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H-QUASI-HEMI-SLANT SUBMERSIONS

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Abstract. In this paper, h-quasi-hemi-slant submersions and almost h-quasi-hemi-slant submersions from almost quaternionic Hermitian manifolds onto Riemannian manifolds are introduced. Fundamental results on h-quasi-hemi-slant submersions: the integrability of distributions, geometry of foliations and the conditions for such submersions to be totally geodesic are investigated. Moreover, some non-trivial examples of the h-quasi-hemi-slant submersion are constructed.

1. Introduction

O'Neill [16] in 1966 and Gray [8] in 1967 independently started work on Riemannian submersions. In 1976, a classification theorem among base manifolds and total manifolds was obtained using the notion of almost Hermitian submersions by Watson [30]. Later, the notion of almost Hermitian submersion has been extended to different kinds of sub-classes, according to the conditions on submersion such as: a Riemannian submersion [27], an anti-invariant Riemannian submersion [24], a semi-invariant Riemannian submersion [25], slant Riemannian Submersion [26], a quaternionic submersion [9], an almost h-slant submersion [17], a h-semi-invariant submersion [18], a conformal hemi-slant submersion ([12, 13]), a conformal semi-slant submersion [14], an almost h-semi-slant Riemannian map [19], a h-semi-slant submersions [20], an almost h-conformal semi-invariant submersion [21] etc. In [29], Tastan and others introduced and studied hemi-slant Riemannian submersions from Hermitian manifolds onto Riemannian manifolds and Akyol and Gündüzalp [1] studied Hemi-slant submersions from almost product Riemannian manifolds. Prasad and others ([22, 23]) introduced the notion of quasi-bi-slant submersions and Longwap, Massamba and Homti in [15] studied quasi-hemi-slant Riemannian submersions.

We also note that Riemannian submersions have applications in physics, mechanics and robotics. Such as: Bedrossian and Spong showed in [5] the

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existence of a class of robotic chains having Riemannian curvature that is locally vanishing, once potential energy and friction phenomena are ignored. C. Altafini [3] commenced using the notion of Riemannian submersions for the modeling and control of redundant robotic chain [5] (redundant means that robotic chain has more than six degrees of freedom). He also showed that Riemannian submersion gives a close relationship between inverse kinematic in robotics and the pull back vectors, called the horizontal lift. On the other hand, Riemannian submersion theory has applications in Kaluza-Klein theory ([6, 10]), the Yang-Mills theory [7], Supergravity and superstring theories ([11, 31]). These broad applications of this topic make it an interesting field of research for geometers.

The present paper is organized as follows. In Section 2, we mention basic definitions and properties of Riemannian submersions which are needed at the following sections. In Section 3, we give the definition of a h-quasi-hemi-slant submersion and obtain some basic results on it. The necessary and sufficient conditions for integrability of distributions and totally geodesicness are also obtained in this section. In the last Section we provide some non-trivial examples of the h-quasi-hemi-slant submersions.

2. Preliminaries

Let $\pi : (N_1, g_1) \to (N_2, g_2)$ be a Riemannian submersions [28].

Define the O'Neill tensors $\mathcal{T}$ and $\mathcal{A}$ by

$$\mathcal{A}_{F_1}F_2 = \mathcal{H}\nabla_{\mathcal{H}F_1}\mathcal{V}F_2 + \mathcal{V}\nabla_{\mathcal{H}F_1}\mathcal{H}F_2, \tag{2.1}$$

$$\mathcal{T}_{F_1}F_2 = \mathcal{H}\nabla_{\mathcal{V}F_1}\mathcal{V}F_2 + \mathcal{V}\nabla_{\mathcal{V}F_1}\mathcal{H}F_2 \tag{2.2}$$

for any vector fields F_1, F_2 on N_1.

Now, from equations (2.1) and (2.2), we get

$$\nabla_{U_1}U_2 = \mathcal{T}_{U_1}U_2 + \mathcal{V}\nabla_{U_1}U_2, \tag{2.3}$$

$$\nabla_{U_1}W_1 = \mathcal{H}\nabla_{U_1}W_1 + \mathcal{T}_{U_1}W_1, \tag{2.4}$$

$$\nabla_{W_1}U_1 = \mathcal{A}_{W_1}U_1 + \mathcal{V}\nabla_{W_1}U_1, \tag{2.5}$$

$$\nabla_{W_1}W_2 = \mathcal{H}\nabla_{W_1}W_2 + \mathcal{A}_{W_1}W_2 \tag{2.6}$$

for all $U_1, U_2 \in \Gamma(\ker \pi_*)$ and $W_1, W_2 \in \Gamma(\ker \pi_*)^{\perp}$, where $\mathcal{V}\nabla_{U_1}U_2 = \widehat{\nabla}_{U_1}U_2$. If W_1 is basic, then $\mathcal{A}_{U_1}W_1 = \mathcal{H}\nabla_{W_1}U_1$.

Let (N_1, g_1) and (N_2, g_2) be Riemannian manifolds and $\pi : (N_1, g_1) \to (N_2, g_2)$ be a C^{∞}-map. The second fundamental form of π is given by

$$(\nabla\pi_*)(Z_1, Z_2) = \nabla^{\pi}_{Z_1}\pi_*Z_2 - \pi_*(\nabla^{N_1}_{Z_1}Z_2) \tag{2.7}$$

for any $Z_1, Z_2 \in \Gamma(TN_1)$, where ∇^{π} is the pullback connection and we denote conveniently by ∇ the Levi-Civita connections of the metrics g_1 and g_2 [4].

Recall that π is said to be a totally geodesic map if $(\nabla\pi_*)(Z_1, Z_2) = 0$ for $Z_1, Z_2 \in \Gamma(TN_1)$ [4].

Now, we recall following definitions for later use:

Definition 2.1 ([28]). Let π be a Riemannian submersion from an almost Hermitian manifold (N_1, g_1, J) onto a Riemannian manifold (N_2, g_2). Then we say that π is an invariant Riemannian submersion if the vertical distribution is invariant with respect to the complex structure J, i.e.,

$$J(\ker\pi_*) = \ker\pi_*.$$

Definition 2.2 ([24]). Let (N_1, g_1, J) be an almost Hermitian manifold and (N_2, g_2) be a Riemannian manifold. Suppose that there exists a Riemannian submersion $\pi : (N_1, g_1, J) \to (N_2, g_2)$ such that $J(\ker\pi_*) \subseteq (\ker\pi_*)^\perp$. Then we say that π is an anti-invariant Riemannian submersion.

Definition 2.3 ([25]). Let $\pi : (N_1, g_1, J) \to (N_2, g_2)$ be a Riemannian submersion from an almost Hermitian manifold onto a Riemannian manifold. Then we say that π is a semi-invariant Riemannian submersion if there is a distribution $\mathfrak{D}_1 \subseteq \ker\pi_*$ such that

$$\ker\pi_* = \mathfrak{D}_1 \oplus \mathfrak{D}_2, J(\mathfrak{D}_1) = \mathfrak{D}_1, J(\mathfrak{D}_2) \subseteq (\ker\pi_*)^\perp,$$

where $\mathfrak{D}_2$ is orthogonal complementary to $\mathfrak{D}_1$ in $\ker\pi_*$.

Definition 2.4 ([26]). Let π be a Riemannian submersion from an almost Hermitian manifold (N_1, g_1, J) onto a Riemannian manifold (N_2, g_2). If for any non-zero vector $Z_1 \in (\ker\pi_*)_q$, $q \in N_1$, the angle $\theta(Z_1)$ between JZ_1 and the space $(\ker\pi_*)_q$ is constant, i.e., it is independent of the choice of the point $q \in N_1$ and the tangent vector Z_1 in $\ker\pi_*$, then we say that π is a slant submersion. In this case, the angle θ is called the slant angle of the submersion.

Definition 2.5 ([28]). Let (N_1, g_1, J) be an almost Hermitian manifold and (N_2, g_2) a Riemannian manifold. A Riemannian submersion $\pi : (N_1, g_1, J) \to (N_2, g_2)$ is called a semi-slant submersion if there is a distribution $\mathcal{D}_1 \subset \ker\pi_*$ such that

$$\ker\pi_* = \mathcal{D} \oplus \mathcal{D}_1, J(\mathcal{D}) = \mathcal{D},$$

and the angle $\theta = \theta(V_1)$ between JV_1 and the space $(\mathcal{D}_1)_x$ is constant for non-zero $V_1 \in (\mathcal{D}_1)_x$ and $x \in N_1$, where $\mathcal{D}_1$ is the orthogonal complement of $\mathcal{D}$ in $\ker\pi_*$.

We call the angle θ a semi-slant angle.

Definition 2.6 ([29]). Let $\pi : (N_1, g_1, J) \to (N_2, g_2)$ be a Riemannian submersion from an almost Hermitian manifold onto a Riemannian manifold. A Riemannian submersion π is called a hemi-slant submersion if the vertical distribution $\ker\pi_*$ of π admits two orthogonal complementary distributions D^θ and $D^\perp$ such that D^θ is slant with angle θ and $D^\perp$ is anti-invariant, i.e., we have

$$\ker\pi_* = D^\theta \oplus D^\perp.$$

In this case, the angle θ is called the hemi-slant angle of the submersion.

Definition 2.7. Let π be a Riemannian submersion from (N_1, g_1, J) an almost Hermitian manifold onto (N_2, g_2) a Riemannian manifold. If the $\ker \pi_*$ has three orthogonal distributions, invariant D, slant D_1 and anti-invariant D_2 i.e.,

$$\ker \pi_* = D \oplus D_1 \oplus D_2, J(D) = D,$$

the angle θ between JD_1 and D_1 is constant and $JD_2 \subseteq (\ker \pi_*)^{\perp}$, then, π is called a quasi-hemi-slant Riemannian submersion, and the angle θ is said to be the quasi-hemi-slant angle [15].

One can easily see that hemi-slant submersions, semi-invariant submersions and semi-slant submersions are particular cases of quasi-hemi-slant submersions.

Let N_1 be a $4k$-dimensional differentiable manifold, g_1 be a Riemannian metric and E be a rank 3 subbundle of $End(TN_1)$ such that for any point $p \in N_1$ with its some neighborhood U, there exists a local basis $\{J_1, J_2, J_3\}$ of sections of E on U satisfying for all $\alpha \in \{1, 2, 3\}$

$$J_\alpha^2 = -id, \ J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2}, \tag{2.8}$$

$$g_1(J_\alpha X_1, J_\alpha X_2) = g_1(X_1, X_2) \tag{2.9}$$

for all vector fields $X_1, X_2 \in \Gamma(TN_1)$, where the indices are taken from $\{1, 2, 3\}$ modulo 3. Then (N_1, E, g_1) is an almost quaternionic Hermitian manifold ([2], [9]). The basis $\{J_1, J_2, J_3\}$ is called a quaternionic Hermitian basis. If there exist locally defined 1-forms $\omega_1, \omega_2, \omega_3$ such that for $\alpha \in \{1, 2, 3\}$

$$\nabla_{X_1} J_\alpha = \omega_{\alpha+2}(X_1) J_{\alpha+1} - \omega_{\alpha+1}(X_1) J_{\alpha+2}, \tag{2.10}$$

where $X_1 \in \Gamma(TN_1)$ and the indices are taken from $\{1, 2, 3\}$ modulo 3, then (N_1, E, g_1) is said to be a quaternionic Kähler manifold. If $\nabla J_\alpha = 0$ for $\alpha \in \{1, 2, 3\}$, then (N_1, E, g_1) is called a hyperkähler manifold. Moreover, $\{J_1, J_2, J_3, g_1\}$ is called a hyperkähler structure on N_1 and g_1 is said to be a hyperkähler metric [20].

Let (N_1, E_1, g_1) and (N_2, E_2, g_2) be almost quaternionic Hermitian manifolds [9]. A map $\pi : N_1 \to N_2$ is said to be a (E_1, E_2)-holomorphic map if given a point $x \in N_1$, for any $J \in (E_1)_x$ there exists $J' \in (E_2)_{\pi(x)}$ that satisfies the condition

$$\pi_* \circ J = J' \circ \pi_*.$$

A Riemannian submersion $\pi : N_1 \to N_2$, which is an (E_1, E_2)-holomorphic map is said to be a quaternionic submersion. In addition, π is said to be a quaternionic Kähler submersion (or a hyperkähler submersion) if (N_1, E_1, g_1) is a quaternionic Kähler manifold.

3. H-quasi-hemi-slant submersions

In this section, h-quasi-hemi-slant submersions π from an almost quaternionic Hermitian manifold (N_1, I, J, K, g_1) onto a Riemannian manifold (N_2, g_2) is defined and studied.

Definition 3.1. $\pi : (N_1, E, g_1) \to (N_2, g_2)$ is said to be an h-quasi-hemi-slant submersion if given a point $p \in N_1$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U so that for any $R \in sp\{I, J, K\}$, there is a distribution $D \subset (\ker \pi_*)$ on U that satisfies the condition

$$\ker \pi_* = D \oplus D_1 \oplus D_2,\ R(D) = D,\ R(D_2) \subset (\ker \pi_*)^{\perp}$$

and the slant angle $\theta_R = \theta_R(Z_1)$ between RZ_1 and the space $(D_1)_p$ is constant for all nonzero vector field $Z_1 \in (D_1)_p$ and $p \in U$, where $\ker \pi_*$ admits three orthogonal complementary distributions D, D_1 and D_2 such that D is invariant, D_1 is slant with angle θ_R and D_2 is anti-invariant.

We call basis $\{I, J, K\}$ is said to be an h-quasi-hemi-slant basis and the angles $\{\theta_I, \theta_J, \theta_K\}$ are said to be h-quasi-hemi-slant angles.

Moreover, if

$$\theta = \theta_I = \theta_J = \theta_K,$$

then we call $\pi : (N_1, E, g_1) \to (N_2, g_2)$ a strictly h-quasi-hemi-slant submersion, $\{I, J, K\}$ a strictly quasi-hemi-slant basis, and θ a strictly quasi-hemi-slant angle.

Definition 3.2. $\pi : (N_1, E, g_1) \to (N_2, g_2)$ is said to be an almost h-quasi-hemi-slant submersion if given a point $p \in N_1$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for any $R \in sp\{I, J, K\}$, there is a distribution $D^R \subset (\ker \pi_*)$ on U such that

$$\ker \pi_* = D^R \oplus D_1^R \oplus D_2^R,\ R(D^R) = D^R,\ R(D_2^R) \subset (\ker \pi_*)^{\perp}$$

and the slant angle $\theta_R = \theta_R(Z_1)$ between RZ_1 and the space $(D_1^R)_p$ is constant for all nonzero vector field $Z_1 \in (D_1^R)_p$ and $p \in U$, where the vertical distribution $\ker \pi_*$ admits three orthogonal complementary distributions D^R, D_1^R and D_2^R such that D^R is invariant, D_1^R is slant with angle θ_R and D_2^R is anti-invariant.

We call such basis $\{I, J, K\}$ an almost h-quasi-hemi-slant basis and the angles $\{\theta_I, \theta_J, \theta_K\}$ almost h-quasi-hemi-slant angles.

We easily observe that

(a) If $\dim D^R \neq 0$, $\dim D_1^R = 0$ and $\dim D_2^R = 0$, then π is an almost h-invariant submersion.

(b) If $\dim D^R \neq 0, \dim D_1^R \neq 0$, $0 < \theta_R < \frac{\pi}{2}$ and $\dim D_2^R = 0$, then π is an almost proper h-semi-slant submersion with semi-slant angle θ_R.

(c) If $\dim D^R = 0$, $\dim D_1^R \neq 0$, $0 < \theta_R < \frac{\pi}{2}$ and $\dim D_2^R = 0$, then π is an almost h-slant submersion with slant angle θ_R.

(d) If $\dim D^R = 0, \dim D_1^R = 0$ and $\dim D_2^R \neq 0$, then π is an almost h-anti-invariant submersion

(e) If $\dim D^R \neq 0, \dim D_1^R = 0$, $\theta_R = \frac{\pi}{2}$ and $\dim D_2^R \neq 0$, then π is an almost h-semi-invariant submersion.

(f) If $\dim D^R = 0, \dim D_1^R \neq 0$, $0 < \theta_R < \frac{\pi}{2}$ and $\dim D_2^R \neq 0$, then π is an almost h-hemi-slant submersion.

We say that the almost h-quasi-hemi-slant submersion $\pi : (N_1, E, g_1) \to (N_2, g_2)$ is proper if $D^R \neq \{0\}$, $D_1^R \neq \{0\}$, $D_2^R \neq \{0\}$ and $\theta_R \neq 0, \frac{\pi}{2}$. Thus, we note that h-hemi-slant submersions, h-semi-invariant submersions and h-semi-slant submersions are examples of h-quasi-hemi-slant submersions.

Moreover, we have

$$TN_1 = (\ker \pi_*) \oplus (\ker \pi_*)^\perp.$$

In addition, $Z_1 \in \Gamma(\ker \pi_*)$, we get

$$Z_1 = P_R Z_1 + Q_R Z_1 + S_R Z_1, \tag{3.1}$$

where $P_R Z_1 \in \Gamma(D^R)$, $Q_R Z_1 \in \Gamma(D_1^R)$, $S_R Z_1 \in \Gamma(D_2^R)$ and $R \in sp\{I, J, K\}$.

For any $Y_1 \in \Gamma(\ker \pi_*)$, we get

$$RY_1 = \phi_R Y_1 + \omega_R Y_1, \tag{3.2}$$

where $\phi_R Y_1 \in \Gamma(\ker \pi_*)$, $\omega_R Y_1 \in \Gamma(\ker \pi_*)^\perp$ and $R \in sp\{I, J, K\}$.

For any $Z_1 \in \Gamma(\ker \pi_*)^\perp$, we have

$$RZ_1 = B_R Z_1 + C_R Z_1, \tag{3.3}$$

where $B_R Z_1 \in \Gamma(\ker \pi_*)$, $C_R Z_1 \in \Gamma(\mu_R)$ and $R \in sp\{I, J, K\}$.

Then, we have

$$(\ker \pi_*)^\perp = \omega_R(D_1^R) \oplus \omega_R(D_2^R) \oplus \mu_R.$$

Obviously μ_R is the orthogonal complement of $\omega_R(D_1^R) \oplus \omega_R(D_2^R)$ in $(\ker \pi_*)^\perp$ and is R-invariant.

We will denote an almost h-quasi-hemi-slant submersion from a hyperkähler manifold (N_1, I, J, K, g_1) onto a Riemannian manifold (N_2, g_2) such that (I, J, K) is an almost h-quasi-hemi-slant basis by π.

The following lemmas are easily obtained.

Lemma 3.3. *Let π be an almost h-quasi-hemi-slant submersion from an almost quaternionic Hermitian manifold (N_1, I, J, K, g_1) onto a Riemannian manifold (N_2, g_2) such that (I, J, K) is an almost h-quasi-hemi-slant basis. Then, we have*

$$\phi_R D^R = D^R, \omega_R D^R = 0, \phi_R D_1^R \subset D_1^R, \phi_R D_2^R = 0, \omega_R D_1^R, \omega_R D_2^R \subset (\ker \pi_*)^\perp.$$

Lemma 3.4. *Let π be an almost h-quasi-hemi-slant submersion from an almost quaternionic Hermitian manifold (N_1, I, J, K, g_1) onto a Riemannian manifold (N_2, g_2) such that (I, J, K) is an almost h-quasi-hemi-slant basis. Then, we have*

$$\phi_R^2 Y_1 + B_R \omega_R Y_1 = -Y_1, \omega_R \phi_R Y_1 + C_R \omega_R Y_1 = 0,$$

$$\phi_R B_R Y_2 + B_R C_R Y_2 = 0, B_R \omega_R Y_2 + C_R^2 Y_2 = -Y_2$$

$\forall Y_1 \in \Gamma(\ker \pi_*), \forall Y_2 \in \Gamma(\ker \pi_*)^\perp$ *and* $R \in sp\{I, J, K\}$.

Lemma 3.5. *Let* π *be an almost h-quasi-hemi-slant submersion from a hyperkähler manifold* (N_1, I, J, K, g_1) *onto a Riemannian manifold* (N_2, g_2) *such that* (I, J, K) *is an almost h-quasi-hemi-slant basis. Then, we obtain*

(i)

$$\mathcal{V}\nabla_{Y_1}\phi_R Y_2 + \mathcal{T}_{Y_1}\omega_R Y_2 = B_R \mathcal{T}_{Y_1} Y_2 + \phi_R \mathcal{V}\nabla_{Y_1} Y_2, \tag{3.4}$$

$$\mathcal{T}_{Y_1}\phi_R Y_2 + \mathcal{H}\nabla_{Y_1}\omega_R Y_2 = C_R \mathcal{T}_{Y_1} Y_2 + \omega_R \mathcal{V}\nabla_{Y_1} Y_2 \tag{3.5}$$

$\forall Y_1, Y_2 \in \Gamma(\ker \pi_*)$ and $R \in sp\{I, J, K\}$.

(ii)

$$\mathcal{T}_{Y_1} B_R V_1 + \mathcal{H}\nabla_{Y_1} C_R V_1 = C_R \mathcal{H}\nabla_{Y_1} V_1 + \omega_R \mathcal{T}_{Y_1} V_1, \tag{3.6}$$

$$\mathcal{V}\nabla_{Y_1} B_R V_1 + \mathcal{T}_{Y_1} C_R V_1 = B_R \mathcal{H}\nabla_{Y_1} V_1 + \phi \mathcal{T}_{Y_1} V_1 \tag{3.7}$$

$\forall Y_1 \in \Gamma(\ker \pi_*), \forall V_1 \in \Gamma(\ker \pi_*)^\perp$ and $R \in sp\{I, J, K\}$.

(iii)

$$\mathcal{V}\nabla_{V_1}\phi_R Y_1 + \mathcal{A}_{V_1}\omega_R Y_1 = B_R \mathcal{A}_{V_1} Y_1 + \phi_R \mathcal{V}\nabla_{V_1} Y_1, \tag{3.8}$$

$$\mathcal{A}_{V_1}\phi_R Y_1 + \mathcal{H}\nabla_{V_1}\omega_R Y_1 = C_R \mathcal{A}_{V_1} Y_1 + \omega_R \mathcal{V}\nabla_{V_1} Y_1 \tag{3.9}$$

$\forall Y_1 \in \Gamma(\ker \pi_*), \forall V_1 \in \Gamma(\ker \pi_*)^\perp$ and $R \in sp\{I, J, K\}$.

(iv)

$$\mathcal{A}_{V_1} B_R V_2 + \mathcal{H}\nabla_{V_1} C_R V_2 = C_R \mathcal{H}\nabla_{V_1} V_2 + \omega_R \mathcal{A}_{V_1} V_2, \tag{3.10}$$

$$\mathcal{V}\nabla_{V_1} B_R V_2 + \mathcal{A}_{V_1} C_R V_2 = B_R \mathcal{H}\nabla_{V_1} V_2 + \phi_R \mathcal{A}_{V_1} V_2 \tag{3.11}$$

for any $V_1, V_2 \in \Gamma(\ker \pi_*)^\perp$ and $R \in sp\{I, J, K\}$.

Proof. The proof of the lemma is obvious. □

Now, we define

$$(\nabla_{Y_1}\phi_R)Y_2 = \mathcal{V}\nabla_{Y_1}\phi_R Y_2 - \phi_R \mathcal{V}\nabla_{Y_1} Y_2, \tag{3.12}$$

$$(\nabla_{Y_1}\omega_R)Y_2 = \mathcal{H}\nabla_{Y_1}\omega_R Y_2 - \omega_R \mathcal{V}\nabla_{Y_1} Y_2, \tag{3.13}$$

$$(\nabla_{V_1} B_R)V_2 = \mathcal{V}\nabla_{V_1} B_R V_2 - B_R \mathcal{H}\nabla_{V_1} V_2, \tag{3.14}$$

$$(\nabla_{V_1} C_R)V_2 = \mathcal{H}\nabla_{V_1} C_R V_2 - C_R \mathcal{H}\nabla_{V_1} V_2 \tag{3.15}$$

for any $Y_1, Y_2 \in \Gamma(\ker \pi_*)$, $V_1, V_2 \in \Gamma(\ker \pi_*)^\perp$ and $R \in sp\{I, J, K\}$.

Lemma 3.6. *Let π be an almost h-quasi-hemi-slant submersion from a hyperkähler manifold (N_1, I, J, K, g_1) onto a Riemannian manifold (N_2, g_2) such that (I, J, K) is an almost h-quasi-hemi-slant basis. Then, we obtain*

$$(\nabla_{Y_1}\phi_R)Y_2 = B_R\mathcal{T}_{Y_1}Y_2 - \mathcal{T}_{Y_1}\omega_R Y_2,$$
$$(\nabla_{Y_1}\omega_R)Y_2 = C_R\mathcal{T}_{Y_1}Y_2 - \mathcal{T}_{Y_1}\phi_R Y_2,$$
$$(\nabla_{V_1}C_R)V_2 = \omega_R\mathcal{A}_{V_1}V_2 - \mathcal{A}_{V_1}B_R V_2,$$
$$(\nabla_{V_1}B_R)V_2 = \phi_R\mathcal{A}_{V_1}V_2 - \mathcal{A}_{V_1}C_R V_2$$

for any $Y_1, Y_2 \in \Gamma(\ker\pi_*)$, $V_1, V_2 \in \Gamma(\ker\pi_*)^\perp$ and $R \in sp\{I, J, K\}$.

Proof. The proof from Eqs. (3.4), (3.5), (3.10), (3.11), (3.12), (3.13), (3.14) and (3.15), is easily done. □

If the tensor ϕ_R and ω_R are parallel, then

$$B_R\mathcal{T}_{V_1}V_2 = \mathcal{T}_{V_1}\omega_R V_2,$$
$$C_R\mathcal{T}_{V_1}V_2 = \mathcal{T}_{V_1}\phi_R V_2$$

for any $V_1, V_2 \in \Gamma(\ker\pi_*)$ and $R \in sp\{I, J, K\}$.

Lemma 3.7. *Let π be an almost h-quasi-hemi-slant submersion from a hyperkähler manifold (N_1, I, J, K, g_1) onto a Riemannian manifold (N_2, g_2). Then we have*

$$\phi_R^2 Y_1 = -\cos^2\theta_R Y_1 \tag{3.16}$$

for any $Y_1 \in \Gamma(D_1^R)$ and $R \in sp\{I, J, K\}$, where (I, J, K) is an almost h-quasi-hemi-slant basis with the almost h-quasi-hemi-slant angles $\{\theta_I, \theta_J, \theta_K\}$.

Proof. For $Y_1 \in \Gamma(D_1^R)$, $Y_1 \neq 0$ and $R \in sp\{I, J, K\}$, we get

$$\cos\theta_R = \frac{\| \phi_R Y_1 \|}{\| RY_1 \|}, \tag{3.17}$$

and

$$\cos\theta_R = \frac{g_1(RY_1, \phi_R Y_1)}{\| \phi_R Y_1 \| \| RY_1 \|}, \tag{3.18}$$

where $\theta_R(Y_1)$ is the h-quasi-hemi-slant angle.

From (2.8) and (3.2), we have

$$\cos\theta_R = -\frac{g_1(Y_1, \phi_R^2 Y_1)}{\| \phi_R Y_1 \| \| RY_1 \|}. \tag{3.19}$$

From equations (3.18) and (3.19), we have

$$\phi_R^2 Y_1 = -(\cos^2\theta_R)Y_1.$$ □

Theorem 3.8. *Let π be an almost h-quasi-hemi-slant submersion from a hyperkähler manifold (N_1, I, J, K, g_1) onto a Riemannian manifold (N_2, g_2) such that (I, J, K) is an almost h-quasi-hemi-slant basis. Then the following conditions are equivalent*:

(i) *the invariant distribution* D^R *is integrable.*
(ii) $g_1(\mathcal{T}_{Y_2}IY_1 - \mathcal{T}_{Y_1}IY_2, \omega_I Q_I V_1 + IS_I V_1) = g_1(\mathcal{V}\nabla_{Y_1}IY_2 - \mathcal{V}\nabla_{Y_2}IY_1, \phi_I Q_I V_1)$ *for* $Y_1, Y_2 \in \Gamma(D^I)$ *and* $V_1 \in \Gamma(D_1^I \oplus D_2^I)$.
(iii) $g_1(\mathcal{T}_{Y_2}JY_1 - \mathcal{T}_{Y_1}JY_2, \omega_J Q_J V_1 + JS_J V_1) = g_1(\mathcal{V}\nabla_{Y_1}JY_2 - \mathcal{V}\nabla_{Y_2}JY_1, \phi_J Q_J V_1)$ *for* $Y_1, Y_2 \in \Gamma(D^J)$ *and* $V_1 \in \Gamma(D_1^J \oplus D_2^J)$.
(iv) $g_1(\mathcal{T}_{Y_2}KY_1 - \mathcal{T}_{Y_1}KY_2, \omega_K Q_K V_1 + KS_K V_1) = g_1(\mathcal{V}\nabla_{Y_1}KY_2 - \mathcal{V}\nabla_{Y_2}KY_1, \phi_K Q_K V_1)$ *for* $Y_1, Y_2 \in \Gamma(D^K)$ *and* $V_1 \in \Gamma(D_1^K \oplus D_2^K)$.

Proof. Let $Y_1, Y_2 \in \Gamma(D^R)$, $V_1 \in \Gamma(D_1^R \oplus D_2^R)$, $V_2 \in (\ker \pi_*)^\perp$ and $R \in sp\{I, J, K\}$. Since $[Y_1, Y_2] \in (\ker \pi_*)$, we have $g_1([Y_1, Y_2], V_2) = 0$. Thus D^R is integrable $\Leftrightarrow$ $g_1([Y_1, Y_2], V_1) = 0$. Now, using equations (2.3), (2.8), (2.9), (3.1) and (3.2), we have

$$\begin{aligned} g_1([Y_1, Y_2], V_1) &= g_1(R\nabla_{Y_1}Y_2, RV_1) - g_1(R\nabla_{Y_2}Y_1, RV_1), \\ &= g_1(\nabla_{Y_1}RY_2, RV_1) - g_1(\nabla_{Y_2}RY_1, RV_1), \\ &= g_1(\mathcal{T}_{Y_1}RY_2 - \mathcal{T}_{Y_2}RY_1, \omega_R Q_R V_1 + RS_R V_1) \\ &\quad - g_1(\mathcal{V}\nabla_{Y_1}RY_2 - \mathcal{V}\nabla_{Y_2}RY_1, \phi_R Q_R V_1). \end{aligned}$$

Since D^R is R-invariant, we have

$$\text{(i)} \Leftrightarrow \text{(ii)},\ \text{(i)} \Leftrightarrow \text{(iii)},\ \text{(i)} \Leftrightarrow \text{(iv)}.$$

Therefore, we get the result. □

Theorem 3.9. *Let* π *be an almost h-quasi-hemi-slant submersion from a hyperkähler manifold* (N_1, I, J, K, g_1) *onto a Riemannian manifold* (N_2, g_2) *such that* (I, J, K) *is an almost h-quasi-hemi-slant basis. Then the following conditions are equivalent*:

(i) *the slant distribution* D_1^R *is integrable.*
(ii)

$$\begin{aligned} &g_1(\mathcal{T}_{X_1}\omega_I\phi_I X_2 - \mathcal{T}_{X_2}\omega_I\phi_I X_1, U_1) \\ &= g_1(\mathcal{T}_{X_1}\omega_I X_2 - \mathcal{T}_{X_2}\omega_I X_1, \phi_I P_I U_1) \\ &\quad + g_1(\mathcal{H}\nabla_{X_1}\omega_I X_2 - \mathcal{H}\nabla_{X_2}\omega_I X_1, \omega_I S_I U_1) \end{aligned}$$

for $X_1, X_2 \in \Gamma(D_1^I)$ *and* $U_1 \in \Gamma(D^I \oplus D_2^I)$.
(iii)

$$\begin{aligned} &g_1(\mathcal{T}_{X_1}\omega_J\phi_J X_2 - \mathcal{T}_{X_2}\omega_J\phi_J X_1, U_1) \\ &= g_1(\mathcal{T}_{X_1}\omega_J X_2 - \mathcal{T}_{X_2}\omega_J X_1, \phi_J P_J U_1) \\ &\quad + g_1(\mathcal{H}\nabla_{X_1}\omega_J X_2 - \mathcal{H}\nabla_{X_2}\omega_J X_1, \omega_J S_J U_1) \end{aligned}$$

for any $X_1, X_2 \in \Gamma(D_1^J)$ *and* $U_1 \in \Gamma(D^J \oplus D_2^J)$.
(iv)

$$\begin{aligned} &g_1(\mathcal{T}_{X_1}\omega_K\phi_K X_2 - \mathcal{T}_{X_2}\omega_K\phi_K X_1, U_1) \\ &= g_1(\mathcal{T}_{X_1}\omega_K X_2 - \mathcal{T}_{X_2}\omega_K X_1, \phi_K P_K U_1) \end{aligned}$$

$$+ g_1(\mathcal{H}\nabla_{X_1}\omega_K X_2 - \mathcal{H}\nabla_{X_2}\omega_K X_1, \omega_K S_K U_1)$$

for any $X_1, X_2 \in \Gamma(D_1^K)$ *and* $U_1 \in \Gamma(D^K \oplus D_2^K)$.

Proof. Let $X_1, X_2 \in \Gamma(D_1^R)$, $U_1 \in \Gamma(D^R \oplus D_2^R)$, $U_2 \in (\ker \pi_*)^\perp$ and $R \in sp\{I, J, K\}$. Since $[X_1, X_2] \in (\ker \pi_*)$, we have $g_1([X_1, X_2], U_2) = 0$. Thus D_1^R is integrable $\Leftrightarrow g_1([X_1, X_2], U_1) = 0$. Using equations (2.4), (2.8), (2.9), (3.1), (3.2) and Lemma 3.7, we have

$$\begin{aligned}
& g_1([X_1, X_2], U_1) \\
= {} & g_1(\nabla_{X_1} R X_2, R U_1) - g_1(\nabla_{X_2} R X_1, R U_1), \\
= {} & g_1(\nabla_{X_1}\phi_R X_2, R U_1) + g_1(\nabla_{X_1}\omega_R X_2, R U_1) \\
& - g_1(\nabla_{X_2}\phi_R X_1, R U_1) - g_1(\nabla_{X_2}\omega_R X_1, R U_1), \\
= {} & \cos^2\theta_R g_1(\nabla_{X_1} X_2, U_1) - \cos^2\theta_R g_1(\nabla_{X_2} X_1, U_1) \\
& - g_1(\mathcal{T}_{X_1}\omega_R\phi_R X_2 - \mathcal{T}_{X_2}\omega_R\phi_R X_1, U_1) \\
& + g_1(\mathcal{H}\nabla_{X_1}\omega_R X_2 + \mathcal{T}_{X_1}\omega_R X_2, R P_R U_1 + \omega_R S_R U_1) \\
& - g_1(\mathcal{H}\nabla_{X_2}\omega_R X_1 + \mathcal{T}_{X_2}\omega_R X_1, R P_R U_1 + \omega_R S_R U_1).
\end{aligned}$$

Now, we have

$$\begin{aligned}
\sin^2\theta_1 g_1([X_1, X_2], U_1) = {} & g_1(\mathcal{T}_{X_1}\omega_R X_2 - \mathcal{T}_{X_2}\omega_R X_1, R P_R U_1) \\
& + g_1(\mathcal{H}\nabla_{X_1}\omega_R X_2 - \mathcal{H}\nabla_{X_2}\omega_R X_1, \omega_R S_R U_1) \\
& - g_1(\mathcal{T}_{X_1}\omega_R\phi_R X_2 - \mathcal{T}_{X_2}\omega_R\phi_R X_1, U_1).
\end{aligned}$$

Since D_1^R is R-slant distribution, we get

$$\text{(i)} \Leftrightarrow \text{(ii)}, \ \text{(i)} \Leftrightarrow \text{(iii)}, \ \text{(i)} \Leftrightarrow \text{(iv)}.$$

□

Theorem 3.10. *Let π be an almost h-quasi-hemi-slant submersion from a hyperkähler manifold (N_1, I, J, K, g_1) onto a Riemannian manifold (N_2, g_2) such that (I, J, K) is an almost h-quasi-hemi-slant basis. Then, anti-invariant distribution D_2^R is always integrable.*

Proof. The proof is easily done for the hemi-slant case given in [29]. □

Theorem 3.11. *Let π be an almost h-quasi-hemi-slant submersion from a hyperkähler manifold (N_1, I, J, K, g_1) onto a Riemannian manifold (N_2, g_2) such that (I, J, K) is an almost h-quasi-hemi-slant basis. Then the following conditions are equivalent:*

(i) *the distribution* $(\ker \pi_*)^\perp$ *is integrable.*

(ii)

$$\begin{aligned}
& g_1(\mathcal{V}\nabla_{Y_1} B_I Y_2 - \mathcal{V}\nabla_{Y_2} B_I Y_1, I V_1) \\
= {} & - g_2(\pi_*(C_I Y_2), (\nabla\pi_*)(Y_1, I V_1)) + g_2(\pi_*(C_I Y_1), (\nabla\pi_*)(Y_2, I V_1)), \\
& g_1(\mathcal{A}_{Y_1} B_I Y_2 - \mathcal{A}_{Y_2} B_I Y_1, \omega_I Q_I V_2) \\
= {} & g_2((\nabla\pi_*)(Y_1, C_I Y_2), \pi_*(\omega_I Q_I V_2)) + g_2((\nabla\pi_*)(Y_2, C_I Y_1), \pi_*(\omega_I Q_I V_2)),
\end{aligned}$$

$$g_1(\mathcal{A}_{Y_1}B_IY_2 - \mathcal{A}_{Y_2}B_IY_1, \omega_I S_I V_3)$$
$$= g_2((\nabla\pi_*)(Y_1, C_IY_2), \pi_*(\omega_I S_I V_3)) + g_2((\nabla\pi_*)(Y_2, C_IY_1), \pi_*(\omega_I S_I V_3))$$

for all $Y_1, Y_2 \in \Gamma(\ker\pi_*)^\perp$, $V_1 \in \Gamma(D^I)$, $V_2 \in \Gamma(D_1^I)$ *and* $V_3 \in \Gamma(D_2^I)$.

(iii)

$$g_1(\mathcal{V}\nabla_{Y_1}B_JY_2 - \mathcal{V}\nabla_{Y_2}B_JY_1, JV_1)$$
$$= -g_2(\pi_*(C_JY_2), (\nabla\pi_*)(Y_1, JV_1)) + g_2(\pi_*(C_JY_1), (\nabla\pi_*)(Y_2, JV_1)),$$
$$g_1(\mathcal{A}_{Y_1}B_JY_2 - \mathcal{A}_{Y_2}BY_1, \omega_J Q_J V_2)$$
$$= g_2((\nabla\pi_*)(Y_1, C_JY_2), \pi_*(\omega_J Q_J V_2)) + g_2((\nabla\pi_*)(Y_2, C_JY_1), \pi_*(\omega_J Q_J V_2)),$$
$$g_1(\mathcal{A}_{Y_1}B_JY_2 - \mathcal{A}_{Y_2}B_JY_1, \omega_J S_J V_3)$$
$$= g_2((\nabla\pi_*)(Y_1, C_JY_2), \pi_*(\omega_J S_J V_3)) + g_2((\nabla\pi_*)(Y_2, C_JY_1), \pi_*(\omega_J S_J V_3))$$

for all $Y_1, Y_2 \in \Gamma(\ker\pi_*)^\perp$, $V_1 \in \Gamma(D^J)$, $V_2 \in \Gamma(D_1^J)$ *and* $V_3 \in \Gamma(D_2^J)$.

(iv)

$$g_1(\mathcal{V}\nabla_{Y_1}B_KY_2 - \mathcal{V}\nabla_{Y_2}B_KY_1, KV_1)$$
$$= -g_2(\pi_*(C_KY_2), (\nabla\pi_*)(Y_1, KV_1)) + g_2(\pi_*(C_KY_1), (\nabla\pi_*)(Y_2, KV_1)),$$
$$g_1(\mathcal{A}_{Y_1}B_KY_2 - \mathcal{A}_{Y_2}B_KY_1, \omega_K Q_K V_2)$$
$$= g_2((\nabla\pi_*)(Y_1, C_KY_2), \pi_*(\omega_K Q_K V_2)) + g_2((\nabla\pi_*)(Y_2, C_KY_1), \pi_*(\omega_K Q_K V_2)),$$
$$g_1(\mathcal{A}_{Y_1}B_KY_2 - \mathcal{A}_{Y_2}B_KY_1, \omega_K S_K V_3)$$
$$= g_2((\nabla\pi_*)(Y_1, C_KY_2), \pi_*(\omega_K S_K V_3)) + g_2((\nabla\pi_*)(Y_2, C_KY_1), \pi_*(\omega_K S_K V_3))$$

for all $Y_1, Y_2 \in \Gamma(\ker\pi_*)^\perp$, $V_1 \in \Gamma(D^K)$, $V_2 \in \Gamma(D_1^K)$ *and* $V_3 \in \Gamma(D_2^K)$.

Proof. Let $Y_1, Y_2 \in \Gamma(\ker\pi_*)^\perp$, $V_1 \in \Gamma(D^R)$, $V_2 \in \Gamma(D_1^R)$, $V_3 \in \Gamma(D_2^R)$ and $R \in sp\{I, J, K\}$. Using equations (2.5), (2.8), (2.9) and (3.3), we have

$$\begin{aligned} g_1([Y_1, Y_2], V_1) &= g_1(\nabla_{Y_1}RY_2, RV_1) - g_1(\nabla_{Y_2}RY_1, RV_1) \\ &= g_1(\mathcal{V}\nabla_{Y_1}B_RY_2 - \mathcal{V}\nabla_{Y_2}B_RY_1, RV_1) \\ &\quad - g_1(C_RY_2, \nabla_{Y_1}RV_1) + g_1(C_RY_1, \nabla_{Y_2}RV_1). \end{aligned}$$

From equation (2.7), we get

$$\begin{aligned} g_1([Y_1, Y_2], V_1) &= g_1(\mathcal{V}\nabla_{Y_1}B_RY_2 - \mathcal{V}\nabla_{Y_2}B_RY_1, RV_1) \\ &\quad + g_2(\pi_*(C_RY_2), (\nabla\pi_*)(Y_1, RV_1)) \\ &\quad - g_2(\pi_*(C_RY_1), (\nabla\pi_*)(Y_2, RV_1)). \end{aligned}$$

Using equations (2.5), (2.6), (2.8), (2.9), (3.1), (3.2), (3.3) and Lemma 3.7, we get

$$\begin{aligned} &g_1([Y_1, Y_2], V_2) \\ &= g_1(R\nabla_{Y_1}Y_2, \phi_R Q_R V_2) + g_1(R\nabla_{Y_1}Y_2, \omega_R Q_R V_2) \\ &\quad - g_1(R\nabla_{Y_2}Y_1, \phi_R Q_R V_2) - g_1(R\nabla_{Y_2}Y_1, \omega_R Q_R V_2), \\ &= \cos^2\theta_R g_1([Y_1, Y_2], V_2) - g_1(\nabla_{Y_1}Y_2, \omega_R\phi_R Q_R V_2) + g_1(\nabla_{Y_2}Y_1, \omega_R\phi_R Q_R V_2) \end{aligned}$$

$$+ g_1(\nabla_{Y_1} B_R Y_2, \omega_R Q_R V_2) + g_1(\nabla_{Y_1} C_R Y_2, \omega_R Q_R V_2)$$
$$- g_1(\nabla_{Y_2} B_R Y_1, \omega_R Q_R V_2) - g_1(\nabla_{Y_2} C_R Y_1, \omega_R Q_R V_2).$$

From equation (2.7), we get

$$\begin{aligned} &\sin^2\theta_R g_1([Y_1, Y_2], V_2) \\ =\ & g_1(\mathcal{A}_{Y_1} B_R Y_2 - \mathcal{A}_{Y_2} B_R Y_1, \omega_R Q_R V_2) - g_2((\nabla\pi_*)(Y_1, C_R Y_2), \pi_*(\omega_R Q_R V_2)) \\ &+ g_2((\nabla\pi_*)(Y_2, C_R Y_1), \pi_*(\omega_R Q_R V_2)). \end{aligned}$$

Similarly, we have

$$\begin{aligned} &g_1([Y_1, Y_2], V_3) \\ =\ & g_1(\mathcal{A}_{Y_1} B_R Y_2 - \mathcal{A}_{Y_2} B_R Y_1,\ \omega_R S_R V_3) - g_2((\nabla\pi_*)(Y_1,\ C_R Y_2),\ \pi_*(\omega_R S_R V_3)) \\ &+ g_2((\nabla\pi_*)(Y_2,\ C_R Y_1),\ \pi_*(\omega_R S_R V_3)). \end{aligned}$$

Thus, we have

$$\text{(i)} \Leftrightarrow \text{(ii)},\ \text{(i)} \Leftrightarrow \text{(iii)},\ \text{(i)} \Leftrightarrow \text{(iv)}. \qquad \square$$

Theorem 3.12. *Let π be an almost h-quasi-hemi-slant submersion from a hyperkähler manifold (N_1, I, J, K, g_1) onto a Riemannian manifold (N_2, g_2) such that (I, J, K) is an almost h-quasi-hemi-slant basis. Then the following conditions are equivalent*:

(i) *the distribution $(\ker\pi_*)^\perp$ defines a totally geodesic foliation on N_1.*

(ii)

$$\begin{aligned} &g_1(\mathcal{A}_{Y_1} Y_2, P_I V_1 + \cos^2\theta_I Q_I V_1) \\ =\ & g_1(\mathcal{H}\nabla_{Y_1} Y_2, \omega_I \phi_I P_I V_1 + \omega_I \phi_I Q_I V_1) - g_1(\mathcal{A}_{Y_1} B_I Y_2 + \mathcal{H}\nabla_{Y_1} C_I Y_2, \omega_I V_1) \end{aligned}$$

for any $Y_1, Y_2 \in \Gamma(\ker\pi_*)^\perp$, $V_1 \in \Gamma(\ker\pi_*)$.

(iii)

$$\begin{aligned} &g_1(\mathcal{A}_{Y_1} Y_2, P_J V_1 + \cos^2\theta_J Q_J V_1) \\ =\ & g_1(\mathcal{H}\nabla_{Y_1} Y_2, \omega_J \phi_J P_J V_1 + \omega_J \phi_J Q_J V_1) - g_1(\mathcal{A}_{Y_1} B_J Y_2 + \mathcal{H}\nabla_{Y_1} C_J Y_2, \omega_J V_1) \end{aligned}$$

for any $Y_1, Y_2 \in \Gamma(\ker\pi_*)^\perp$, $V_1 \in \Gamma(\ker\pi_*)$.

(iv)

$$\begin{aligned} &g_1(\mathcal{A}_{Y_1} Y_2, P_K V_1 + \cos^2\theta_K Q_K V_1) \\ =\ & g_1(\mathcal{H}\nabla_{Y_1} Y_2, \omega_K \phi_K P_K V_1 + \omega_K \phi_K Q_K V_1) - g_1(\mathcal{A}_{Y_1} B_K Y_2 + \mathcal{H}\nabla_{Y_1} C_K Y_2, \omega_K V_1) \end{aligned}$$

for any $Y_1, Y_2 \in \Gamma(\ker\pi_*)^\perp$, $V_1 \in \Gamma(\ker\pi_*)$.

Proof. Let $Y_1, Y_2 \in \Gamma(\ker\pi_*)^\perp$, $V_1 \in \Gamma(\ker\pi_*)$ and $R \in sp\{I, J, K\}$. Using equations (2.5), (2.6), (2.8), (2.9), (3.1), (3.2), (3.3) and Lemma 3.7, we have

$$\begin{aligned} &g_1(\nabla_{Y_1} Y_2, V_1) \\ =\ & g_1(R\nabla_{Y_1} Y_2, RV_1) \\ =\ & g_1(R\nabla_{Y_1} Y_2, \phi_R P_R V_1 + \phi_R Q_R V_1 + \omega_R Q_R V_1 + \omega_R S_R V_1) \end{aligned}$$

$$= -g_1(\nabla_{Y_1}Y_2, \phi_R^2 P_R V_1 + \omega_R\phi_R P_R V_1 + \omega_R\phi_R Q_R V_1)$$
$$+ g_1(\nabla_{Y_1}B_R Y_2, \omega_R Q_R V_1 + \omega_R S_R V_1) + g_1(\nabla_{Y_1}C_R Y_2, \omega_R Q_R V_1 + \omega_R S_R V_1)$$
$$= g_1(\mathcal{A}_{Y_1}Y_2, P_R V_1 + \cos^2\theta_R Q_R V_1) - g_1(\mathcal{H}\nabla_{Y_1}Y_2, \omega_R\phi_R P_R V_1 + \omega_R\phi_R Q_R V_1)$$
$$+ g_1(\mathcal{A}_{Y_1}B_R Y_2, \omega_R Q_R V_1 + \omega_R S_R V_1) + g_1(\mathcal{H}\nabla_{Y_1}C_R Y_2, \omega_R Q_R V_1 + \omega_R S_R V_1).$$

Thus, we have

$$\text{(i)} \Leftrightarrow \text{(ii)},\ \text{(i)} \Leftrightarrow \text{(iii) and (i)} \Leftrightarrow \text{(iv)}.$$ □

Theorem 3.13. *Let π be an almost h-quasi-hemi-slant submersion from a hyperkähler manifold (N_1, I, J, K, g_1) onto a Riemannian manifold (N_2, g_2) such that (I, J, K) is an almost h-quasi-hemi-slant basis. Then the following conditions are equivalent*:

(i) *the distribution* $(\ker\pi_*)$ *defines a totally geodesic foliation on* N_1.

(ii)
$$\begin{aligned} &g_1(\mathcal{T}_{V_1}P_I V_2 + \cos^2\theta_I \mathcal{T}_{V_1}Q_I V_2, V_3)\\ &= g_1(\mathcal{H}\nabla_{V_1}\omega_I\phi_I P_I V_2 + \mathcal{H}\nabla_{V_1}\omega_I\phi_I Q_I V_2, V_3)\\ &\quad - g_1(\mathcal{H}\nabla_{V_1}\omega_I Q_I V_2 + \mathcal{H}\nabla_{V_1}\omega_I S_I V_2, C_I V_3)\\ &\quad - g_1(\mathcal{T}_{V_1}\omega_I Q_I V_2 + \mathcal{T}_{V_1}\omega_I S_I V_2, B_I V_3)\end{aligned}$$

for any $V_1, V_2 \in \Gamma(\ker\pi_*)$ *and* $V_3 \in \Gamma(\ker\pi_*)^\perp$.

(iii)
$$\begin{aligned} &g_1(\mathcal{T}_{V_1}P_J V_2 + \cos^2\theta_J \mathcal{T}_{V_1}Q_J V_2, V_3)\\ &= g_1(\mathcal{H}\nabla_{V_1}\omega_J\phi_J P_J V_2 + \mathcal{H}\nabla_{V_1}\omega_J\phi_J Q_J V_2, V_3)\\ &\quad - g_1(\mathcal{H}\nabla_{V_1}\omega_J Q_J V_2 + \mathcal{H}\nabla_{V_1}\omega_J S_J V_2, C_J V_3)\\ &\quad - g_1(\mathcal{T}_{V_1}\omega_J Q_J V_2 + \mathcal{T}_{V_1}\omega_J S_J V_2, B_J V_3)\end{aligned}$$

for any $V_1, V_2 \in \Gamma(\ker\pi_*)$ *and* $V_3 \in \Gamma(\ker\pi_*)^\perp$.

(iv)
$$\begin{aligned} &g_1(\mathcal{T}_{V_1}P_K V_2 + \cos^2\theta_K \mathcal{T}_{V_1}Q_K V_2, V_3)\\ &= g_1(\mathcal{H}\nabla_{V_1}\omega_K\phi_K P_K V_2 + \mathcal{H}\nabla_{V_1}\omega_K\phi_K Q_K V_2, V_3)\\ &\quad - g_1(\mathcal{H}\nabla_{V_1}\omega_K Q_K V_2 + \mathcal{H}\nabla_{V_1}\omega_K S_K V_2, C_K V_3)\\ &\quad - g_1(\mathcal{T}_{V_1}\omega_K Q_K V_2 + \mathcal{T}_{V_1}\omega_K S_K V_2, B_K V_3)\end{aligned}$$

for any $V_1, V_2 \in \Gamma(\ker\pi_*)$ *and* $V_3 \in \Gamma(\ker\pi_*)^\perp$

Proof. Let $V_1, V_2 \in \Gamma(\ker\pi_*)$, $V_3 \in \Gamma(\ker\pi_*)^\perp$ and $R \in sp\{I, J, K\}$. Using equations (2.3), (2.4), (2.8), (2.9), (3.1), (3.2), (3.3) and Lemma 3.7, we have

$$\begin{aligned} &g_1(\nabla_{V_1}V_2, V_3)\\ &= g_1(R\nabla_{V_1}V_2, RV_3)\\ &= g_1(\nabla_{V_1}\phi_R P_R V_2, RV_3) + g_1(\nabla_{V_1}\phi_R Q_R V_2, RV_3)\end{aligned}$$

$$
\begin{aligned}
&+ g_1(\nabla_{V_1}\omega_R Q_R V_2, RV_3) + g_1(\nabla_{V_1}\omega_R S_R V_2, RV_3)\\
&= g_1(\mathcal{T}_{V_1} P_R V_2, V_3) + \cos^2\theta_R g_1(\mathcal{T}_{V_1} Q_R V_2, V_3) - g_1(\mathcal{H}\nabla_{V_1}\omega_R\phi_R P_R V_2, V_3)\\
&- g_1(\mathcal{H}\nabla_{V_1}\omega_R\phi_R Q_R V_2, V_3) + g_1(\mathcal{H}\nabla_{V_1}\omega_R Q_R V_2 + \mathcal{H}\nabla_{V_1}\omega_R S_R V_2, C_R V_3)\\
&+ g_1(\mathcal{T}_{V_1}\omega_R Q_R V_2 + \mathcal{T}_{V_1}\omega_R S_R V_2, B_R V_3).
\end{aligned}
$$

Thus, we get

$$\text{(i)} \Leftrightarrow \text{(ii)},\ \text{(i)} \Leftrightarrow \text{(iii)},\ \text{(i)} \Leftrightarrow \text{(iv)}. \qquad \square$$

Theorem 3.14. *Let π be an almost h-quasi-hemi-slant submersion from a hyperkähler manifold (N_1, I, J, K, g_1) onto a Riemannian manifold (N_2, g_2) such that (I, J, K) is an almost h-quasi-hemi-slant basis. Then the following conditions are equivalent*:

(i) *the distribution D^R defines a totally geodesic foliation on N_1.*

(ii)

$$
\begin{aligned}
g_1(\mathcal{T}_{V_1} IP_I V_2, \omega_I Q_I U_1 + \omega_I S_I U_1) &= -g_1(\mathcal{V}\nabla_{V_1} IP_I V_2, \phi_I U_1),\\
g_1(\mathcal{T}_{V_1} IP_I V_2, C_I U_2) &= -g_1(\mathcal{V}\nabla_{V_1} IP_I V_2, B_I U_2)
\end{aligned}
$$

for any $V_1, V_2 \in \Gamma(D^I)$, $U_1 \in \Gamma(D_1^I \oplus D_2^I)$ and $U_2 \in \Gamma(\ker \pi_*)^\perp$.

(iii)

$$
\begin{aligned}
g_1(\mathcal{T}_{V_1} JP_J V_2, \omega_J Q_J U_1 + \omega_J S_J U_1) &= -g_1(\mathcal{V}\nabla_{V_1} JP_J V_2, \phi_J U_1),\\
g_1(\mathcal{T}_{V_1} JP_J V_2, C_J U_2) &= -g_1(\mathcal{V}\nabla_{V_1} JP_J V_2, B_J U_2)
\end{aligned}
$$

for any $V_1, V_2 \in \Gamma(D^J)$, $U_1 \in \Gamma(D_1^J \oplus D_2^J)$ and $U_2 \in \Gamma(\ker \pi_*)^\perp$.

(iv)

$$
\begin{aligned}
g_1(\mathcal{T}_{V_1} KP_K V_2, \omega_K Q_K U_1 + \omega_K S_K U_1) &= -g_1(\mathcal{V}\nabla_{V_1} KP_K V_2, \phi_K U_1),\\
g_1(\mathcal{T}_{V_1} KP_K V_2, C_K U_2) &= -g_1(\mathcal{V}\nabla_{V_1} KP_K V_2, B_K U_2)
\end{aligned}
$$

for any $V_1, V_2 \in \Gamma(D^K)$, $U_1 \in \Gamma(D_1^K \oplus D_2^K)$ and $U_2 \in \Gamma(\ker \pi_*)^\perp$.

Proof. Let $V_1, V_2 \in \Gamma(D^R)$, $U_1 \in \Gamma(D_1^R \oplus D_2^R)$, $U_2 \in \Gamma(\ker \pi_*)^\perp$ and $R \in sp\{I, J, K\}$. Using equations (2.3), (2.8), (2.9), (3.1) and (3.2), we have

$$
\begin{aligned}
&g_1(\nabla_{V_1} V_2, U_1)\\
&= g_1(\nabla_{V_1} RV_2, RU_1)\\
&= g_1(\nabla_{V_1} RP_R V_2, RQ_R U_1 + RS_R U_1)\\
&= g_1(\mathcal{T}_{V_1} RP_R V_2, \omega_R Q_R U_1 + \omega_R S_R U_1) + g_1(\mathcal{V}\nabla_{V_1} RP_R V_2, \phi_R Q_R U_1).
\end{aligned}
$$

Again, using equations (2.9), (3.1) and (3.3), we have

$$
\begin{aligned}
&g_1(\nabla_{V_1} V_2, U_2)\\
&= g_1(\nabla_{V_1} RV_2, RU_2)\\
&= g_1(\nabla_{V_1} RP_R V_2, B_R U_2 + C_R U_2)\\
&= g_1(\mathcal{V}\nabla_{V_1} RP_R V_2, B_R U_2) + g_1(\mathcal{T}_{V_1} JP_R V_2, C_R U_2).
\end{aligned}
$$

Thus, we get

$$\text{(i)} \Leftrightarrow \text{(ii)},\ \text{(i)} \Leftrightarrow \text{(iii)},\ \text{(i)} \Leftrightarrow \text{(iv)}. \qquad \square$$

Theorem 3.15. *Let π be an almost h-quasi-hemi-slant submersion from a hyperkähler manifold (N_1, I, J, K, g_1) onto a Riemannian manifold (N_2, g_2) such that (I, J, K) is an almost h-quasi-hemi-slant basis. Then the following conditions are equivalent*:

(i) *the distribution D_1^R defines a totally geodesic foliation on N_1.*

(ii)

$$g_1(\mathcal{T}_{V_1}\omega_I\phi_I V_2, U_1) = g_1(\mathcal{T}_{V_1}\omega_I V_2, \phi_I P_I U_1) + g_1(\mathcal{H}\nabla_{V_1}\omega_I V_2, \omega_I S_I U_1),$$
$$g_1(\mathcal{H}\nabla_{V_1}\omega_I\phi_I V_2, U_2) = g_1(\mathcal{H}\nabla_{V_1}\omega_I V_2, C_I U_2) + g_1(\mathcal{T}_{V_1}\omega_I V_2, B_I U_2)$$

for any $V_1, V_2 \in \Gamma(D_1^I)$, $U_1 \in \Gamma(D^I \oplus D_2^I)$, $U_2 \in \Gamma(\ker \pi_*)^\perp$.

(iii)

$$g_1(\mathcal{T}_{V_1}\omega_J\phi_J V_2, U_1) = g_1(\mathcal{T}_{V_1}\omega_J V_2, \phi_J P_J U_1) + g_1(\mathcal{H}\nabla_{V_1}\omega_J V_2, \omega_J S_J U_1),$$
$$g_1(\mathcal{H}\nabla_{V_1}\omega_J\phi_J V_2, U_2) = g_1(\mathcal{H}\nabla_{V_1}\omega_J V_2, C_J U_2) + g_1(\mathcal{T}_{V_1}\omega_J V_2, B_J U_2)$$

for any $V_1, V_2 \in \Gamma(D_1^J)$, $U_1 \in \Gamma(D^J \oplus D_2^J)$, $U_2 \in \Gamma(\ker \pi_*)^\perp$.

(iv)

$$g_1(\mathcal{T}_{V_1}\omega_K\phi_K V_2, U_1) = g_1(\mathcal{T}_{V_1}\omega_K V_2, \phi_K P_K U_1) + g_1(\mathcal{H}\nabla_{V_1}\omega_K V_2, \omega_K S_K U_1),$$
$$g_1(\mathcal{H}\nabla_{V_1}\omega_K\phi_K V_2, U_2) = g_1(\mathcal{H}\nabla_{V_1}\omega_K V_2, C_K U_2) + g_1(\mathcal{T}_{V_1}\omega_K V_2, B_K U_2)$$

for any $V_1, V_2 \in \Gamma(D_1^K)$, $U_1 \in \Gamma(D^K \oplus D_2^K)$, $U_2 \in \Gamma(\ker \pi_*)^\perp$.

Proof. Let $V_1, V_2 \in \Gamma(D_1^R)$, $U_1 \in \Gamma(D^R \oplus D_2^R)$, $U_2 \in \Gamma(\ker \pi_*)^\perp$ and $R \in sp\{I, J, K\}$. Using equations (2.4), (2.8), (2.9), (3.1), (3.2) and Lemma 3.7, we have

$$\begin{aligned}
& g_1(\nabla_{V_1} V_2, U_1) \\
&= g_1(\nabla_{V_1} R V_2, R U_1) \\
&= g_1(\nabla_{V_1}\phi_R V_2, R U_1) + g_1(\nabla_{V_1}\omega_R V_2, R U_1) \\
&= \cos^2\theta_R g_1(\nabla_{V_1} V_2, U_1) - g_1(\mathcal{T}_{V_1}\omega_R\phi_R V_2, U_1) \\
&\quad + g_1(\mathcal{T}_{V_1}\omega_R V_2, \phi_R P_R U_1) + g_1(\mathcal{H}\nabla_{V_1}\omega_R V_2, \omega_R S_R U_1).
\end{aligned}$$

So, we have

$$\begin{aligned}
& \sin^2\theta_R g_1(\nabla_{V_1} V_2, U_1) \\
&= -g_1(\mathcal{T}_{V_1}\omega_R\phi_R V_2, U_1) + g_1(\mathcal{T}_{V_1}\omega_R V_2, R P_R U_1) + g_1(\mathcal{H}\nabla_{V_1}\omega_R V_2, \omega_R S_R U_1).
\end{aligned}$$

From equations (2.4), (2.8), (2.9), (3.2), (3.3) and Lemma 3.7, we get

$$\begin{aligned}
& g_1(\nabla_{V_1} V_2, U_2) \\
&= g_1(\nabla_{V_1} R V_2, R U_2), \\
&= g_1(\nabla_{V_1}\phi_R V_2, R U_2) + g_1(\nabla_{V_1}\omega_R V_2, R U_2), \\
&= \cos^2\theta_R g_1(\nabla_{V_1} V_2, U_2) - g_1(\mathcal{H}\nabla_{V_1}\omega_R\phi_R V_2, U_2)
\end{aligned}$$

$$+ g_1(\mathcal{H}\nabla_{V_1}\omega_R V_2, C_R U_2) + g_1(\mathcal{T}_{V_1}\omega_R V_2, B_R U_2).$$

Thus, we have

$$\begin{aligned}&\sin^2\theta_R g_1(\nabla_{V_1}V_2, U_2)\\ =& -g_1(\mathcal{H}\nabla_{V_1}\omega_R\phi_R V_2, U_2) + g_1(\mathcal{H}\nabla_{V_1}\omega_R V_2, C_R U_2) + g_1(\mathcal{T}_{V_1}\omega_R V_2, B_R U_2).\end{aligned}$$

Then, we get

$$\text{(i)} \Leftrightarrow \text{(ii)},\ \text{(i)} \Leftrightarrow \text{(iii)},\ \text{(i)} \Leftrightarrow \text{(iv)}. \qquad \square$$

Theorem 3.16. *Let π be an almost h-quasi-hemi-slant submersion from a hyperkähler manifold (N_1, I, J, K, g_1) onto a Riemannian manifold (N_2, g_2) such that (I, J, K) is an almost h-quasi-hemi-slant basis. Then the following conditions are equivalent*:

(i) *the distribution D_2^R defines a totally geodesic foliation on N_1.*

(ii)

$$g_1(\mathcal{H}\nabla_{V_1}\omega_I V_2, \omega_I Q_I X_1) = -g_1(\mathcal{T}_{V_1}\omega_I S_I V_2, \phi_I P_I X_1 + \phi_I Q_I X_1),$$
$$g_1(\mathcal{H}\nabla_{V_1}\omega_I S_I V_2, C_I X_2) = -g_1(\mathcal{T}_{V_1}\omega_I S_I V_2, B_I X_2)$$

for any $V_1, V_2 \in \Gamma(D_2^I)$, $X_1 \in \Gamma(D^I \oplus D_1^I)$, $X_2 \in \Gamma(\ker\pi_*)^\perp$.

(iii)

$$g_1(\mathcal{H}\nabla_{V_1}\omega_J V_2, \omega_J Q_J X_1) = -g_1(\mathcal{T}_{V_1}\omega_J S_J V_2, \phi_J P_J X_1 + \phi_J Q_J X_1),$$
$$g_1(\mathcal{H}\nabla_{V_1}\omega_J S V_2, C_J X_2) = -g_1(\mathcal{T}_{V_1}\omega_J S V_2, B_J X_2)$$

for any $V_1, V_2 \in \Gamma(D_2^J)$, $X_1 \in \Gamma(D^J \oplus D_1^J)$, $X_2 \in \Gamma(\ker\pi_*)^\perp$.

(iv)

$$g_1(\mathcal{H}\nabla_{V_1}\omega_K V_2, \omega_K Q_K X_1) = -g_1(\mathcal{T}_{V_1}\omega_K S V_2, \phi_K P_K X_1 + \phi_K Q_K X_1),$$
$$g_1(\mathcal{H}\nabla_{V_1}\omega_K S V_2, C_K X_2) = -g_1(\mathcal{T}_{V_1}\omega_K S_K V_2, B_K X_2)$$

for any $V_1, V_2 \in \Gamma(D_2^K)$, $X_1 \in \Gamma(D^K \oplus D_1^K)$, $X_2 \in \Gamma(\ker\pi_*)^\perp$.

Proof. Let $V_1, V_2 \in \Gamma(D_2^R)$, $X_1 \in \Gamma(D^R \oplus D_1^R)$, $X_2 \in \Gamma(\ker\pi_*)^\perp$ and $R \in sp\{I, J, K\}$. Using equations (2.4), (2.9), (3.1) and (3.2), we have

$$\begin{aligned}&g_1(\nabla_{V_1}V_2, X_1)\\ &= g_1(\nabla_{V_1}RV_2, RX_1)\\ &= g_1(\nabla_{V_1}\omega_R S_R V_2, \phi_R P_R X_1 + \phi_R Q_R X_1 + \omega_R Q_R X_1),\\ &= g_1(\mathcal{T}_{V_1}\omega_R S_R V_2, \phi_R P_R X_1 + \phi_R Q_R X_1) + g_1(\mathcal{H}\nabla_{V_1}\omega_R S_R V_2, \omega_R Q_R X_1).\end{aligned}$$

Similarly, using equations (2.4), (2.9), (3.1), (3.2) and (3.3), we get

$$\begin{aligned}g_1(\nabla_{V_1}V_2, X_2) &= g_1(\nabla_{V_1}RV_2, RX_2)\\ &= g_1(\nabla_{V_1}\omega_R S_R V_2, B_R X_2 + C_R X_2)\\ &= g_1(\mathcal{T}_{V_1}\omega_R S_R V_2, B_R X_2) + g_1(\mathcal{H}\nabla_{V_1}\omega_R R V_2, C_R X_2).\end{aligned}$$

Thus, we get

$$\text{(i)} \Leftrightarrow \text{(ii)},\ \text{(i)} \Leftrightarrow \text{(iii)},\ \text{(i)} \Leftrightarrow \text{(iv)}. \qquad \square$$

Theorem 3.17. *Let π be an almost h-quasi-hemi-slant submersion from a hyperkähler manifold (N_1, I, J, K, g_1) onto a Riemannian manifold (N_2, g_2) such that (I, J, K) is an almost h-quasi-hemi-slant basis. Then the following conditions are equivalent*:

(i) *π is a totally geodesic map.*

(ii)

$$\begin{aligned} & g_1(\mathcal{T}_{X_1}P_IX_2 + \cos^2\theta_I\mathcal{T}_{X_1}Q_IX_2 - \mathcal{H}\nabla_{X_1}\omega_I\phi_IP_IX_2 - \mathcal{H}\nabla_{X_1}\omega_I\phi_IQ_IX_2, Z_1) \\ = {} & g_1(\mathcal{T}_{X_1}\omega_IQ_IX_2 + \mathcal{T}_{X_1}\omega_IS_IX_2, B_IZ_1) \\ & + g_1(\mathcal{H}\nabla_{X_1}\omega_I\phi_IQ_IX_2 + \mathcal{H}\nabla_{X_1}\omega_I\phi_IS_IX_2, Z_1), \\ & g_1(\mathcal{A}_{Z_1}P_IX_1 + \cos^2\theta_I\mathcal{A}_{Z_1}Q_IX_1 - \mathcal{H}\nabla_{Z_1}\omega_I\phi_IP_IX_1 - \mathcal{H}\nabla_{Z_1}\omega_I\phi_IQ_IX_1, Z_2) \\ = {} & g_1(\mathcal{A}_{Z_1}\omega_IQIX_1 + \mathcal{A}_{Z_1}\omega_IS_IX_1, B_IZ_2) \\ & + g_1(\mathcal{H}\nabla_{Z_1}\omega_IQ_IX_1 + \mathcal{H}\nabla_{Z_1}\omega_IS_IX_1, C_IZ_2) \end{aligned}$$

for any $X_1, X_2 \in \Gamma(\ker\pi_*)$ and $Z_1, Z_2 \in \Gamma(\ker\pi_*)^\perp$.

(iii)

$$\begin{aligned} & g_1(\mathcal{T}_{X_1}P_JX_2 + \cos^2\theta_J\mathcal{T}_{X_1}Q_JX_2 - \mathcal{H}\nabla_{X_1}\omega_J\phi_JP_JX_2 - \mathcal{H}\nabla_{X_1}\omega_J\phi_JQ_JX_2, Z_1) \\ = {} & g_1(\mathcal{T}_{X_1}\omega_JQ_JX_2 + \mathcal{T}_{X_1}\omega_JS_JX_2, B_JZ_1) \\ & + g_1(\mathcal{H}\nabla_{X_1}\omega_J\phi_JQ_JX_2 + \mathcal{H}\nabla_{X_1}\omega_J\phi_JS_JX_2, Z_1), \\ & g_1(\mathcal{A}_{Z_1}P_JX_1 + \cos^2\theta_J\mathcal{A}_{Z_1}Q_JX_1 - \mathcal{H}\nabla_{Z_1}\omega_J\phi_JP_JX_1 - \mathcal{H}\nabla_{Z_1}\omega_J\phi_JQ_JX_1, Z_2) \\ = {} & g_1(\mathcal{A}_{Z_1}\omega_JQ_JX_1 + \mathcal{A}_{Z_1}\omega_JS_JX_1, B_JZ_2) \\ & + g_1(\mathcal{H}\nabla_{Z_1}\omega_JQ_JX_1 + \mathcal{H}\nabla_{Z_1}\omega_JS_JX_1, C_IZ_2) \end{aligned}$$

for any $X_1, X_2 \in \Gamma(\ker\pi_*)$ and $Z_1, Z_2 \in \Gamma(\ker\pi_*)^\perp$.

(iv)

$$\begin{aligned} & g_1(\mathcal{T}_{X_1}P_KX_2 + \cos^2\theta_K\mathcal{T}_{X_1}Q_KX_2 - \mathcal{H}\nabla_{X_1}\omega_K\phi_KP_KX_2 - \mathcal{H}\nabla_{X_1}\omega_K\phi_KQ_KX_2, Z_1) \\ = {} & g_1(\mathcal{T}_{X_1}\omega_KQ_KX_2 + \mathcal{T}_{X_1}\omega_KS_KX_2, B_KZ_1) \\ & + g_1(\mathcal{H}\nabla_{X_1}\omega_K\phi_KQ_KX_2 + \mathcal{H}\nabla_{X_1}\omega_K\phi_KS_KX_2, Z_1), \\ & g_1(\mathcal{A}_{Z_1}P_KX_1 + \cos^2\theta_K\mathcal{A}_{Z_1}Q_KX_1 - \mathcal{H}\nabla_{Z_1}\omega_K\phi_KP_KX_1 - \mathcal{H}\nabla_{Z_1}\omega_K\phi_KQ_KX_1, Z_2) \\ = {} & g_1(\mathcal{A}_{Z_1}\omega_KQ_KX_1 + \mathcal{A}_{Z_1}\omega_KS_KX_1, B_KZ_2) \\ & + g_1(\mathcal{H}\nabla_{Z_1}\omega_KQ_KX_1 + \mathcal{H}\nabla_{Z_1}\omega_KS_KX_1, C_KZ_2) \end{aligned}$$

for any $X_1, X_2 \in \Gamma(\ker\pi_*)$ and $Z_1, Z_2 \in \Gamma(\ker\pi_*)^\perp$.

Proof. Since π is a Riemannian submersion, we have

$$(\nabla\pi_*)(Z_1, Z_2) = 0$$

for any $Z_1, Z_2 \in \Gamma(\ker\pi_*)^\perp$.

For any $X_1, X_2 \in \Gamma(\ker\pi_*)$, $Z_1, Z_2 \in \Gamma(\ker\pi_*)^\perp$ and $R \in sp\{I, J, K\}$. Using equations (2.3), (2.4), (2.7), (2.8), (2.9), (3.1), (3.2), (3.3) and Lemma 3.7, we have

$$
\begin{aligned}
& g_2((\nabla\pi_*)(X_1,X_2),\pi_*(Z_1)) \\
= & -g_1(\nabla_{X_1}X_2,Z_1), \\
= & -g_1(\nabla_{X_1}RX_2,RZ_1), \\
= & -g_1(\nabla_{X_1}RP_RX_2,RZ_1)-g_1(\nabla_{X_1}RQ_RX_2,RZ_1)-g_1(\nabla_{X_1}RS_RX_2,RZ_1), \\
= & -g_1(\nabla_{X_1}\phi_RP_RX_2,RZ_1)-g_1(\nabla_{X_1}\phi_RQ_RX_2,RZ_1) \\
& -g_1(\nabla_{X_1}\omega_RQ_RX_2,RZ_1)-g_1(\nabla_{X_1}\omega_RS_RX_2,RZ_1), \\
= & -g_1(\mathcal{T}_{X_1}P_RX_2+\cos^2\theta_R\mathcal{T}_{X_1}Q_RX_2-\mathcal{H}\nabla_{X_1}\omega_R\phi_RP_RX_2-\mathcal{H}\nabla_{X_1}\omega_R\phi_RQ_RX_2,Z_1) \\
& -g_1(\mathcal{T}_{X_1}\omega_RQ_RX_2+\mathcal{T}_{X_1}\omega_RS_RX_2,B_RZ_1) \\
& -g_1(\mathcal{H}\nabla_{X_1}\omega_R\phi_RQ_RX_2+\mathcal{H}\nabla_{X_1}\omega_R\phi_RS_RX_2,Z_1).
\end{aligned}
$$

Next, using equations (2.5), (2.6), (2.7), (2.8), (2.9), (3.1), (3.2), (3.3) and Lemma 3.7, we have

$$
\begin{aligned}
& g_2((\nabla\pi_*)(Z_1,X_1),\pi_*(Z_2)) \\
= & -g_1(\nabla_{Z_1}X_1,Z_2), \\
= & -g_1(\nabla_{Z_1}RX_1,RZ_2), \\
= & -g_1(\nabla_{Z_1}RP_RX_1,RZ_2)-g_1(\nabla_{Z_1}RQ_RX_1,RZ_2)-g_1(\nabla_{Z_1}RS_RX_1,RZ_2), \\
= & -g_1(\nabla_{Z_1}\phi_RP_RX_1,RZ_2)-g_1(\nabla_{Z_1}\phi_RQ_RX_1,RZ_2) \\
& -g_1(\nabla_{Z_1}\omega_RQ_RX_1,RZ_2)-g_1(\nabla_{Z_1}\omega_RS_RX_1,RZ_2), \\
= & -g_1(\mathcal{A}_{Z_1}P_RX_1+\cos^2\theta_R\mathcal{A}_{Z_1}Q_RX_1-\mathcal{H}\nabla_{Z_1}\omega_R\phi_RP_RX_1-\mathcal{H}\nabla_{Z_1}\omega_R\phi_RQ_RX_1,Z_2) \\
& -g_1(\mathcal{A}_{Z_1}\omega_RQ_RX_1+\mathcal{A}_{Z_1}\omega_RS_RX_1,B_RZ_2) \\
& -g_1(\mathcal{H}\nabla_{Z_1}\omega_RQ_RX_1+\mathcal{H}\nabla_{Z_1}\omega_RS_RX_1,C_RZ_2).
\end{aligned}
$$

Thus, we get

$$\text{(i)}\Leftrightarrow\text{(ii)},\ \text{(i)}\Leftrightarrow\text{(iii) and (i)}\Leftrightarrow\text{(iv)}. \qquad \square$$

4. Example

Note that given an Euclidean space $\mathbb{R}^{4k}$ with coordinates $(x_1,x_2,\dots,x_{4k})$, we can canonically choose complex structures I,J,K on R^{4k} as follows:

$$
\begin{aligned}
& I(\frac{\partial}{\partial x_{4s+1}})=\frac{\partial}{\partial x_{4s+2}},I(\frac{\partial}{\partial x_{4s+2}})=-\frac{\partial}{\partial x_{4s+1}},I(\frac{\partial}{\partial x_{4s+3}})=\frac{\partial}{\partial x_{4s+4}}, \\
& I(\frac{\partial}{\partial x_{4s+4}})=-\frac{\partial}{\partial x_{4s+3}},J(\frac{\partial}{\partial x_{4s+1}})=\frac{\partial}{\partial x_{4s+3}},J(\frac{\partial}{\partial x_{4s+2}})=-\frac{\partial}{\partial x_{4s+4}}, \\
& J(\frac{\partial}{\partial x_{4s+3}})=-\frac{\partial}{\partial x_{4s+1}},J(\frac{\partial}{\partial x_{4s+4}})=\frac{\partial}{\partial x_{4s+2}},K(\frac{\partial}{\partial x_{4s+1}})=\frac{\partial}{\partial x_{4s+4}}, \\
& K(\frac{\partial}{\partial x_{4s+2}})=\frac{\partial}{\partial x_{4s+3}},K(\frac{\partial}{\partial x_{4s+3}})=-\frac{\partial}{\partial x_{4s+2}},K(\frac{\partial}{\partial x_{4s+4}})=-\frac{\partial}{\partial x_{4s+1}}
\end{aligned}
$$

for $s \in \{0, 1, 2, \ldots, k-1\}$. Thus, $(\mathbb{R}^{4k}, I, J, K, \langle\ ,\ \rangle)$ is a hyperkähler manifold where $\langle\ ,\ \rangle$ denotes the Euclidean metric on $\mathbb{R}^{4k}$.

These notations are used throughout this section.

Example 4.1. Define a map $\pi : \mathbb{R}^{12} \to \mathbb{R}^4$ by

$$\pi(x_1, x_2, \ldots, x_{12}) = (x_3, \frac{x_4 + x_5}{\sqrt{2}}, x_7, x_{11}).$$

The map π is a strictly h-quasi-hemi-slant Riemannian submersion such that

$$\ker \pi_* = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_5}), \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_8}, \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{12}} \right\rangle,$$

$$(\ker \pi_*)^\perp = \left\langle \frac{\partial}{\partial x_3}, \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5}), \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_{11}} \right\rangle,$$

$$D^I = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{10}} \right\rangle, D_1^I = \left\langle \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_5}), \frac{\partial}{\partial x_6} \right\rangle,$$

$$D_2^I = \left\langle \frac{\partial}{\partial x_8}, \frac{\partial}{\partial x_{12}} \right\rangle, D^J = \left\langle \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_8}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{12}} \right\rangle,$$

$$D_1^J = \left\langle \frac{\partial}{\partial x_2}, \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_5}) \right\rangle, D_2^J = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_9} \right\rangle,$$

$$D^K = \left\langle \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{12}} \right\rangle, D_1^K = \left\langle \frac{\partial}{\partial x_1}, \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_5}), \frac{\partial}{\partial x_8} \right\rangle,$$

$$D_2^K = \left\langle \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_{10}} \right\rangle,$$

with the almost h-quasi-hemi-slant angles $\{\theta_I = \frac{\pi}{4}, \theta_J = \frac{\pi}{4}, \theta_K = \frac{\pi}{4}\}$.

Example 4.2. Define a map $\pi : \mathbb{R}^{16} \to \mathbb{R}^6$ by

$$\pi(x_1, x_2, \ldots, x_{16}) = (\frac{x_1 - \sqrt{3}\ x_5}{2}, x_2, x_7, x_8, x_{13}, x_{16}).$$

Then the map π is an almost h-quasi-hemi-slant Riemannian submersion such that

$$\ker \pi_* = \left\langle \frac{1}{2}(\sqrt{3}\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_5}), \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}}, \frac{\partial}{\partial x_{14}}, \frac{\partial}{\partial x_{15}} \right\rangle,$$

$$(\ker \pi_*)^\perp = \left\langle \frac{1}{2}(\frac{\partial}{\partial x_1} - \sqrt{3}\frac{\partial}{\partial x_5}), \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8}, \frac{\partial}{\partial x_{13}}, H_4 = \frac{\partial}{\partial x_{16}} \right\rangle,$$

$$D^I = \left\langle \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}} \right\rangle, D_1^I = \left\langle \frac{1}{2}(\sqrt{3}\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_5}), \frac{\partial}{\partial x_6} \right\rangle,$$

$$D_2^I = \left\langle \frac{\partial}{\partial x_{14}}, \frac{\partial}{\partial x_{15}} \right\rangle, D^J = \left\langle \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}} \right\rangle,$$

$$D_1^J = \left\langle \frac{1}{2}(\sqrt{3}\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_5}), \frac{\partial}{\partial x_3} \right\rangle, D_2^J = \left\langle \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_{14}}, \frac{\partial}{\partial x_{15}} \right\rangle,$$

$$D^K = \left\langle \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}}, \frac{\partial}{\partial x_{14}}, \frac{\partial}{\partial x_{15}} \right\rangle, D_1^K = \left\langle \frac{1}{2}(\sqrt{3}\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_5}), \frac{\partial}{\partial x_4} \right\rangle,$$
$$D_2^K = \left\langle \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_6} \right\rangle,$$

with the almost h-quasi-hemi-slant angles $\{\theta_I = \frac{\pi}{3}, \theta_J = \frac{\pi}{6}, \theta_K = \frac{\pi}{6}\}$.

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