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# GENERALIZED HYPERBOLIC GEOMETRIC FLOW

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Abstract. In the present paper, we consider a kind of generalized hyperbolic geometric flow which has a gradient form. Firstly, we establish the existence and uniqueness for the solution of this flow on an $n$-dimensional closed Riemannian manifold. Then, we give the evolution of some geometric structures of the manifold along this flow.

## 1. Introduction

A geometric flow on a manifold $M$ is a type of partial differential equation for a geometric structure such as metric tensor or an embedding under a differential equation with a functional on a manifold which has a geometric interpreting, usually associated with some extrinsic or intrinsic curvature. Such flows are fundamentally related to the calculus of variations and are important tool in differential geometry and physics. Let $M$ be an $n$-dimensional complete Riemannian manifold with the Riemannian metric $g = (g_{ij})$. Intrinsic geometric flows independent of any embedding or immersion and usually are flows on the metric tensor. The first important intrinsic geometric flow is Ricci flow which for the first time introduced by R. Hamilton in 1982 [9] as follows:

$$\frac{\partial}{\partial t}g = -2Ric, \quad g(0) = g_0, \tag{1}$$

where $Ric$ is the Ricci curvature of $g$ and Ricci flow evolves a Riemannian metric by its Ricci curvature. R. Hamilton (see [9]) and D. DeTurck (see [8]) proved the short-time existence and uniqueness for the solution of the Ricci flow on compact Riemannian manifolds. Also, evolution equations for geometric objects dependant to the metric investigated by some researchers along the geometric flow, in especially Ricci flow (see [6]).

The second important geometric flow which is a generalization of Ricci flow is called the Ricci-Bourguignon flow and is defined by

$$\frac{\partial}{\partial t}g = -2Ric + 2\rho Rg = -2(Ric - \rho Rg), \quad g(0) = g_0. \tag{2}$$

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Here $R$ and $\rho$ are the scalar curvatures of $g$ and a real constant, respectively. For the first time, Bourguignon (see [4]) introduced Ricci-Bourguignon flow and then Catino et al. [5] showed the unique short-time existence for solution to the Ricci-Bourguignon flow on $[0,T)$ for $\rho < \frac{1}{2(n-1)}$.

Another important geometric flow which is also the generalization of the Ricci flow called the harmonic-Ricci flow [12] and is defined by

$$\begin{cases} \frac{\partial}{\partial t}g(t) = -2Ric + 2\alpha(t)\nabla\phi\otimes\nabla\phi, & g(0)=g_0,\\ \frac{\partial}{\partial t}\phi = \tau_g\phi, & \phi(0)=\phi_0, \end{cases} \tag{3}$$

where $\alpha(t) > 0$ is a smooth function and depends only on $m$ and $t$, $\phi(t) : (M, g(t)) \to (N, h)$ is a family of smooth maps from $(M, g(t))$ to a fixed compact Riemannian manifold $(N, h)$, and $\tau_g\phi$ is the tension field of the map $\phi$ given by the evolving metric $g(t)$. If $(N, h) = (\mathbb{R}, dr^2)$, then this flow reduces to the Bernhard List's flow [11],

$$\begin{cases} \frac{\partial}{\partial t}g(t) = -2Ric + 2\eta(t)\nabla\phi\otimes\nabla\phi, & g(0)=g_0,\\ \frac{\partial}{\partial t}\phi = \Delta_g\phi, & \phi(0)=\phi_0. \end{cases} \tag{4}$$

Then Azami [2] generalized harmonic-Ricci flow to Ricci-Bourguignon flow coupled with harmonic map as follows:

$$\begin{cases} \frac{\partial}{\partial t}g(t) = -2Ric + 2\rho Rg + 2\eta\nabla\phi\otimes\nabla\phi, & g(0)=g_0,\\ \frac{\partial}{\partial t}\phi = \tau_g\phi, & \phi(0)=\phi_0, \end{cases} \tag{5}$$

and showed the unique short-time existence for solution of the flow (5) on $(0,T)$ for $\rho < \frac{1}{2(n-1)}$ on any smooth $m$-dimensional compact Riemannian manifold $M$ with initial condition $(g_0, \phi_0)$.

J. Y. Wu in [13] studied a generalization of harmonic-Ricci flow as follows:

$$\begin{cases} \frac{\partial g}{\partial t}(x,t) = -2Ric(x,t) + h(x,t) + 2\tau du\otimes du(x,t), \quad g(x,0)=g(x),\\ \frac{\partial}{\partial t}H(x,t) = \Delta_{HL,g(x,t)}H(x,t), \ \ H(x,0)=H(x),\\ \frac{\partial}{\partial t}u(x,t) = \Delta u(x,t), \ \ u(x,0)=u_0(x), \end{cases} \tag{6}$$

where $Ric$ is the Ricci tensor of the manifold $M$, $h$ is a two-form with components $h_{ij} = \frac{1}{2}H_{ikl}H_j^{kl}$, $\Delta_{HL} = -(dd^* + d^*d)$ is the Hodge-Laplace operator, and $\tau$ is a positive constant.

Another important geometric flow introduced by Kong and Liu [10] which is the generalized hyperbolic geometric flow and is defined by

$$\frac{\partial^2 g}{\partial t^2} + 2Ric + \mathcal{F}(g, \frac{\partial g}{\partial t}) = 0, \tag{7}$$

where $\mathcal{F}$ is a smooth function of the Riemannian metric $g$ and its first derivative with respect to $t$. The hyperbolic geometric flow same as the Einstein equation

$$\frac{\partial^2}{\partial t^2}g_{ij} = -2R_{ij} - \frac{1}{2}g^{pq}\frac{\partial g_{ij}}{\partial t}\frac{\partial g_{pq}}{\partial t} + g^{pq}\frac{\partial g_{ip}}{\partial t}\frac{\partial g_{jq}}{\partial t},$$

is similar to the wave equation flow metrics and is defined by

$$\frac{\partial^2}{\partial t^2}g = -2Ric, \quad g(0) = g_0, \quad \frac{\partial g}{\partial t}(0) = k_0, \tag{8}$$

where $k_0$ is a $(0,2)$-type symmetric tensor field on $M$. The existences and uniqueness of (8) investigated in [7].

Let $(M, g)$ be an $n$-dimensional closed Riemannian manifold and $H = \{H_{ijk}\}$ be a three-form on $M$. Motivated by the above works in present paper, we consider the following generalized hyperbolic geometric flow (GHGF) on $M$:

$$\begin{cases} \frac{\partial^2 g_{ij}}{\partial t^2}(x,t) = -2Ric(x,t) + 2\rho R g(x,t) + h(x,t) + 2\tau du \otimes du, \\ \qquad g(x,0) = g(x), \ \frac{\partial}{\partial t} g(x,0) = \kappa_0(x), \\ \frac{\partial}{\partial t} H(x,t) = \Delta_{HL,g(x,t)} H(x,t), \ \ H(x,0) = H(x), \\ \frac{\partial}{\partial t} u(x,t) = \Delta u(x,t), \ u(x,0) = u_0(x), \end{cases} \tag{9}$$

where $Ric$ is the Ricci tensor of the manifold $M$, $h$ is a two-form with components $h_{ij} = \frac{1}{2} H_{ikl} H_j^{kl}$, $\kappa_0$ is a symmetric $(0,2)$-tensor field on $M$, $\rho$ is a real constant, $\Delta_{HL} = -(dd^* + d^*d)$ denotes the Hodge-Laplace operator, and $\tau$ is a positive constant. In above system if the form $H$ is closed, then the corresponding system is called the refined generalized hyperbolic geometric flow (RGHGF):

$$\begin{cases} \frac{\partial^2 g_{ij}}{\partial t^2}(x,t) = -2Ric(x,t) + 2\rho R g(x,t) + h(x,t) + 2\tau du \otimes du, \\ \qquad g(x,0) = g(x), \ \frac{\partial}{\partial t} g(x,0) = \kappa_0(x), \\ \frac{\partial}{\partial t} H(x,t) = -dd^* H(x,t), \ \ H(x,0) = H(x), \\ \frac{\partial}{\partial t} u(x,t) = \Delta u(x,t), \ u(x,0) = u_0(x). \end{cases} \tag{10}$$

The coupled geometric flow (9) is related to the harmonic-hyperbolic geometric flow [1]. In this paper, we mainly discuss the short-time existence, evolution of some geometric structure for this general hyperbolic geometric flow system on closed Riemannian manifolds. Many of our results obviously extend previous results in [1, 3, 7, 10]. If $H = 0$ and $u = 0$, the GHGF system (9) reduces to the hyperbolic Ricci-Bourguignon flow [3]. Also, when $H = 0$ and $\rho = 0$ the GHGF system (9) reduces to the harmonic-hyperbolic geometric flow [1].

## 2. Short-time existence and uniqueness of the GHGF flow

In this section, by a similar argument with the unique short-time existence for solution to geometric flows such as Ricci flow, Ricci-Bourguignon flow, harmonic-Ricci flow, and hyperbolic geometric flow, we show the short-time existence and uniqueness for the GHGF system on a compact $n$-dimensional Riemannian manifold. Firstly, we show that if $H(x)$ is closed, then we have the following results.

**Proposition 2.1.** *Along the RGHGF, the form $H(x,t)$ is closed if the initial value $H(x)$ is closed.*

*Proof.* The exterior derivative $d$ is independent of the metric, so we get

$$\frac{\partial}{\partial t}dH(x,t) = d\frac{\partial}{\partial t}H(x,t) = d\left(-dd^*_{g(x,t)}H(x,t)\right) = 0.$$

Therefore, $dH(x,t)$ is independent of time variable $t$ and $dH(x,t) = dH(x)$. Since $H(x)$ is a closed form, we conclude that $dH(x,t) = 0$. □

**Proposition 2.2.** *If $(g(x,t), H(x,t))$ is a solution to RGHGF and the initial value $H(x)$ is a closed form, then $(g(x,t), H(x,t))$ is also a solution to GHGF.*

*Proof.* From Proposition 2.1, since $H(x)$ is closed then $H(x,t)$ is also closed under the RGRBF. Therefore,

$$\Delta_{HL,g(x,t)}H(x,t) = -dd^*_{g(x,t)}H(x,t).$$

This completes the proof of proposition. □

**Theorem 2.3.** *Let $\rho < \frac{1}{2(n-1)}$. Then the evolution GHGF system has a unique solution for a short time on any smooth, n-dimensional, closed Riemannian manifold $M$.*

*Proof.* Our method to establish the theorem will be applying the DeTurk trick in Ricci flow to prove its short time existence. Let $(g(x,t), H(x,t), u(x,t))$ be the solution of the GHGF and $\phi_t : M \to M$ be a family of smooth diffeomorphisms of $M$. Suppose

$$\hat{g}(x,t) = \phi_t^* g(x,t)$$

is the pull-back metrics of the metric $g(x,t)$. We consider local coordinates system $x = \{x^1, x^2, \dots, x^n\}$ for $M$ and for computing the evolution equation for the metric $\hat{g}(x,t)$, let

$$y(x,t) = \phi_t(x) = \{y^1(x,t), y^2(x,t), \dots, y^n(x,t)\}.$$

We follow the same calculation of [1,3], we have

$$\hat{g}_{ij}(x,t) = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j} g_{\alpha\beta}(y,t),$$

and

$$\begin{aligned}\frac{\partial}{\partial t}\hat{g}_{ij}(x,t) = {}& \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\left(\frac{\partial}{\partial t}g_{\alpha\beta}(y,t)\right) + \frac{\partial}{\partial x^i}\left(\frac{\partial y^\alpha}{\partial t}\right)\frac{\partial y^\beta}{\partial x^j}g_{\alpha\beta}(y,t) \\ &+ \frac{\partial y^\alpha}{\partial x^i}\frac{\partial}{\partial x^j}\left(\frac{\partial y^\beta}{\partial t}\right)g_{\alpha\beta}(y,t).\end{aligned} \tag{11}$$

As

$$\begin{aligned}&\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial^2 y^\gamma}{\partial t^2}\frac{\partial g_{\alpha\beta}}{\partial y^\gamma} \\ &= \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial^2 y^\gamma}{\partial t^2}\hat{g}_{kl}(x,t)\frac{\partial}{\partial y^\gamma}\left(\frac{\partial x^k}{\partial y^\alpha}\frac{\partial x^l}{\partial y^\beta}\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{\partial^2 y^\alpha}{\partial t^2}\frac{\partial}{\partial x^i}\left(\frac{\partial x^k}{\partial y^\alpha}\right)\hat{g}_{jk}(x,t) + \frac{\partial^2 y^\beta}{\partial t^2}\frac{\partial}{\partial x^j}\left(\frac{\partial x^k}{\partial y^\beta}\right)\hat{g}_{ik}(x,t)\\
&= \frac{\partial}{\partial x^i}\left(\frac{\partial^2 y^\alpha}{\partial t^2}\frac{\partial x^k}{\partial y^\alpha}\hat{g}_{jk}(x,t)\right) + \frac{\partial}{\partial x^j}\left(\frac{\partial^2 y^\beta}{\partial t^2}\frac{\partial x^k}{\partial y^\beta}\hat{g}_{ik}(x,t)\right)\\
&\quad - \frac{\partial}{\partial x^i}\left(\frac{\partial^2 y^\alpha}{\partial t^2}\right)\frac{\partial x^k}{\partial y^\alpha}\hat{g}_{jk}(x,t) - \frac{\partial}{\partial x^j}\left(\frac{\partial^2 y^\beta}{\partial t^2}\right)\frac{\partial x^k}{\partial y^\beta}\hat{g}_{ik}(x,t),
\end{aligned}
$$

we get

$$
\begin{aligned}
&\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial^2 y^\gamma}{\partial t^2}\frac{\partial g_{\alpha\beta}}{\partial y^\gamma}\\
&= \frac{\partial}{\partial x^i}\left(\frac{\partial^2 y^\alpha}{\partial t^2}\frac{\partial x^k}{\partial y^\alpha}\hat{g}_{jk}(x,t)\right) + \frac{\partial}{\partial x^j}\left(\frac{\partial^2 y^\beta}{\partial t^2}\frac{\partial x^k}{\partial y^\beta}\hat{g}_{ik}(x,t)\right)\\
&\quad - \frac{\partial}{\partial x^i}\left(\frac{\partial^2 y^\alpha}{\partial t^2}\right)\frac{\partial y^\beta}{\partial x^j}g_{\alpha\beta}(y,t) - \frac{\partial}{\partial x^j}\left(\frac{\partial^2 y^\beta}{\partial t^2}\right)\frac{\partial y^\alpha}{\partial x^i}g_{\alpha\beta}(y,t).
\end{aligned}
$$

Thus, by direct computing, we infer

$$
\begin{aligned}
(12)\quad &\frac{\partial^2 \hat{g}_{ij}}{\partial t^2}(x,t)\\
&= \frac{d^2 g_{\alpha\beta}}{dt^2}(y,t)\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j} + \frac{\partial^2 g_{\alpha\beta}}{\partial y^\gamma \partial y^\lambda}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial t}\frac{\partial y^\lambda}{\partial t}\\
&\quad + 2\frac{\partial^2 g_{\alpha\beta}}{\partial y^\gamma \partial t}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial t} + \frac{\partial}{\partial x^i}(g_{\alpha\beta}\frac{\partial y^\beta}{\partial x^j}\frac{\partial^2 y^\alpha}{\partial t^2}) + \frac{\partial}{\partial x^j}(g_{\alpha\beta}\frac{\partial y^\beta}{\partial x^i}\frac{\partial^2 y^\alpha}{\partial t^2})\\
&\quad + \left[\frac{\partial g_{\alpha\beta}}{\partial y^\gamma}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j} - \frac{\partial}{\partial x^i}(g_{\beta\gamma}\frac{\partial y^\beta}{\partial x^j}) - \frac{\partial}{\partial x^j}(g_{\beta\gamma}\frac{\partial y^\beta}{\partial x^i})\right]\frac{\partial^2 y^\gamma}{\partial t^2}\\
&\quad + 2\frac{\partial}{\partial x^i}(\frac{\partial y^\alpha}{\partial t})\frac{\partial y^\beta}{\partial x^j}(\frac{\partial g_{\alpha\beta}}{\partial t} + \frac{\partial g_{\alpha\beta}}{\partial y^\gamma}\frac{\partial y^\gamma}{\partial t})\\
&\quad + 2\frac{\partial y^\alpha}{\partial x^i}\frac{\partial}{\partial x^j}(\frac{\partial y^\beta}{\partial t})(\frac{\partial g_{\alpha\beta}}{\partial t} + \frac{\partial g_{\alpha\beta}}{\partial y^\gamma}\frac{\partial y^\gamma}{\partial t}) + 2g_{\alpha\beta}\frac{\partial}{\partial x^i}(\frac{\partial y^\alpha}{\partial t})\frac{\partial}{\partial x^j}(\frac{\partial y^\beta}{\partial t}).
\end{aligned}
$$

For a fixed point $p \in M$, we assume a normal coordinate $\{x^i\}$ around point $p$ such that $\frac{\partial \hat{g}_{ij}}{\partial x^k} = 0$ at $p$. We obtain

$$
(13)\quad \frac{\partial g_{\alpha\beta}}{\partial y^\gamma}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j} - \frac{\partial}{\partial x^i}(g_{\beta\gamma}\frac{\partial y^\beta}{\partial x^j}) - \frac{\partial}{\partial x^j}(g_{\beta\gamma}\frac{\partial y^\beta}{\partial x^i}) = 0,\ \forall i,j,\gamma = 1,2,\ldots,n.
$$

On the other hand, we have

$$
\frac{dg_{\alpha\beta}}{dt}(y,t) = \frac{\partial g_{\alpha\beta}}{\partial y^\gamma}\frac{\partial y^\gamma}{\partial t} + \frac{\partial g_{\alpha\beta}}{\partial t},
$$

and

$$
\frac{d^2 g_{\alpha\beta}}{dt^2}(y,t) = \frac{\partial^2 g_{\alpha\beta}}{\partial y^\gamma \partial y^\lambda}\frac{\partial y^\gamma}{\partial t}\frac{\partial y^\lambda}{\partial t} + 2\frac{\partial^2 g_{\alpha\beta}}{\partial y^\gamma \partial t}\frac{\partial y^\gamma}{\partial t} + \frac{\partial^2 g_{\alpha\beta}}{\partial t^2} + \frac{\partial g_{\alpha\beta}}{\partial y^\gamma}\frac{\partial^2 y^\gamma}{\partial t^2}.
$$

Since $g(x,t)$ is the solution of GHGF we infer

$$\frac{\partial^2}{\partial t^2} g_{\alpha\beta}(y,t) = -2R_{\alpha\beta}(y,t) + 2\rho R g_{\alpha\beta}(y,t) + h_{\alpha\beta}(y,t) + 2\tau du \otimes du,$$

where $h_{\alpha\beta} = \frac{1}{2} H_{\alpha kl} H_{\beta}^{kl}(x,t)$. Hence,

$$\begin{aligned}\frac{d^2 g_{\alpha\beta}}{dt^2}(y,t) = & -2R_{\alpha\beta}(y,t) + 2\rho R g_{\alpha\beta}(y,t) + h_{\alpha\beta}(y,t) + 2\tau du \otimes du \\ & + \frac{\partial^2 g_{\alpha\beta}}{\partial y^\gamma \partial y^\lambda} \frac{\partial y^\gamma}{\partial t} \frac{\partial y^\lambda}{\partial t} + 2 \frac{\partial^2 g_{\alpha\beta}}{\partial y^\gamma \partial t} \frac{\partial y^\gamma}{\partial t} + \frac{\partial g_{\alpha\beta}}{\partial y^\gamma} \frac{\partial^2 y^\gamma}{\partial t^2}.\end{aligned}$$

Substituting the above equation in (12) we conclude

$$\begin{aligned}\frac{\partial^2 \hat{g}_{ij}}{\partial t^2}(x,t) = & -2R_{\alpha\beta}(y,t) \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} + 2\rho R(y,t) g_{\alpha\beta}(y,t) \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \\ & + h_{\alpha\beta}(y,t) \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} + 2\tau du \otimes du(y,t) \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \\ & + \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \frac{\partial^2 g_{\alpha\beta}}{\partial y^\gamma \partial y^\lambda} \frac{\partial y^\gamma}{\partial t} \frac{\partial y^\lambda}{\partial t} + 2 \frac{\partial^2 g_{\alpha\beta}}{\partial y^\gamma \partial t} \frac{\partial y^\gamma}{\partial t} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \\ & + \frac{\partial}{\partial x^i} \left( \hat{g}_{\alpha\beta} \frac{\partial y^\beta}{\partial x^j} \frac{\partial^2 y^\alpha}{\partial t^2} \right) + \frac{\partial}{\partial x^j} \left( \hat{g}_{\alpha\beta} \frac{\partial y^\beta}{\partial x^i} \frac{\partial^2 y^\alpha}{\partial t^2} \right) \\ & + 2 \frac{\partial}{\partial x^i} (\frac{\partial y^\alpha}{\partial t}) \frac{\partial y^\beta}{\partial x^j} \left( \frac{\partial g_{\alpha\beta}}{\partial y^\gamma} \frac{\partial y^\gamma}{\partial t} + \frac{\partial g_{\alpha\beta}}{\partial t} \right) \\ & + 2 \frac{\partial}{\partial x^j} (\frac{\partial y^\beta}{\partial t}) \frac{\partial y^\alpha}{\partial x^i} \left( \frac{\partial g_{\alpha\beta}}{\partial y^\gamma} \frac{\partial y^\gamma}{\partial t} + \frac{\partial g_{\alpha\beta}}{\partial t} \right) \\ & + 2 \frac{\partial}{\partial x^i} (\frac{\partial y^\alpha}{\partial t}) \frac{\partial}{\partial x^j} (\frac{\partial y^\beta}{\partial t}) g_{\alpha\beta}.\end{aligned} \tag{14}$$

Since

$$\begin{aligned}& \hat{R}_{ij}(x,t) = \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} R_{\alpha\beta}(y,t), \quad \hat{R}(x,t) = R(y,t), \\ & \hat{h}_{ij}(x,t) = \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} h_{\alpha\beta}(y,t), \quad \tau d\hat{u} \otimes d\hat{u}(x,t) = \tau du \otimes du(y,t) \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j},\end{aligned}$$

we obtain

$$\begin{aligned}\frac{\partial^2}{\partial t^2} \hat{g}_{ij}(x,t) = & -2\hat{R}_{ij}(x,t) + 2\rho R \hat{g}_{ij}(x,t) + \hat{h}_{ij}(x,t) + 2\tau d\hat{u} \otimes d\hat{u} \\ & + \nabla_i \left( \frac{\partial^2 y^\alpha}{\partial t^2} \frac{\partial x^k}{\partial y^\alpha} \hat{g}_{jk}(x,t) \right) + \nabla_j \left( \frac{\partial^2 y^\beta}{\partial t^2} \frac{\partial x^k}{\partial y^\beta} \hat{g}_{ik}(x,t) \right) \\ & + \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \frac{\partial^2 g_{\alpha\beta}}{\partial y^\gamma \partial y^\lambda} \frac{\partial y^\gamma}{\partial t} \frac{\partial y^\lambda}{\partial t} + 2 \frac{\partial^2 g_{\alpha\beta}}{\partial y^\gamma \partial t} \frac{\partial y^\gamma}{\partial t} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \\ & + 2 \frac{\partial}{\partial x^i} (\frac{\partial y^\alpha}{\partial t}) \frac{\partial y^\beta}{\partial x^j} \left( \frac{\partial g_{\alpha\beta}}{\partial y^\gamma} \frac{\partial y^\gamma}{\partial t} + \frac{\partial g_{\alpha\beta}}{\partial t} \right)\end{aligned} \tag{15}$$

$$+ 2\frac{\partial}{\partial x^j}(\frac{\partial y^\beta}{\partial t})\frac{\partial y^\alpha}{\partial x^i}\left(\frac{\partial g_{\alpha\beta}}{\partial y^\gamma}\frac{\partial y^\gamma}{\partial t} + \frac{\partial g_{\alpha\beta}}{\partial t}\right)$$
$$+ 2\frac{\partial}{\partial x^i}(\frac{\partial y^\alpha}{\partial t})\frac{\partial}{\partial x^j}(\frac{\partial y^\beta}{\partial t})g_{\alpha\beta}.$$

According to DeTurk trick, we define $y(x,t) = \phi_t(x)$ by the equations

$$\frac{\partial^2 y^\alpha}{\partial t^2} = \frac{\partial y^\alpha}{\partial x^k}\hat{g}^{jl}\left(\hat{\Gamma}^k_{jl} - \mathring{\Gamma}^k_{jl}\right), \quad y^\alpha(x,0) = x^\alpha, \quad \frac{\partial}{\partial t}y^\alpha(x,0) = y_1^\alpha(x), \tag{16}$$

and $V_i = \hat{g}_{ik}\hat{g}^{jl}\left(\hat{\Gamma}^k_{jl} - \mathring{\Gamma}^k_{jl}\right)$, where $\hat{\Gamma}^k_{jl}$ and $\bar{\Gamma}^k_{jl}$ are the connection coefficients corresponding to the metrics $\hat{g}(t)$ and $g_0$, respectively and $y_1^\alpha(x)$ for $\alpha = 1, 2, \ldots, n$, are arbitrary smooth functions on the manifold $M$. Then (15) becomes

$$\begin{aligned}\frac{\partial}{\partial t}\hat{g}_{ij}(x,t) = & -2\hat{R}_{ij}(x,t) + 2\rho R\hat{g}_{ij}(x,t) + \hat{h}_{ij}(x,t) + 2\tau d\hat{u}\otimes d\hat{u} \\ & + \nabla_i V_j + \nabla_j V_i + F(Dy, D_tD_xy), \quad \hat{g}_{ij}(x,0) = \hat{g}(x),\end{aligned} \tag{17}$$

where

$$Dy = (\frac{\partial y^\alpha}{\partial t}, \frac{\partial y^\alpha}{\partial x^i}), \quad D_tD_xy = (\frac{\partial^2 y^\alpha}{\partial x^i \partial t}), \quad \alpha, i = 1, 2, \ldots, n.$$

Also,

$$\frac{\partial \hat{H}}{\partial t} = \phi_t^*\left(\frac{\partial H}{\partial t} + L_V H\right) = \Delta_{LB}\hat{H} - d\langle \hat{H}, V\rangle, \quad \hat{H}(x,0) = \hat{H}(x), \tag{18}$$

and

$$\begin{aligned}\frac{\partial \hat{u}}{\partial t} &= \phi_*(\frac{\partial u}{\partial t}) + L_V\hat{u} = \Delta_{\hat{g}}\hat{u} + \langle \nabla\hat{u}, V\rangle = \Delta_{\hat{g}}\hat{u} + d\hat{u}(V) \\ &= \Delta_{\hat{g}}\hat{u} + d\hat{u}(V) = \hat{g}^{kl}(\frac{\partial^2}{\partial x^k \partial x^l}\hat{u}^\lambda - \hat{\Gamma}^j_{kl}\nabla_j\hat{u}^\lambda) + \nabla_j\hat{u}^\lambda\hat{g}^{kl}(\hat{\Gamma}^j_{kl} - \mathring{\Gamma}^k_{jl}) \\ &= \hat{g}^{kl}(\partial_k\partial_l\hat{u}^\lambda - \mathring{\Gamma}^k_{jl}\nabla_j\hat{u}^\lambda)\end{aligned} \tag{19}$$

which is a strictly hyperbolic equation. Since

$$\hat{\Gamma}^k_{ij} = \frac{\partial y^\alpha}{\partial x^j}\frac{\partial y^\beta}{\partial x^i}\frac{\partial x^k}{\partial y^\gamma}\Gamma^\gamma_{\alpha\beta} + \frac{\partial x^k}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^j \partial x^i},$$

the initial value problem (16) can be becomes

$$\begin{cases}\frac{\partial^2 y^\alpha}{\partial t^2} = \hat{g}^{jl}\left(\Gamma^\beta_{\alpha\gamma}\frac{\partial y^\gamma}{\partial x^i}\frac{\partial y^\beta}{\partial x^j} + \frac{\partial^2 y^\alpha}{\partial x^j \partial x^l} - \mathring{\Gamma}^k_{jl}\frac{\partial y^\alpha}{\partial x^k}\right), \\ y^\alpha(x,0) = x^\alpha, \qquad \frac{\partial}{\partial t}y^\alpha(x,0) = y_1^\alpha(x).\end{cases} \tag{20}$$

Equation (20) is a strictly parabolic system. The theory of parabolic equations implies that the system (20) has a unique smooth solution for a short time since the manifold $M$ is compact. At the same time, we have

$$\frac{\partial}{\partial t}\hat{g}_{ij}(x,t) = \hat{g}^{kl}\frac{\partial^2 \hat{g}_{ij}}{\partial x^k \partial x^l}(x,t) - 2\rho\hat{g}_{ij}\hat{g}^{pq}\hat{g}^{kl}\frac{\partial^2 \hat{g}_{kl}}{\partial x^p \partial x^q}(x,t) \tag{21}$$

$$+ 2\rho\hat{g}_{ij}\hat{g}^{pq}\hat{g}^{kl}\frac{\partial^2 \hat{g}_{ql}}{\partial x^p \partial x^k}(x,t) + \hat{h}_{ij}(x,t) + 2\tau\nabla_i u\nabla_j u + F(Dy, D_t D_y) + G(g, D_x g),$$

and

$$\frac{\partial}{\partial t}\hat{H}_{ijk}(x,t) = \hat{g}^{rs}\frac{\partial^2 \hat{H}_{ijk}}{\partial x^r \partial x^s}(x,t) + \text{lower order terms}, \tag{22}$$

where $D_x g = (\frac{\partial g_{ij}}{\partial x^k})$, $i,j,k = 1,2,\ldots,n$. From [3, 5], for $\rho < \frac{1}{2(n-1)}$ the equations (21) and (22) are strictly parabolic systems. Since the manifold $M$ is compact, then by the standard theory of parabolic equation the systems (21) and (22) have a unique smooth solution for a short time. From the solution of (21) and (22) we can obtain a solution of the GHGF system (9). Now, we show the uniqueness of the solution.

For any two solutions $g_{ij}^{(1)}(x,t)$ and $g_{ij}^{(2)}(x,t)$ of the GHGF system (9) with the same initial data, we solve the initial value problem (20) and find two families $\phi_t^{(1)}$ and $\phi_t^{(2)}$ of diffeomorphisms of $M$. Therefore we get two solutions, $\hat{g}_{ij}^{(1)}(x,t) = (\phi_t^{(1)})^* g_{ij}^{(1)}(x,t)$ and $\hat{g}_{ij}^{(2)}(x,t) = (\phi_t^{(2)})^* g_{ij}^{(2)}(x,t)$, to the modified evolution equation (21) with same initial data $\hat{g}_{ij}(x,0) = g_{ij}(x)$. The uniqueness result for the strictly parabolic equation (21) implies that $\hat{g}_{ij}^{(1)}(x,t) = \hat{g}_{ij}^{(2)}(x,t)$ and then by system (20) and the standard uniqueness result of PDE system, the corresponding solutions $\phi_t^{(1)}$ and $\phi_t^{(2)}$ of (20) must agree. Consequently the metrics $g_{ij}^{(1)}(x,t)$ and $g_{ij}^{(2)}(x,t)$ must agree also. Therefore we have showed the uniqueness for the solution of the GHGF system (9). □

## 3. Evolution equations of curvature tensor along the GHGF system

In the following, we applying the techniques and ideas to deal with the evolution equation under the Ricci flow by Brendle (see [6]) and the evolution equation along the hyperbolic geometric flow by W. R. Dai and et al. (see [7]) to find the evolution formula for some geometric objects of $(M,g)$ under the GHGF system.

**Theorem 3.1.** *Under the GHGF system, the evolution of the Riemannian curvature tensor $R_{ijkl}$ of $(M,g)$ is as follows:*

$$\begin{aligned}\frac{\partial^2}{\partial t^2}R_{ijkl} &= \Delta R_{ijkl} + 2g_{pq}\Big(\frac{\partial}{\partial t}\Gamma^p_{il}\cdot\frac{\partial}{\partial t}\Gamma^q_{jk} - \frac{\partial}{\partial t}\Gamma^p_{jl}\cdot\frac{\partial}{\partial t}\Gamma^q_{ik}\Big)\\ &\quad - g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj} - R_{ijqk}R_{pl} + R_{ijql}R_{pk} + 2R_{ilqj}R_{kp})\\ &\quad + 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl})\\ &\quad - \rho\left[\nabla_i\nabla_k R g_{jl} - \nabla_i\nabla_l R g_{jk} - \nabla_j\nabla_k R g_{il} + \nabla_j\nabla_l R g_{ik}\right] + 2\rho R R_{ijkl}\end{aligned} \tag{23}$$

$$+\tau\left[\frac{\partial^2(\nabla_k\phi\nabla_j\phi)}{\partial x^i\partial x^l}-\frac{\partial^2(\nabla_j\phi\nabla_l\phi)}{\partial x^i\partial x^k}-\frac{\partial^2(\nabla_k\phi\nabla_i\phi)}{\partial x^j\partial x^l}-\frac{\partial^2(\nabla_i\phi\nabla_l\phi)}{\partial x^j\partial x^k}\right]$$
$$+\left[\frac{\partial^2 h_{kj}}{\partial x^i\partial x^l}-\frac{\partial^2 h_{jl}}{\partial x^i\partial x^k}-\frac{\partial^2 h_{ki}}{\partial x^j\partial x^l}+\frac{\partial^2 h_{il}}{\partial x^j\partial x^k}\right].$$

*Here $B_{ijkl}=g^{pr}g^{qs}R_{piqj}R_{rksl}$ and $\Delta$ is the Laplacian with respect to the metric $g(t)$.*

*Proof.* The second order evolution of the Christoffel symbol of the evolving metric $g$ that is $\Gamma^q_{jl}=\frac{1}{2}g^{qm}\big(\frac{\partial g_{mj}}{\partial x^l}+\frac{\partial g_{ml}}{\partial x^j}-\frac{\partial g_{jl}}{\partial x^m}\big)$, satisfies the evolution equation

$$\begin{aligned}&\frac{\partial^2}{\partial t^2}\Gamma^q_{jl}\\&=\frac{1}{2}\frac{\partial^2 g^{qm}}{\partial t^2}\Big(\frac{\partial g_{mj}}{\partial x^l}+\frac{\partial g_{ml}}{\partial x^j}-\frac{\partial g_{jl}}{\partial x^m}\Big)+\frac{\partial g^{qm}}{\partial t}\Big(\frac{\partial^2 g_{mj}}{\partial x^l\partial t}+\frac{\partial^2 g_{ml}}{\partial x^j\partial t}-\frac{\partial^2 g_{jl}}{\partial x^m\partial t}\Big)\\&\quad+\frac{1}{2}g^{qm}\Big(\frac{\partial}{\partial x^l}(\frac{\partial^2 g_{mj}}{\partial t^2})+\frac{\partial}{\partial x^j}(\frac{\partial^2 g_{ml}}{\partial t^2})-\frac{\partial}{\partial x^m}(\frac{\partial^2 g_{jl}}{\partial t^2})\Big).\end{aligned}$$

In local coordinate we have $R^k_{ijl}=\frac{\partial\Gamma^k_{jl}}{\partial x^i}-\frac{\partial\Gamma^k_{il}}{\partial x^j}+\Gamma^k_{ip}\Gamma^p_{jl}-\Gamma^k_{jp}\Gamma^p_{il}$ and the Riemannian curvature tensor of $(M,g)$ is $R_{ijkl}=g_{pk}R^p_{ijl}$, hence with a double differentiation respect to $t$ we infer

$$\begin{aligned}\frac{\partial^2}{\partial t^2}R_{ijkl}&=g_{qk}\left[\frac{\partial}{\partial x^i}(\frac{\partial^2\Gamma^q_{jl}}{\partial t^2})-\frac{\partial}{\partial x^j}(\frac{\partial^2\Gamma^q_{il}}{\partial t^2})+\frac{\partial^2}{\partial t^2}(\Gamma^q_{ip}\Gamma^p_{jl}-\Gamma^h_{jp}\Gamma^p_{il})\right]\\&\quad+2\frac{\partial g_{qk}}{\partial t}\left[\frac{\partial}{\partial x^i}(\frac{\partial\Gamma^q_{jl}}{\partial t})-\frac{\partial}{\partial x^j}(\frac{\partial\Gamma^q_{il}}{\partial t})+\frac{\partial}{\partial t}(\Gamma^q_{ip}\Gamma^p_{jl}-\Gamma^q_{jp}\Gamma^p_{il})\right]\\&\quad+R^q_{ijl}\frac{\partial^2 g_{qk}}{\partial t^2}.\end{aligned}\tag{24}$$

For a fixed point $p\in M$, we suppose the normal coordinates $\{x^1,\dots,x^n\}$ around point $p$ on $M$. At this point, we have $\frac{\partial g_{ij}}{\partial x^k}(p)=0$, $\Gamma^k_{ij}(p)=0$, and

$$\begin{aligned}\frac{\partial^2}{\partial t^2}R_{ijkl}&=\frac{1}{2}\left[\frac{\partial^2}{\partial x^i\partial x^l}(\frac{\partial^2 g_{kj}}{\partial t^2})+\frac{\partial^2}{\partial x^i\partial x^j}(\frac{\partial^2 g_{kl}}{\partial t^2})-\frac{\partial^2}{\partial x^i\partial x^k}(\frac{\partial^2 g_{jl}}{\partial t^2})\right]\\&\quad-\frac{1}{2}\left[\frac{\partial^2}{\partial x^j\partial x^l}(\frac{\partial^2 g_{kj}}{\partial t^2})+\frac{\partial^2}{\partial x^j\partial x^i}(\frac{\partial^2 g_{kl}}{\partial t^2})-\frac{\partial^2}{\partial x^j\partial x^k}(\frac{\partial^2 g_{il}}{\partial t^2})\right]\\&\quad-g^{pm}\frac{\partial^2 g_{kp}}{\partial x^i\partial t}\Big(\frac{\partial^2 g_{mj}}{\partial x^l\partial t}+\frac{\partial^2 g_{ml}}{\partial x^j\partial t}-\frac{\partial^2 g_{jl}}{\partial x^m\partial t}\Big)\\&\quad+g^{pm}\frac{\partial^2 g_{kp}}{\partial x^j\partial t}\Big(\frac{\partial^2 g_{mi}}{\partial x^l\partial t}+\frac{\partial^2 g_{ml}}{\partial x^i\partial t}-\frac{\partial^2 g_{il}}{\partial x^m\partial t}\Big)\\&\quad+2g_{qk}\Big(\frac{\partial}{\partial t}\Gamma^q_{ip}\cdot\frac{\partial}{\partial t}\Gamma^p_{jl}-\frac{\partial}{\partial t}\Gamma^q_{jp}\cdot\frac{\partial}{\partial t}\Gamma^p_{il}\Big).\end{aligned}\tag{25}$$

Plugging $\frac{\partial^2}{\partial t^2}g = -2Ric + 2\rho Rg + 2\tau\nabla u \otimes \nabla u + h$ into (25) we conclude

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2}R_{ijkl} = & \frac{1}{2}\frac{\partial^2}{\partial x^i \partial x^l}\big( -2R_{kj} + 2\rho R g_{kj} + 2\tau\nabla_k u \nabla_j u + h_{kj}\big) \\
& - \frac{1}{2}\frac{\partial^2}{\partial x^i \partial x^k}\big( -2R_{jl} + 2\rho R g_{jl} + 2\tau\nabla_j u \nabla_l u + h_{jl}\big) \\
& - \frac{1}{2}\frac{\partial^2}{\partial x^j \partial x^l}\big( -2R_{ki} + 2\rho R g_{ki} + 2\tau\nabla_k u \nabla_i u + h_{ki}\big) \\
& + \frac{1}{2}\frac{\partial^2}{\partial x^j \partial x^k}\big( -2R_{il} + 2\rho R g_{il} + 2\tau\nabla_i u \nabla_l u + h_{il}\big) \\
& + \frac{1}{2}\frac{\partial^2}{\partial x^i \partial x^j}\big( -2R_{kl} + 2\rho R g_{kl} + 2\tau\nabla_k u \nabla_l u + h_{kl}\big) \\
& - \frac{1}{2}\frac{\partial^2}{\partial x^j \partial x^i}\big( -2R_{kl} + 2\rho R g_{kl} + 2\tau\nabla_k u \nabla_l u + h_{kl}\big) \\
& - g^{pm}\frac{\partial^2 g_{kp}}{\partial x^i \partial t}\Big(\frac{\partial^2 g_{mj}}{\partial x^l \partial t} + \frac{\partial^2 g_{ml}}{\partial x^j \partial t} - \frac{\partial^2 g_{jl}}{\partial x^m \partial t}\Big) \\
& + g^{pm}\frac{\partial^2 g_{kp}}{\partial x^j \partial t}\Big(\frac{\partial^2 g_{mi}}{\partial x^l \partial t} + \frac{\partial^2 g_{ml}}{\partial x^i \partial t} - \frac{\partial^2 g_{il}}{\partial x^m \partial t}\Big) \\
& + 2g_{qk}\Big(\frac{\partial}{\partial t}\Gamma^q_{ip}\cdot\frac{\partial}{\partial t}\Gamma^p_{jl} - \frac{\partial}{\partial t}\Gamma^q_{jp}\cdot\frac{\partial}{\partial t}\Gamma^p_{il}\Big).
\end{aligned} \tag{26}
$$

On the other hand, for a given symmetric $(0,2)$-tensor $v$ we have

$$
\frac{\partial^2}{\partial x^i \partial x^l}v_{jk} = \nabla_i\nabla_l v_{jk} - R_{jp}\nabla_i\Gamma^p_{lk} - v_{kp}\nabla_i\Gamma^p_{lj}. \tag{27}
$$

Also, the following identity holds.

$$
\begin{aligned}
& - g^{pm}\frac{\partial^2 g_{kp}}{\partial x^i \partial t}\Big(\frac{\partial^2 g_{mj}}{\partial x^l \partial t} + \frac{\partial^2 g_{ml}}{\partial x^j \partial t} - \frac{\partial^2 g_{jl}}{\partial x^m \partial t}\Big) \\
& + g^{pm}\frac{\partial^2 g_{kp}}{\partial x^j \partial t}\Big(\frac{\partial^2 g_{mi}}{\partial x^l \partial t} + \frac{\partial^2 g_{ml}}{\partial x^i \partial t} - \frac{\partial^2 g_{il}}{\partial x^m \partial t}\Big) \\
& + 2g_{qk}\Big(\frac{\partial}{\partial t}\Gamma^q_{ip}\cdot\frac{\partial}{\partial t}\Gamma^p_{jl} - \frac{\partial}{\partial t}\Gamma^q_{jp}\cdot\frac{\partial}{\partial t}\Gamma^p_{il}\Big) \\
= & \; 2g_{pq}\Big(\frac{\partial}{\partial t}\Gamma^p_{il}\cdot\frac{\partial}{\partial t}\Gamma^q_{jk} - \frac{\partial}{\partial t}\Gamma^p_{jl}\cdot\frac{\partial}{\partial t}\Gamma^q_{ik}\Big).
\end{aligned} \tag{28}
$$

Hence

$$
\begin{aligned}
& \frac{\partial^2 v_{kj}}{\partial x^i \partial x^l} - \frac{\partial^2 v_{jl}}{\partial x^i \partial x^k} - \frac{\partial^2 v_{ki}}{\partial x^j \partial x^l} + \frac{\partial^2 v_{il}}{\partial x^j \partial x^k} + \frac{\partial^2 v_{kl}}{\partial x^i \partial x^j} + \frac{\partial^2 v_{kl}}{\partial x^j \partial x^i} \\
= & \nabla_i\nabla_l v_{kj} - \nabla_i\nabla_k v_{jl} - \nabla_j\nabla_l v_{ki} + \nabla_j\nabla_k v_{il} \\
& - g^{pq}\Big(R_{ijqk}v_{pl} + R_{ijql}v_{pk} + 2R_{ilqj}v_{kp}\Big).
\end{aligned} \tag{29}
$$

Therefore, using (29) and inserting (27) and (28) into (26) we deduce

$$
\frac{\partial^2}{\partial t^2}R_{ijkl}
$$

$$
\begin{aligned}
= & -\nabla_i\nabla_l R_{jk} + \nabla_i\nabla_k R_{jl} + \nabla_j\nabla_l R_{ki} - \nabla_j\nabla_k R_{il} \\
& - g^{pq}(-R_{ijqk}R_{pl} + R_{ijql}R_{kp} + 2R_{ilqj}R_{kp}) \\
& + 2g_{pq}\Big(\frac{\partial}{\partial t}\Gamma^p_{il}\cdot\frac{\partial}{\partial t}\Gamma^q_{jk} - \frac{\partial}{\partial t}\Gamma^p_{jl}\cdot\frac{\partial}{\partial t}\Gamma^q_{ik}\Big) \\
& + \rho\left[\frac{\partial^2 R}{\partial x^i\partial x^l}g_{kj} - \frac{\partial^2 R}{\partial x^i\partial x^k}g_{jl} - \frac{\partial^2 R}{\partial x^j\partial x^l}g_{ki} + \frac{\partial^2 R}{\partial x^j\partial x^k}g_{il}\right] \\
& + \rho\left[\frac{\partial^2 g_{kj}}{\partial x^i\partial x^l}R - \frac{\partial^2 g_{jl}}{\partial x^i\partial x^k}R - \frac{\partial^2 g_{ki}}{\partial x^j\partial x^l}R + \frac{\partial^2 g_{il}}{\partial x^j\partial x^k}R\right] \\
& + \tau\left[\frac{\partial^2(\nabla_k u\nabla_j u)}{\partial x^i\partial x^l} - \frac{\partial^2(\nabla_j u\nabla_l u)}{\partial x^i\partial x^k} - \frac{\partial^2(\nabla_k u\nabla_i u)}{\partial x^j\partial x^l} - \frac{\partial^2(\nabla_i u\nabla_l u)}{\partial x^j\partial x^k}\right] \\
& + \left[\frac{\partial^2 h_{kj}}{\partial x^i\partial x^l} - \frac{\partial^2 h_{jl}}{\partial x^i\partial x^k} - \frac{\partial^2 h_{ki}}{\partial x^j\partial x^l} + \frac{\partial^2 h_{il}}{\partial x^j\partial x^k}\right]
\end{aligned}
$$

and equivalently

$$
\begin{aligned}
& \frac{\partial^2}{\partial t^2}R_{ijkl} \\
= & \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) \\
& - g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj} - R_{ijqk}R_{pl} + R_{ijql}R_{pk} + 2R_{ilqj}R_{kp}) \\
& + 2g_{pq}\Big(\frac{\partial}{\partial t}\Gamma^p_{il}\cdot\frac{\partial}{\partial t}\Gamma^q_{jk} - \frac{\partial}{\partial t}\Gamma^p_{jl}\cdot\frac{\partial}{\partial t}\Gamma^q_{ik}\Big) \\
& - \rho\,[\nabla_i\nabla_k R g_{jl} - \nabla_i\nabla_l R g_{jk} - \nabla_j\nabla_k R g_{il} + \nabla_j\nabla_l R g_{ik}] + 2\rho R R_{ijkl} \\
& + \tau\left[\frac{\partial^2(\nabla_k\phi\nabla_j\phi)}{\partial x^i\partial x^l} - \frac{\partial^2(\nabla_j\phi\nabla_l\phi)}{\partial x^i\partial x^k} - \frac{\partial^2(\nabla_k\phi\nabla_i\phi)}{\partial x^j\partial x^l} - \frac{\partial^2(\nabla_i\phi\nabla_l\phi)}{\partial x^j\partial x^k}\right] \\
& + \left[\frac{\partial^2 h_{kj}}{\partial x^i\partial x^l} - \frac{\partial^2 h_{jl}}{\partial x^i\partial x^k} - \frac{\partial^2 h_{ki}}{\partial x^j\partial x^l} + \frac{\partial^2 h_{il}}{\partial x^j\partial x^k}\right],
\end{aligned}
$$

where $B_{ijkl} = g^{pr}g^{qs}R_{piqj}R_{rksl}$, so the proof of the theorem is complete. □

**Theorem 3.2.** *Under the GHGF system, the Ricci curvature tensor of $(M, g)$ satisfies the evolution equation*

$$
\begin{aligned}
(30)\quad & \frac{\partial^2}{\partial t^2}R_{ij} \\
= & \Delta R_{ij} - (n-2)\rho\nabla_i\nabla_j R - \rho\Delta R g_{ij} - 2g^{pq}R_{pi}R_{qj} - 2g^{kl}g^{pq}R_{ilqj}R_{kp} \\
& + 2g^{kl}g_{pq}\Big(\frac{\partial}{\partial t}\Gamma^p_{il}\cdot\frac{\partial}{\partial t}\Gamma^q_{jk} - \frac{\partial}{\partial t}\Gamma^p_{jl}\cdot\frac{\partial}{\partial t}\Gamma^q_{ik}\Big) - 2g^{kp}g^{lq}\frac{\partial g_{pq}}{\partial t}\frac{\partial R_{ikjl}}{\partial t} \\
& - g^{kp}g^{lq}(2\rho R g_{pq} + h_{pq} + 2\tau\nabla_p u\nabla_q u)R_{ikjl} + 2g^{kp}g^{rq}g^{sl}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}R_{ikjl} \\
& + \tau g^{kl}\left[\frac{\partial^2(\nabla_k u\nabla_j u)}{\partial x^i\partial x^l} - \frac{\partial^2(\nabla_j u\nabla_l u)}{\partial x^i\partial x^k} - \frac{\partial^2(\nabla_k u\nabla_i u)}{\partial x^j\partial x^l} - \frac{\partial^2(\nabla_i u\nabla_l u)}{\partial x^j\partial x^k}\right]
\end{aligned}
$$

$$+ g^{kl}\left[\frac{\partial^2 h_{kj}}{\partial x^i \partial x^l} - \frac{\partial^2 h_{jl}}{\partial x^i \partial x^k} - \frac{\partial^2 h_{ki}}{\partial x^j \partial x^l} + \frac{\partial^2 h_{il}}{\partial x^j \partial x^k}\right].$$

*Proof.* For a fixed point $p \in M$, we choose a normal coordinate system at point $p$ as $\{x^1, \ldots, x^n\}$. At this point, we have

$$\frac{\partial^2}{\partial t^2} R_{ij} = \frac{\partial^2}{\partial t^2}(g^{kl} R_{ikjl}) = g^{kl}\frac{\partial^2}{\partial t^2} R_{ikjl} + 2\frac{\partial g^{kl}}{\partial t}\frac{\partial R_{ikjl}}{\partial t} + R_{ikjl}\frac{\partial^2 g^{kl}}{\partial t^2}.$$

The equations

$$\frac{\partial g^{kl}}{\partial t} = -g^{kp} g^{lq}\frac{\partial g_{pq}}{\partial t} \text{ and } \frac{\partial^2 g^{kl}}{\partial t^2} = -g^{kp} g^{lq}\frac{\partial^2 g_{pq}}{\partial t^2} + 2g^{kp} g^{rq} g^{sl}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}$$

imply that

$$\frac{\partial^2}{\partial t^2} R_{ij} = g^{kl}\frac{\partial^2}{\partial t^2} R_{ikjl} - 2g^{kp} g^{lq}\frac{\partial g_{pq}}{\partial t}\frac{\partial R_{ikjl}}{\partial t} - g^{kp} g^{lq}\frac{\partial^2 g_{pq}}{\partial t^2} R_{ikjl} + 2g^{kp} g^{rq} g^{sl}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t} R_{ikjl}. \tag{31}$$

Substituting (23) and (9) into (31) we obtain

$$\begin{aligned}
\frac{\partial^2}{\partial t^2} R_{ij} &= \Delta R_{ij} + 2g^{kl}(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) \\
&\quad - g^{kl} g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj} - R_{ijqk}R_{pl} + R_{ijql}R_{pk} + 2R_{ilqj}R_{kp}) \\
&\quad + 2g^{kl} g_{pq}\Big(\frac{\partial}{\partial t}\Gamma^p_{il}\cdot\frac{\partial}{\partial t}\Gamma^q_{jk} - \frac{\partial}{\partial t}\Gamma^p_{jl}\cdot\frac{\partial}{\partial t}\Gamma^q_{ik}\Big) \\
&\quad - (n-2)\rho\nabla_i\nabla_j R - \rho\Delta R g_{ij} - 2g^{kp} g^{lq}\frac{\partial g_{pq}}{\partial t}\frac{\partial R_{ikjl}}{\partial t} \\
&\quad - g^{kp} g^{lq}(-2R_{pq} + 2\rho R g_{pq} + h_{pq} + 2\tau\nabla_p u\nabla_q u) R_{ikjl} \\
&\quad + 2g^{kp} g^{rq} g^{sl}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t} R_{ikjl} \\
&\quad + \tau g^{kl}\left[\frac{\partial^2(\nabla_k u\nabla_j u)}{\partial x^i \partial x^l} - \frac{\partial^2(\nabla_j u\nabla_l u)}{\partial x^i \partial x^k} - \frac{\partial^2(\nabla_k u\nabla_i u)}{\partial x^j \partial x^l} - \frac{\partial^2(\nabla_i u\nabla_l u)}{\partial x^j \partial x^k}\right] \\
&\quad + g^{kl}\left[\frac{\partial^2 h_{kj}}{\partial x^i \partial x^l} - \frac{\partial^2 h_{jl}}{\partial x^i \partial x^k} - \frac{\partial^2 h_{ki}}{\partial x^j \partial x^l} + \frac{\partial^2 h_{il}}{\partial x^j \partial x^k}\right],
\end{aligned} \tag{32}$$

where

$$2g^{kl}(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) = 2g^{kl}(B_{ikjl} - 2B_{iklj}) + 2g^{pr} g^{qs} R_{piqj} R_{rs}$$

and

$$\begin{aligned}
&g^{kl} g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj} - R_{ijqk}R_{pl} + R_{ijql}R_{pk} + 2R_{ilqj}R_{kp}) \\
&= 2g^{pq} R_{pi} R_{qj} + 2g^{kl} g^{pq} R_{ilqj} R_{kp}.
\end{aligned}$$

Since $g^{kl}(B_{ikjl} - 2B_{iklj}) = 0$, the proof of the theorem is complete by plugging the last equations in (32). $\square$

Since $R = g^{ij}R_{ij}$ we get

$$\frac{\partial^2}{\partial t^2}R = \frac{\partial^2}{\partial t^2}(g^{ij}R_{ij}) = g^{ij}\frac{\partial^2 R_{ij}}{\partial t^2} + 2\frac{\partial g^{ij}}{\partial t}\frac{\partial R_{ij}}{\partial t} + R_{ij}\frac{\partial^2 g^{ij}}{\partial t^2}.$$

Inserting (9) and (30) in the above equation we obtain the following corollary.

**Corollary 3.3.** *Under the GHGF system, the scalar curvature of* $(M, g)$ *satisfies the evolution equation*

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2}R \\
&= (1-2(n-1)\rho)\Delta R + 2|Ric|^2 - 4\rho R^2 - 2h^{ij}R_{ij} - 4\tau g^{kp}g^{lq}\nabla_p u\nabla_q u R_{kl} \\
&\quad + 2g^{ij}g^{kl}g_{pq}\Big(\frac{\partial}{\partial t}\Gamma^p_{il}\cdot\frac{\partial}{\partial t}\Gamma^q_{kj} - \frac{\partial}{\partial t}\Gamma^p_{kl}\cdot\frac{\partial}{\partial t}\Gamma^q_{ij}\Big) \\
&\quad - 2g^{ij}g^{kp}g^{lq}\frac{\partial g_{pq}}{\partial t}\frac{\partial R_{ikjl}}{\partial t} + 4g^{kp}g^{rq}g^{sl}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}R_{kl} - 2g^{ip}g^{jq}\frac{\partial g_{pq}}{\partial t}\frac{\partial R_{ij}}{\partial t} \\
&\quad + \tau g^{ij}g^{kl}\left[\frac{\partial^2(\nabla_k u\nabla_j u)}{\partial x^i\partial x^l} - \frac{\partial^2(\nabla_j u\nabla_l u)}{\partial x^i\partial x^k} - \frac{\partial^2(\nabla_k u\nabla_i u)}{\partial x^j\partial x^l} - \frac{\partial^2(\nabla_i u\nabla_l u)}{\partial x^j\partial x^k}\right] \\
&\quad + g^{ij}g^{kl}\left[\frac{\partial^2 h_{kj}}{\partial x^i\partial x^l} - \frac{\partial^2 h_{jl}}{\partial x^i\partial x^k} - \frac{\partial^2 h_{ki}}{\partial x^j\partial x^l} + \frac{\partial^2 h_{il}}{\partial x^j\partial x^k}\right].
\end{aligned}$$

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