\mathrm{e}^{-2\tau}\omega \wedge \phi. \tag{2.9}$$

From (2.8) and (2.9), one can get

$$\mathrm{d}\Omega = 2\omega \wedge \Omega. \tag{2.10}$$

On other hand, from (1.2), we have

$$\begin{aligned}N_J(X, Y) &= N_\varphi(X, Y) + 2(\mathrm{d}\eta - \omega \wedge \eta)(X, Y)\xi\\ &\quad + \mathrm{e}^{\tau}\Big(\eta\big((\nabla_Y\varphi)X - \eta(Y)(\omega(\varphi X)\xi + \eta(X)\varphi\psi)\big)\\ &\quad - \eta\big((\nabla_X\varphi)Y - \eta(X)(\omega(\varphi Y)\xi + \eta(Y)\varphi\psi)\big)\\ &\quad + (\nabla_{\varphi X})(Y) - (\nabla_{\varphi Y})(X)\Big)\partial t,\end{aligned}$$

and

$$\begin{aligned}N_J(X, \partial t) &= \mathrm{e}^{-\tau}\Big((\nabla_\xi\varphi)X - \omega(\varphi X)\xi + \varphi\nabla_X\xi - \varphi\nabla_{\varphi X}\xi\Big)\\ &\quad + g\big(\nabla_\xi\xi + \psi - \omega(\xi)\xi, X\big)\partial t.\end{aligned}$$

Easily we can see that $N_J = 0$ if and only if M is a C_{12}-manifold.

Therefore, summing up the arguments above, we have the following main theorem:

Theorem 2.4. *The almost Hemitian manifold $(\tilde{M}, J, \tilde{g})$ defined as above is a conformally Kähler structure if and only if $(M, \varphi, \xi, \psi, \eta, \omega, g)$ is a C_{12}-structure with $\omega = \mathrm{d}\tau$.*

Example 2.5. For this concrete example, we use the product of the Kählerian manifold $(\mathbb{R}^2, J, h)$ by the real line $\mathbb{R}$, with $h = dx^2 + dy^2$ and $J\partial_x = \partial_y$, $J\partial_y = -\partial_x$. Then by using Theorem 2.2 we have

$$g = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \mathrm{e}^{2\rho}\end{pmatrix}, \quad \varphi = \begin{pmatrix} 0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad \xi = \mathrm{e}^{-\rho}\partial t, \quad \eta = \mathrm{e}^{\rho}dt,$$

is a C_{12}-structure on $M = \mathbb{R}^2 \times \mathbb{R}_t$ with $\omega = \mathrm{d}\rho$.

Again, by Theorem 2.4, the structure

$$\tilde{g} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{2\rho} & 0 \\ 0 & 0 & 0 & \mathrm{e}^{2\rho} \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

is l.c. Kählerian structure on $\tilde{M} = \mathbb{R}_r \times M$.

This example can be generalized to other dimensions.

Other examples we can build them starting from a β-Kenmotsu manifolds in particular cosymplectic manifolds (for $\beta = 0$) with a certain deformation.

Let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold. We put,

$$\tilde{\varphi} = \varphi, \quad \tilde{\xi} = \mathrm{e}^{-\rho-\tau}\xi, \quad \tilde{\eta} = \mathrm{e}^{\rho+\tau}\eta, \quad \tilde{g} = \mathrm{e}^{2\rho}g + \mathrm{e}^{2\rho}(\mathrm{e}^{2\tau} - 1)\eta \otimes \eta, \tag{2.11}$$

where ρ and τ are two smooth functions on M. Then, $(M, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is also an almost contact metric manifold [1]. If we denote by $\tilde{\phi}$ the fundamental 2-form of $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$, then we have

$$\begin{cases} \mathrm{d}\tilde{\eta} = \mathrm{d}(\rho + \tau) \wedge \tilde{\eta} + \mathrm{e}^{\rho+\tau}\mathrm{d}\eta, \\ \mathrm{d}\tilde{\phi} = 2\mathrm{d}\rho \wedge \tilde{\phi} + \mathrm{e}^{2\rho}\mathrm{d}\phi. \end{cases} \tag{2.12}$$

Let us know that the manifold $(M, \varphi, \xi, \eta, g)$ is said to be β-Kenmotsu means that $\mathrm{d}\eta = N^{(1)} = 0$ and $\mathrm{d}\phi = 2\beta\eta \wedge \phi$. Then, we have

$$\begin{cases} \mathrm{d}\tilde{\eta} = \mathrm{d}(\rho + \tau) \wedge \tilde{\eta}, \\ \mathrm{d}\tilde{\phi} = 2(\mathrm{d}\rho + \beta\eta) \wedge \tilde{\phi}. \end{cases} \tag{2.13}$$

From (2.13) and formula (1.6) we get the following proposition:

Proposition 2.6. *Let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold.*

1. *If $(M, \varphi, \xi, \eta, g)$ is a cosymplectic manifold (i.e., $\beta = 0$) and ρ is a constant, then the manifold $(M, \tilde{\varphi}, \tilde{\xi}, \tilde{\psi}, \tilde{\eta}, \tilde{\omega}, \tilde{g})$ defined by* (2.11) *is a C_{12}-manifold with $\tilde{\omega} = \mathrm{d}\tau$ and $\tilde{\psi} = \mathrm{grad}\tau$.*
2. *If $(M, \varphi, \xi, \eta, g)$ is a β-Kenmotsu manifold with $\eta = -\frac{1}{\beta}\mathrm{d}\rho$, then $(M, \tilde{\varphi}, \tilde{\xi}, \tilde{\psi}, \tilde{\eta}, \tilde{\omega}, \tilde{g})$ is a C_{12}-manifold with $\tilde{\omega} = \mathrm{d}\tau$ and $\tilde{\psi} = \mathrm{grad}\tau$.*

3. Generalized C_{12}-manifolds

Let $(M, \varphi, \xi, \psi, \eta, \omega, g)$ be an almost C_{12}-manifold. For any function σ on M, we put

$$\tilde{\varphi} = \varphi, \quad \tilde{\xi} = \mathrm{e}^{-\sigma}\xi, \quad \tilde{\eta} = \mathrm{e}^{\sigma}\eta, \quad \tilde{g} = \mathrm{e}^{2\sigma}g. \tag{3.1}$$

Then, $(M, \varphi, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is also an almost C_{12}-manifold. First, notice that

$$\begin{aligned} \mathrm{d}\tilde{\eta} &= \mathrm{e}^{\sigma}\mathrm{d}\sigma \wedge \eta + \mathrm{e}^{\sigma}\mathrm{d}\eta \\ &= (\theta + \omega) \wedge \tilde{\eta}, \end{aligned}$$

where $\theta = \mathrm{d}\sigma$. Second, let $\tilde{\phi}$ denote the fundamental 2-form of the almost contact metric structure $(\varphi, \tilde{\xi}, \tilde{\eta}, \tilde{g})$, we have $\tilde{\phi} = \mathrm{e}^{2\sigma}\phi$ which implies

$$\begin{aligned} \mathrm{d}\tilde{\phi} &= 2\mathrm{e}^{2\sigma}\mathrm{d}\sigma \wedge \phi \\ &= 2\theta \wedge \tilde{\phi}. \end{aligned}$$

Based on these facts, we give the following definition

Definition 3.1. Let $(M^{2n+1}, \varphi, \xi, \psi, \eta, \omega, g)$ be an almost C_{12}-manifold. M is called a generalized almost C_{12}-manifold if there exists a closed one-form θ which satisfies

$$\mathrm{d}\phi = 2\theta \wedge \phi \quad \text{and} \quad \mathrm{d}\eta = (\omega + \theta) \wedge \eta.$$

Moreover, if $N_\varphi = 0$ we say that M is a generalized C_{12}-manifold and we denote it by $(M, \varphi, \theta, \omega, \eta, g)$.

In particular, if $\theta = 0$, then we have an almost C_{12}-manifold.

Now, suppose that $(M^{2n+1}, \varphi, \xi, \psi, \eta, \omega, g)$ is an almost C_{12}-manifold. Apply the deformation (3.1) twice successively for the function σ then for the function μ, we obtain

$$\overline{\phi} = \mathrm{e}^{2(\sigma+\mu)}\phi \quad \text{and} \quad \overline{\eta} = \mathrm{e}^{\sigma+\mu}\eta,$$

which implies

$$\mathrm{d}\overline{\eta} = (\omega + \theta_1 + \theta_2) \wedge \overline{\eta} \quad \text{and} \quad \mathrm{d}\overline{\phi} = 2(\theta_1 + \theta_2) \wedge \overline{\phi},$$

where $\theta_1 = \mathrm{d}\sigma$ and $\theta_2 = \mathrm{d}\mu$.

Continuing the current method exactly $n-1$ times, we produce generalized C_{12}-manifold of order n given by the following definition:

Definition 3.2. Let $(M^{2n+1}, \varphi, \xi, \psi, \eta, \omega, g)$ be an almost C_{12}-manifold. M is called a generalized C_{12}-manifold of order p if there exist p closed one-forms θ_i which satisfies

$$\mathrm{d}\phi = 2\sum_{i=1}^{p} \theta_i \wedge \phi \quad \text{and} \quad \mathrm{d}\eta = \Big(\omega + \sum_{i=1}^{p} \theta_i\Big) \wedge \eta,$$

where $0 \leq p \leq n-1$.

Moreover, if $N_\varphi = 0$ we say that M is a generalized C_{12}-manifold of order p and we denote it by $(M, \varphi, \theta_1, \dots, \theta_p, \omega, \eta, g)$.

4. Curvature formulas and main result

Now, we denote by R the curvature tensor and by S the Ricci curvature, which are defined for all $X, Y, Z \in \mathfrak{X}(M)$ by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{4.1}$$

$$S(X,Y) = \sum_{i=1}^{2n+1} g\big(R(e_i, X)Y, e_i\big), \tag{4.2}$$

with $\{e_1, \dots, e_{2n+1}\}$ is a local orthonormal basis. Then, we have:

Proposition 4.1. *Let $(M, \varphi, \xi, \psi, \eta, \omega, g)$ be a C_{12}-manifold. Then*

$$R(X,Y)\xi = -2\mathrm{d}\eta(X,Y)\psi - \eta(Y)\nabla_X\psi + \eta(X)\nabla_Y\psi, \tag{4.3}$$

$$g\big(R(X,\xi)Y,Z\big) = 2\mathrm{d}\eta(Y,Z)\omega(X) + g(\nabla_Y\psi, X)\eta(Z) - g(\nabla_Z\psi, X)\eta(Y), \tag{4.4}$$

$$S(X,\xi) = -\eta(X)\mathrm{div}\psi. \tag{4.5}$$

Proof. The relation (4.3) follows from (4.1) with $Z = \xi$ and formula (2.3).

For the second relation (4.4), we have for all vectors fields X, Y, Z on M

$$g\big(R(X,\xi)Y,Z\big) = g\big(R(Z,Y)\xi, X\big),$$

and using (4.3). Finally, knowing that

$$S(X,Y) = \sum_{i=1}^{2n+1} g\big(R(e_i, X)Y, e_i\big),$$

then,

$$S(X,\xi) = \sum_{i=1}^{2n+1} g\big(R(e_i, X)\xi, e_i\big),$$

using (4.3) with $\mathrm{div}\psi = \sum_i g(\nabla_{e_i}\psi, e_i)$, we obtain (4.5). This completes the proof of the proposition. □

Proposition 4.2. *Let $(M, \varphi, \xi, \psi, \eta, \omega, g)$ be a C_{12}-manifold. Then, we have*

$$\begin{aligned} R(X,Y)\varphi Z = {}& \varphi R(X,Y)Z + 2(\omega\wedge\eta)(X,Y)\big(\omega(\varphi Z)\xi + \eta(Z)\varphi\psi\big) \\ & + \eta(Y)g(\nabla_X\varphi\psi, Z)\xi - \eta(X)g(\nabla_Y\varphi\psi, Z)\xi, \end{aligned} \tag{4.6}$$

$$\begin{aligned} R(\varphi X,Y)Z = {}& -R(X,\varphi Y)Z - 2(\eta\wedge\omega\circ\varphi)(X,Y)\big(\omega(Z)\xi - \eta(Z)\psi\big) \\ & + \eta(Y)g(\nabla_{\varphi X}\psi, Z)\xi - \eta(Y)\eta(Z)\nabla_{\varphi X}\psi, \end{aligned} \tag{4.7}$$

$$\begin{aligned} R(\varphi X,\varphi Y)Z = {}& R(X,Y)Z - 2(\omega\wedge\eta)(X,Y)\big(\omega(Z)\xi - \eta(Z)\psi\big) \\ & - \eta(Y)g(\nabla_X\psi, Z)\xi + \eta(Y)\eta(Z)\nabla_X\psi. \end{aligned} \tag{4.8}$$

Proof. (4.6) follows from (4.1) and (2.2). For (4.7), we have

$$g\big(R(\varphi X,Y)Z,W\big) = -g\big(R(Z,W)\varphi X, Y\big).$$

So, using (4.6), we can conclude our formula directly. The last formula comes from (4.6) and the formula

$$g\big(R(\varphi X,\varphi Y)Z,W\big) = -g\big(\varphi R(Z,W)\varphi X, Y\big),$$

taking into consideration the formula (4.3). □

Definition 4.3. Let (M, g) be a complete Riemannian manifold, M is called a space form if its sectional curvature is constant.

It is known that the sectional curvature of the plane section spanned by the unit tangent vector field X orthogonal to ξ and φX is called a φ-sectional curvature. If any Sasakian manifold has a constant φ-sectional curvature c, then it is called a Sasakian space form. The Riemannian curvature tensor of Sasakian space form is given by the following formula [3, 13]:

$$R(X,Y) = X \wedge Y + \frac{c-1}{4}\big(\varphi^2 X \wedge \varphi^2 Y + \varphi X \wedge \varphi Y + 2g(X,\varphi Y)\varphi\big), \tag{4.9}$$

where $(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y$ for all X, Y, Z vector fields on M. Also, the Riemannian curvature tensor of Kenmotsu space form is given by [9]:

$$R(X,Y) = -X \wedge Y + \frac{c+1}{4}\big(\varphi^2 X \wedge \varphi^2 Y + \varphi X \wedge \varphi Y + 2g(X,\varphi Y)\varphi\big), \tag{4.10}$$

and the Riemannian curvature tensor of cosymplectic space form is given by [10]:

$$R(X,Y) = \frac{c}{4}\big(\varphi^2 X \wedge \varphi^2 Y + \varphi X \wedge \varphi Y + 2g(X,\varphi Y)\varphi\big). \tag{4.11}$$

In our case we have:

Theorem 4.4. *Let $(M, \varphi, \xi, \psi, \eta, \omega, g)$ be a C_{12}-manifold. The necessary and sufficient condition for M to have constant φ-holomorphic sectional curvature c is*

$$\begin{aligned} R(X,Y) = {}& \frac{c}{4}\big((\varphi^2 X \wedge \varphi^2 Y) + (\varphi X \wedge \varphi Y) + 2g(X,\varphi Y)\varphi\big) \\ & + 2\mathrm{d}\eta(X,Y)(\xi \wedge \psi) - \eta(X)(\xi \wedge \nabla_Y \psi) + \eta(Y)(\xi \wedge \nabla_X \psi). \end{aligned} \tag{4.12}$$

Proof. Suppose that H is the φ-sectional curvature of C_{12}-manifold. That is, for any vector field X orthogonal to ξ

$$H = K(X,\varphi X) = \frac{g\big(R(X,\varphi X)\varphi X, X\big)}{g(X,X)^2}, \tag{4.13}$$

or, equivalently

$$-Hg(X,X)^2 = g\big(R(X,\varphi X)X, \varphi X\big). \tag{4.14}$$

Firstly, let $\mathcal{D} := \{X \in \Gamma(TM) : \eta(X) = 0\}$ be the contact distribution of M. From which any vector field X on M can be uniquely written as follows:

$$X = \overline{X} + \eta(X)\xi, \tag{4.15}$$

where $\overline{X} \in \mathcal{D}$. So, for any vector fields $\overline{X}, \overline{Y}$ and $\overline{Z}$ on $\mathcal{D}$, the equations (4.6)-(4.8) becomes

$$R(\overline{X},\overline{Y})\varphi\overline{Z} = \varphi R(\overline{X},\overline{Y})\overline{Z}, \tag{4.16}$$

$$R(\varphi\overline{X},\overline{Y})\overline{Z} = -R(\overline{X},\varphi\overline{Y})\overline{Z}, \tag{4.17}$$

$$R(\varphi\overline{X},\varphi\overline{Y})\overline{Z} = R(\overline{X},\overline{Y})\overline{Z}. \tag{4.18}$$

By (4.16) we get

$$g\big(R(\overline{X},\varphi\overline{Y})\overline{X},\varphi\overline{Y}\big)=g\big(R(\overline{X},\varphi\overline{Y})\overline{Y},\varphi\overline{X}\big), \tag{4.19}$$

and

$$g\big(R(\overline{X},\varphi\overline{X})\overline{Y},\varphi\overline{X}\big)=g\big(R(\overline{X},\varphi\overline{X})\overline{X},\varphi\overline{Y}\big). \tag{4.20}$$

Substituting $\overline{X}+\overline{Y}$ in (4.14), we get

$$\begin{aligned}&-H\big(2g(\overline{X},\overline{Y})^2+2g(\overline{X},\overline{X})g(\overline{X},\overline{Y})+2g(\overline{X},\overline{Y})g(\overline{Y},\overline{Y})+g(\overline{X},\overline{X})g(\overline{Y},\overline{Y})\big)\\ =&\ \frac{1}{2}g\big(R(\overline{X}+\overline{Y},\varphi\overline{X}+\varphi\overline{Y})(\overline{X}+\overline{Y}),\varphi\overline{X}+\varphi\overline{Y}\big)\\ &+\frac{1}{2}H\big(g(\overline{X},\overline{X})^2+g(\overline{Y},\overline{Y})^2\big),\end{aligned}$$

With the help of (4.16), (4.20) and the Bianchi identity. It then turns to

$$\begin{aligned}&-H\big(2g(\overline{X},\overline{Y})^2+2g(\overline{X},\overline{X})g(\overline{X},\overline{Y})+2g(\overline{X},\overline{Y})g(\overline{Y},\overline{Y})+g(\overline{X},\overline{X})g(\overline{Y},\overline{Y})\big)\\ =&\ g\big(R(\overline{Y},\varphi\overline{X})\overline{X},\varphi\overline{X}\big)+g\big(R(\overline{X},\varphi\overline{X})\overline{X},\varphi\overline{Y}\big)+g\big(R(\overline{Y},\varphi\overline{Y})\overline{X},\varphi\overline{X}\big)\\ &+g\big(R(\overline{Y},\varphi\overline{Y})\overline{Y},\varphi\overline{X}\big)+g\big(R(\overline{X},\varphi\overline{Y})\overline{Y},\varphi\overline{X}\big)+g\big(R(\overline{X},\varphi\overline{Y})\overline{Y},\varphi\overline{Y}\big)\\ &+g\big(R(\overline{X},\varphi\overline{Y})\overline{X},\varphi\overline{Y}\big)\\ =&\ 2g\big(R(\overline{X},\varphi\overline{X})\overline{X},\varphi\overline{Y}\big)+2g\big(R(\overline{Y},\varphi\overline{Y})\overline{Y},\varphi\overline{X}\big)-g\big(R(\varphi\overline{Y},\overline{X})\overline{Y},\varphi\overline{X}\big)\\ &-g\big(R(\overline{X},\overline{Y})\varphi\overline{Y},\varphi\overline{X}\big)+g\big(R(\overline{X},\varphi\overline{Y})\overline{Y},\varphi\overline{X}\big)+g\big(R(\overline{X},\varphi\overline{Y})\overline{X},\varphi\overline{Y}\big),\end{aligned}$$

using (4.18) and (4.19) we get

$$\begin{aligned}&-H\big(2g(\overline{X},\overline{Y})^2+2g(\overline{X},\overline{X})g(X,\overline{Y})+2g(\overline{X},\overline{Y})g(\overline{Y},\overline{Y})+g(\overline{X},\overline{X})g(\overline{Y},\overline{Y})\big)\\ &=2g\big(R(\overline{X},\varphi\overline{X})\overline{X},\varphi\overline{Y}\big)+2g\big(R(\overline{Y},\varphi\overline{Y})\overline{Y},\varphi\overline{X}\big)+2g\big(R(\overline{X},\varphi\overline{Y})\overline{Y},\varphi\overline{X}\big)\\ &\quad+g\big(R(\varphi\overline{X},\varphi\overline{Y})\overline{X},\overline{Y}\big)+g\big(R(\overline{X},\varphi\overline{Y})\overline{X},\varphi\overline{Y}\big)\\ &=2g\big(R(\overline{X},\varphi\overline{X})\overline{X},\varphi\overline{Y}\big)+2g\big(R(\overline{Y},\varphi\overline{Y})\overline{Y},\varphi\overline{X}\big)+3g\big(R(\overline{X},\varphi\overline{Y})\overline{Y},\varphi\overline{X}\big)\\ &\quad+g\big(R(\overline{X},\overline{Y})\overline{X},\overline{Y}\big).\end{aligned} \tag{4.21}$$

Replacing $\overline{Y}$ by $-\overline{Y}$ in (4.21) and summing it to (4.21) we have

$$\begin{aligned}&3g\big(R(\overline{X},\varphi\overline{Y})\overline{Y},\varphi\overline{X}\big)+g\big(R(\overline{X},\overline{Y})\overline{X},\overline{Y}\big)\\ &=-H\big(2g(\overline{X},\overline{Y})^2+g(\overline{X},\overline{X})g(\overline{Y},\overline{Y})\big).\end{aligned} \tag{4.22}$$

Replacing $\overline{Y}$ by $\varphi\overline{Y}$ in (4.22) and using (4.18) with (4.20) we get

$$\begin{aligned}&-H\big(2g(\overline{X},\varphi\overline{Y})^2+g(\overline{X},\overline{X})g(\varphi\overline{Y},\varphi\overline{Y})\big)\\ =&-3g\big(R(\overline{X},\overline{Y})\varphi\overline{Y},\varphi\overline{X}\big)+g\big(R(\overline{X},\varphi\overline{Y})\overline{X},\varphi\overline{Y}\big)\\ =&\ 3g\big(R(\varphi\overline{X},\varphi\overline{Y})\overline{X},\overline{Y}\big)+g\big(R(\overline{X},\varphi\overline{Y})\overline{X},\varphi\overline{Y}\big)\\ =&\ 3g\big(R(\overline{X},\overline{Y})\overline{X},\overline{Y}\big)+g\big(R(\overline{X},\varphi\overline{Y})\overline{Y},\varphi\overline{X}\big)+2g(\overline{X},\overline{Y})^2.\end{aligned}$$

By virtue of (4.22) we obtain

$$\begin{aligned}&-H\big(2g(\overline{X},\varphi\overline{Y})^2+g(\overline{X},\overline{X})g(\varphi\overline{Y},\varphi\overline{Y})\big)\\ &=3g\big(R(\overline{X},\overline{Y})\overline{X},\overline{Y}\big)-\frac{1}{3}g\big(R(\overline{X},\overline{Y})\overline{X},\overline{Y}\big)\\ &\quad-\frac{H}{3}\big(2g(\overline{X},\overline{Y})^2+g(\overline{X},\overline{X})g(\overline{Y},\overline{Y})\big).\end{aligned}$$

After simplification, we get

$$4g\big(R(\overline{X},\overline{Y})\overline{X},\overline{Y}\big)=H\big(g(\overline{X},\overline{Y})^2-3g(\overline{X},\varphi\overline{Y})^2-g(\overline{X},\overline{X})g(\overline{Y},\overline{Y})\big).\tag{4.23}$$

Now, we calculate $g\big(R(\overline{X}+\overline{Z},\overline{Y}+\overline{W})(\overline{X}+\overline{Z}),\overline{Y}+\overline{W}\big)$. Using (4.23) and (4.6)-(4.8), we obtain

$$\begin{aligned}&4g\big(R(\overline{X},\overline{Y})\overline{Z},\overline{W}\big)+4g\big(R(\overline{X},\overline{W})\overline{Y},\overline{Z}\big)\\ &=H\big(g(\overline{X},\overline{Z})g(\overline{Y},\overline{W})+g(\overline{X},\overline{W})g(\overline{Y},\overline{Z})-3g(\overline{X},\varphi\overline{Z})g(\overline{Y},\varphi\overline{W})\\ &\quad-3g(\overline{X},\varphi\overline{W})g(\overline{Y},\varphi\overline{Z})-2g(\overline{X},\overline{Y})g(\overline{Z},\overline{W})\big),\end{aligned}\tag{4.24}$$

in the equation (4.24), we substitute between $\overline{Y}$ and $\overline{Z}$ and then subtract the resulting equation from (4.24), we get

$$\begin{aligned}&g\big(R(\overline{X},\overline{Y})\overline{Z},\overline{W}\big)-g\big(R(\overline{X},\overline{Z})\overline{Y},\overline{W}\big)+2g\big(R(\overline{X},\overline{W})\overline{Z},\overline{Y}\big)\\ &=\frac{3H}{4}\big(g(\overline{X},\overline{Y})g(\overline{Z},\overline{W})-g(\overline{X},\overline{Z})g(\overline{Y},\overline{W})+g(\overline{X},\varphi\overline{Z})g(\overline{Y},\varphi\overline{W})\\ &\quad-g(\overline{X},\varphi\overline{Y})g(\overline{Z},\varphi\overline{W})-2g(\overline{X},\varphi\overline{W})g(\overline{Z},\varphi\overline{Y})\big),\end{aligned}$$

using the Bianchi identity we obtain

$$\begin{aligned}&g\big(R(\overline{X},\overline{Y})\overline{Z},\overline{W}\big)-g\big(R(\overline{X},\overline{Z})\overline{Y},\overline{W}\big)+2g\big(R(\overline{X},\overline{W})\overline{Z},\overline{Y}\big)\\ &=3g\big(R(\overline{X},\overline{W})\overline{Z},\overline{Y}\big),\end{aligned}$$

therefore

$$\begin{aligned}g\big(R(\overline{X},\overline{W})\overline{Z},\overline{Y}\big)=\frac{H}{4}\big(&g(\overline{X},\overline{Y})g(\overline{Z},\overline{W})-g(\overline{X},\overline{Z})g(\overline{Y},\overline{W})\\ &+g(\overline{X},\varphi\overline{Z})g(\overline{Y},\varphi\overline{W})-g(\overline{X},\varphi\overline{Y})g(\overline{Z},\varphi\overline{W})\\ &-2g(\overline{X},\varphi\overline{W})g(\overline{Z},\varphi\overline{Y})\big),\end{aligned}\tag{4.25}$$

finally, we get

$$\begin{aligned}R(\overline{X},\overline{Y})\overline{Z}=\frac{H}{4}\big(&g(\overline{Y},\overline{Z})\overline{X}-g(\overline{X},\overline{Z})\overline{Y}+g(\varphi\overline{Y},\overline{Z})\varphi\overline{X}-g(\varphi\overline{X},\overline{Z})\varphi\overline{Y}\\ &+2g(\overline{X},\varphi\overline{Y})\varphi\overline{Z}\big),\end{aligned}\tag{4.26}$$

where $\overline{X},\overline{Y}$ and $\overline{Z}$ are orthogonal to ξ. For any vector fields X,Y,Z on M using (4.15) and (4.3)-(4.4), one can get

$$\begin{aligned}R(\overline{X},\overline{Y})\overline{Z}=R(X,Y)Z&-\big(\omega(X)\eta(Y)-\omega(Y)\eta(Y)\big)\eta(Z)\psi\\ &+\big(\omega(X)\eta(Y)-\omega(Y)\eta(Y)\big)\omega(Z)\xi\end{aligned}\tag{4.27}$$

$$
\begin{aligned}
&- \eta(Y)\big(\eta(Z)\nabla_X\psi - g(\nabla_X\psi, Z)\xi\big)\\
&+ \eta(X)\big(\eta(Z)\nabla_Y\psi - g(\nabla_Y\psi, Z)\xi\big)\\
=\ & R(X,Y)Z - 2\mathrm{d}\eta(X,Y)(\xi\wedge\psi)Z\\
&+ \eta(X)(\xi\wedge\nabla_Y\psi)Z - \eta(Y)(\xi\wedge\nabla_X\psi)Z,
\end{aligned}
$$

finally, replacing in (4.26) we get our formula.

Conversely, just replace (4.12) in (4.27) and using (4.26). □

A C_{12}-manifold of constant φ-sectional curvature c will be called a C_{12}-space form and denoted by C_{12}(c).

Proposition 4.5. *For any C_{12}-space form the Ricci curvature S and scalar curvature r are given by*:

$$
\begin{aligned}
S(X,Y) = &-\frac{c}{2}(n+1)g(X,Y) + \big(\mathrm{div}\psi + \omega(\psi) + \frac{c}{2}(n+1)\big)\eta(X)\eta(Y)\\
&+\ \omega(X)\omega(Y) + g(\nabla_X\psi, Y),
\end{aligned}
\tag{4.28}
$$

$$
r = -nc(n+1) + 2\big(\mathrm{div}\psi + \omega(\psi)\big). \tag{4.29}
$$

Proof. The proof is direct, just use the formulas

$$
S(X,Y) = \sum_{i=1}^{2n+1} g\big(R(e_i,X)Y, e_i\big) \quad \text{and} \quad r = \sum_{i=1}^{2n+1} S(e_i,e_i).
$$

□

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