Commun. Korean Math. Soc. **38** (2023), No. 2, pp. 561–573
https://doi.org/10.4134/CKMS.c220095
pISSN: 1225-1763 / eISSN: 2234-3024

GENERALIZED VARIGNON'S AND MEDIAL TRIANGLE THEOREMS

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Abstract. In this paper, we extend the medial triangle theorem and Varignon's theorem to generic two-dimensional polygons and highlight the role played by diagonals in this process. One of the results is a synthetic definition of the concept of median for an n-sided polygon.

1. Introduction

The "medial triangle theorem" (see [1, 1.4, p. 10] and [5, Theorem 36 c, p. 24]) and Varignon's theorem (see [17, Corollaire IV, Livre Quartieme, p. 62] and [2, Theorem 3.11, p. 53]) are among the simplest and, at the same time, most interesting results in Euclidean geometry. As is well known (see Figure 1), the first theorem can be briefly stated as follows:

$\boldsymbol{\mathcal{M}}$: *The midpoints of the sides of an arbitrary triangle form a triangle with sides that are half and parallel to the sides of the initial triangle.*
While the second theorem states that:

$\boldsymbol{\mathcal{V}}$: *The midpoints of the sides of an arbitrary quadrilateral form a parallelogram with sides that are half and parallel to the diagonals of the initial quadrilateral.*

The immediate consequences of the abovementioned theorems are that the perimeter of the medial triangle $M_1M_2M_3$ is half that of the perimeter of the triangle $A_1A_2A_3$, whereas the perimeter of the Varignon parallelogram $N_1N_2N_3N_4$ equals the sum of the diagonals of the initial quadrilateral $B_1B_2B_3B_4$. Figure 2 shows that the previous statements cannot be extended sic et simpliciter to polygons with more than 4 sides.

In this paper, we extend Varignon's theorem and the medial triangle theorem to generic two-dimensional polygons. To prove these extensions synthetically, we first provide a synthetic definition by induction of the concept of the median of a generic polygon (Definition 1). Indeed, the "centre" of a polygon is often defined in terms of the average of the coordinates of the vertices (see

Received April 6, 2022; Accepted July 28, 2022.
2020 *Mathematics Subject Classification.* Primary 51M05, 97G80.
Key words and phrases. Median point, median, medial triangle theorem, Varignon's theorem.

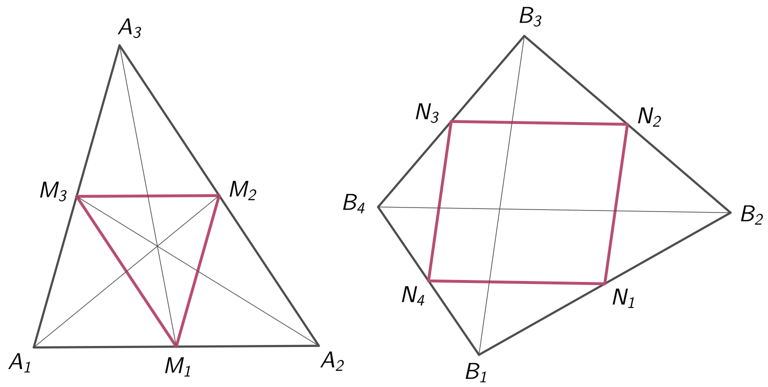

Figure 1. The medial triangle theorem and Varignon's theorem.

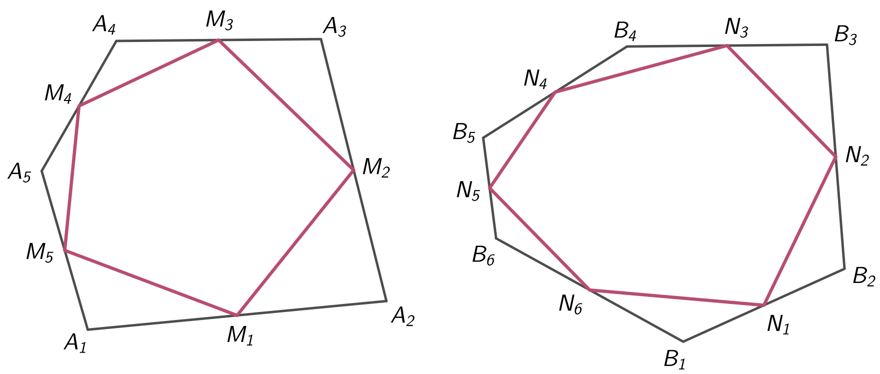

Figure 2. The sides of $M_1M_2M_3M_4M_5$ are not parallel to the sides of $A_1A_2A_3A_4A_5$. The opposite sides of $N_1N_2N_3N_4N_5N_6$ are neither parallel nor equal.

[8, Definition 1, p. 650] and [10, p. 122]), and the medians are then defined as the segments joining the vertices to the "centre" of the opposite polygon. The same definition can be applied to a triangle, by considering the sides as degenerate polygons (see, for example, [3, p. 21] and [16, p. 72]).

Various terms have been used in the literature to refer to the intersection point of the medians. In particular, we note the use of "centroid" ([1, p. 10], [10, p. 122] and [7, p. 637]), "barycenter" ([7, p. 637] and [16, p. 72]), "centre of gravity" ([6], [9] and [7, p. 637]) and "center of mass" ([16, p. 72] and [7, p. 637]). The use of these terms is understandable; indeed, as the author has noted in [16, p. 72]:

The vector average of two or more points is physically significant because it is the barycenter or center of mass of the system obtained by placing equal masses at the given points.

However, it is desirable to adopt a unique term for this intersection point, that does not reference physical characteristics. Moreover, as noted in [15, Theorem 1, p. 42] and [7, Proposition 1.1 and Theorem A, pp. 637–638], different centres of mass can be associated with a generic n-gon, because the mass can be distributed over the entire region delimited by the sides, only on the sides or only on the vertices. Therefore, to avoid these ambiguities, we use the term "median point" that was adopted in [5, p. 161 and p. 173] in reference to triangles.

In Section 2, starting from a synthetic definition of the concept of median and "median point" (Definition 1), we provide an extension of the medial triangle theorem that is valid for two-dimensional convex and nonintertwined polygons (Theorem 2.2).

In Section 3, we define the *Varignon polygon* of a generic $2k$-gon and provide an extension of Varignon's theorem (Theorem 3.3).

In Section 4, we define the *medial polygons* of a generic n-gon, then we show that the medial triangle theorem and Varignon's theorem are special cases of a general theorem that holds for any n-gon (Theorem 4.1). Finally, we use this general theorem to derive natural extensions for the classic sentences of the medial triangle theorem and Varignon's theorem (Theorem 4.3, Theorem 4.4 and Theorem 4.5).

2. A generalization of medial triangle theorem

In this section we will extend the medial triangle theorem, that we can detail as follows (see [5, Theorem 36 c., p. 24]), [1, 1.4, p. 10], [16, p. 73] and [8, Corollary 1, p. 650]).

Proposition 2.1. a) *The segment that connects the midpoints of two sides of a triangle is half and parallel to the third side.*

b) *The medians of any triangle pass through the same point that divides them into two parts of which the one containing the vertex is* 2*-times the other.*

Consequently, the triangle whose vertices are the midpoints of the sides of a given triangle is similar to the latter, with ratio $\frac{1}{2}$ [5, Theorem 36 c., p. 24].

In the following analysis, the definition of a polygon corresponds to that adopted in [11, 4.4, p. 70] for quadrilaterals.

We denote by the symbol $\mathscr{P}_n = A_1A_2\cdots A_n$ the convex (notintertwined) polygon with consecutive vertices $A_1, A_2, \dots, A_n$, where the indices $1, 2, \dots, n$ are assigned modulo n. Therefore, the polygon $\mathscr{P}_n$ can be indicated with any of the n symbols $A_iA_{i+1}\cdots A_{i+n-1}$, with $i \in \{1, 2, \dots, n\}$.

We call *k-subpolygon* of $\mathscr{P}_n = A_1A_2\cdots A_n$ any k-gon $A_{i+1}A_{i+2}\cdots A_{i+k}$, with $i, k \in \{1, 2, \dots, n\}$.

In particular, the $(n-1)$-sided polygon $A_{i+1}A_{i+2}\cdots A_{i+n-1}$ is defined as the *sub-polygon opposite vertex* A_i.

Moreover, we define *k-diagonal* of $\mathscr{P}_n$ every segment of type A_iA_{i+k}. Note that the 1-diagonals and n-diagonals are the sides and vertices of $\mathscr{P}_n$, respectively.

To extend Proposition 2.1 to any n-gon, we define the concept of median and "median point" of a generic n-gon. Considering a segment as a degenerate polygon and assuming the existence and uniqueness of the midpoint of a segment (see [11, Postulate C-5., p. 104]), we give the following definition, for any $n \geq 3$.

Definition 1. Let $\mathscr{P}_n = A_1A_2\cdots A_n$ be an n-sided polygon.

- A median of $\mathscr{P}_n$ is a segment joining a vertex A_i of $\mathscr{P}_n$ with a median point of the $(n-1)$-subpolygon opposite to A_i if such a median point exists.
- A median point of $\mathscr{P}_n$ is any (possible) intersection point between two consecutive medians of $\mathscr{P}_n$.

In Figure 3 below, the segment joining the vertex A_2 with the median point $M_{3;4}$ of the triangle opposite to it, is a median of quadrilateral $\mathscr{P}_4 = A_1A_2A_3A_4$.

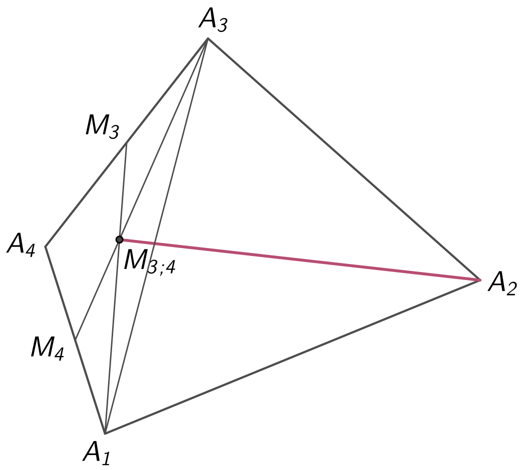

FIGURE 3. Definition of median of $\mathscr{P}_n$.

Note that Definition 1 does not use the Euclidean metric, as it refers exclusively to the existence and uniqueness of the midpoint of a segment, which can be defined through the congruence relation between segments. Therefore, Definition 1 could be extended to spaces without a metric.

The following theorem extends Proposition 2.1 to any n-sided polygon.

Theorem 2.2. *Let $\mathscr{P}_n = A_1A_2\cdots A_n$ be an n-sided polygon. Then,*

a) *each $(n-1)$-subpolygon of $\mathscr{P}_n$ has a unique median point.*
Moreover, for each $i \in \{1, 2, \ldots, n\}$, the segment that connects the median points M_i and M_{i+1} of the $(n-1)$-subpolygons of $\mathscr{P}_n$ opposite to vertices A_i and A_{i+1} is parallel to the side A_iA_{i+1} and $M_iM_{i+1} = \frac{1}{n-1}A_iA_{i+1}$,

b) *the medians of $\mathscr{P}_n$ pass through the same point that divides them into two parts, of which the part containing the vertex is $(n-1)$-times the other part.*

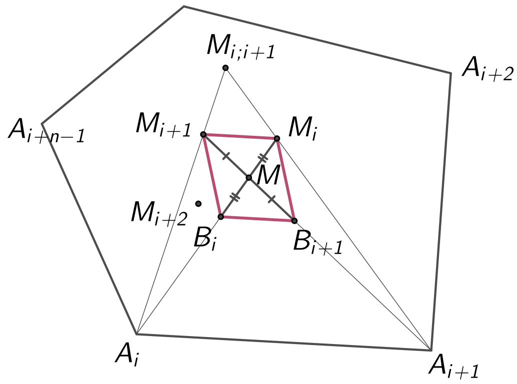

FIGURE 4. Segments joining median points opposite to consecutive vertices of $\mathscr{P}_n$.

Proof. We proceed by induction on n. For $n = 3$ the statements a) and b) are true, for the uniqueness of the midpoint and for Proposition 2.1. Let $P_n = A_1A_2\cdots A_n$ be an n-sided polygon with $n > 3$. By the inductive hypothesis, we can assume that the statements a) and b) are true for each k-gon $\mathscr{P}_k$ with $3 \leq k < n$. Therefore, the $(n-1)$-sided polygon $A_{i+1}A_{i+2}\cdots A_{i+n-1}$, opposite to the vertex A_i, has exactly one median point M_i. Similarly, the $(n-1)$-sided polygon $A_iA_{i+2}A_{i+3}\cdots A_{i+n-1}$, opposite to the vertex A_{i+1}, has exactly one median point M_{i+1}, and the $(n-2)$-sided polygon $A_{i+2}A_{i+3}\cdots A_{i+n-1}$ has exactly one median point $M_{i;i+1}$ (see Figure 4). On the other hand, the segment $A_iM_{i;i+1}$ is a median of $A_iA_{i+2}A_{i+3}\cdots A_{i+n-1}$, and M_{i+1} is its median point, i.e., by Definition 1, the intersection point of its medians. Then M_{i+1} belongs to the segment $A_iM_{i;i+1}$. In the same way follows that M_i belongs to $A_{i+1}M_{i;i+1}$. Moreover, from the inductive hypothesis *a*), we have $A_iM_{i+1} = (n-2)M_{i+1}M_{i;i+1}$, and, similarly, $A_{i+1}M_i = (n-2)M_iM_{i;i+1}$. Hence, the triangles $A_iA_{i+1}M_{i;i+1}$ and $M_iM_{i+1}M_{i;i+1}$ are similar. Then, the segment M_iM_{i+1} is parallel to the side A_iA_{i+1}. In addiction, being $A_iM_{i+1} = (n-2)M_{i+1}M_{i;i+1}$, we have $A_iM_{i;i+1} = A_iM_{i+1} + M_{i+1}M_{i;i+1} =$

$(n-2)M_{i+1}M_{i;i+1} + M_{i+1}M_{i;i+1} = (n-1)M_{i+1}M_{i;i+1}$. Therefore, since the similarity between the previous triangles, we have $A_iA_{i+1}(n-1) = M_{i+1}M_i$. Hence the statement a) is proved.

Moreover, since M_i and M_{i+1} are respectively internal to sides $A_{i+1}M_{i;i+1}$ and $A_iM_{i;i+1}$, the segments A_iM_i and $A_{i+1}M_{i+1}$ meet at a point M internal to the triangle $A_iA_{i+1}M_{i;i+1}$. Let points B_i and B_{i+1} be symmetric to points M_i and M_{i+1}, respectively, through the point M. Then, the quadrilateral $B_iB_{i+1}M_iM_{i+1}$ is a parallelogram, thus $B_iB_{i+1} = M_iM_{i+1} = \frac{1}{n-1}A_iA_{i+1}$ and $B_iB_{i+1} \| M_iM_{i+1} \| A_iA_{i+1}$. Hence, the triangles $A_iA_{i+1}M$ and $B_iB_{i+1}M$ are similar with ratio $n-1$. Therefore, $A_iM = (n-1)B_iM = (n-1)MM_i$.

Since the median $A_{i+2}M_{i+2}$ intersect the median $A_{i+i}M_{i+1}$ at a point M' such that $A_{i+1}M' = (n-1)M'M_{i+1}$, it follows that M' coincides with M, and so for all the successive medians. Hence the statement b) is proved. □

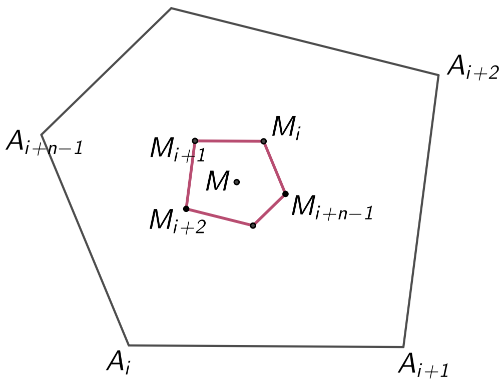

Figure 5. The n-gon of the median points opposite to the vertices of $\mathscr{P}_n$

Obviously, for $n = 3$ Theorem 2.2 returns the medial triangle theorem.

Moreover, for point b) of Theorem 2.2, M is internal to A_iM_i and $A_iM = (n-1)MM_i$ for each $i \in \{1, 2, \ldots, n\}$. Then, each M_i is the image of A_i in the homothety of centre M and ratio $-\frac{1}{n-1}$. Therefore, the following corollary holds (see Figure 5).

Corollary 2.3. *The median points of the $(n-1)$-gons opposite to the vertices of an arbitrary n-gon form a polygon similar to the initial n-gon, with ratio $\frac{1}{n-1}$.*

It seems appropriate to observe that, using a Cartesian coordinates system, the median point M of $\mathscr{P}_n = A_1A_2\cdots A_n$, can be obtained by calculating the arithmetic average of the coordinates of the vertices. Indeed, let x_i, x_M and x_{M_i} be, respectively, the coordinates of A_i, of the median point M and of the median point M_i of the $(n-1)$-sub-polygon opposite to A_i. Theorem 2.2 ensures that the vectors $\overrightarrow{A_iM}$ and $\overrightarrow{MM_i}$ are proportional. Precisely, we have $\overrightarrow{A_iM} = (n-1)\overrightarrow{MM_i}$, i.e., $x_M - x_i = (n-1)(x_{M_i} - x_M)$. Therefore, by induction on n, $nx_M = (n-1)x_{M_i} + x_i = (n-1)\left(\frac{x_{i+1}+x_{i+2}+\cdots+x_{i+n-1}}{n-1}\right) + x_i$, i.e., $x_M = \frac{1}{n}\sum_{j=1}^{n} x_j$.

3. A generalization of Varignon's theorem

In this section we extend Varignon's theorem, that we summarize below (see [17, Corollaire IV, Livre Quartieme, p. 62] and [1, Theorem 3.11, p. 53]).

Theorem 3.1. *Let $\mathscr{P}_4 = A_1A_2A_3A_4$ be a quadrilateral and $\mathscr{V}(\mathscr{P}_4)$ the polygon obtained by joining the midpoints of its sides. Then,*
a) *each side of $\mathscr{V}(\mathscr{P}_4)$ is parallel to a 2-diagonal of $\mathscr{P}_4$,*
b) *$\mathscr{V}(\mathscr{P}_4)$ is a parallelogram,*
c) *the perimeter of $\mathscr{V}(\mathscr{P}_4)$ is equal to the sum of the diagonals of $\mathscr{P}_4$.*

In addition, the area of $\mathscr{V}(\mathscr{P}_4)$ is half that of the area of$\mathscr{P}_4$ (see, for example [12, p. 407]).

The points a) and c) of Theorem 3.1 can be easily extended to all n-sided polygons. Indeed, the parallelism between A_iA_{i+2} and M_iM_{i+1} is obtained by applying the converse of Thales' intercept theorem ([13, Theorem 1.1, p. 5]). Moreover, we have $M_iM_{i+1} = \frac{A_iA_{i+2}}{2}$ for each $i \in \{1, 2, \ldots, n\}$ (see Figure 6). Then, we can state the proposition below, that provides a first extension of Varignon's theorem.

Proposition 3.2. *Let $\mathscr{P}_n = A_1A_2\cdots A_n$ be an n-sided polygon and $\mathscr{V}_2(\mathscr{P}_n)$ be the polygon obtained by joining the midpoints of the sides of $\mathscr{P}_n$. Then,*
a) *each side of $\mathscr{V}_2(\mathscr{P}_n)$ is half and parallel to a 2-diagonal of $\mathscr{P}_n$,*
b) *the perimeter of $\mathscr{V}_2(\mathscr{P}_n)$ is equal to $\frac{1}{2}\sum_{i=1}^{n} A_iA_{i+2}$.*

We note that, in the case $n = 4$, the expression $\sum_{i=1}^{n} A_iA_{i+2}$ doubles the sum of the 2-diagonals. Indeed, since in this case $A_iA_{i+2} = A_{i+2}A_{i+4} = A_{i+2}A_i$, each 2-diagonal of $\mathscr{P}_4$ appears twice in the above sum. Therefore, the perimeter of Varignon parallelogram $\mathscr{V}_2(\mathscr{P}_4)$ is equal to the sum of the diagonals of $\mathscr{P}_4$ ([12, p. 407]).

We can see that this situation occurs only if $n = 4$. In fact, if the 2-diagonals A_iA_{i+2} and A_jA_{j+2} of $\mathscr{P}_n$ with $1 \le i < j \le n$ coincide, we have: $i \equiv_n j+2$ and $i+2 \equiv_n j$. By adding a cross, it follows that $i+j \equiv_n i+j+4$, i.e., $4 \equiv_n 0$. Then $n = 4$.

Hence, for $n \neq 4$, the expression $\sum_{i=1}^{n} A_iA_{i+2}$ corresponds to the sum of the 2-diagonals of $\mathscr{P}_n$. Therefore, we can add statement c), given below, to Proposition 3.2.

If $n \neq 4$ the perimeter of the $\mathscr{V}_2(\mathscr{P}_n)$ is equals to half the sum of the 2-diagonals of $\mathscr{P}_n$.

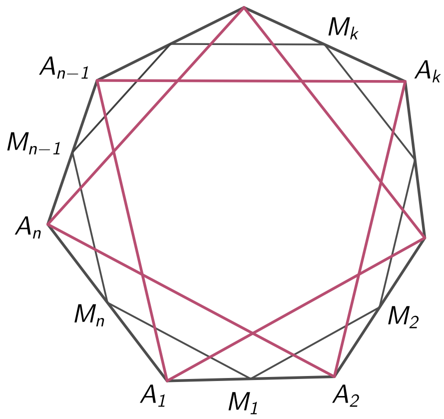

FIGURE 6. Relation between the perimeter of $\mathscr{V}_2(\mathscr{P}_n)$ and the 2-diagonals of $\mathscr{P}_n$.

To provide a further generalization of Varignon's theorem, we need the following definition.

Definition 2. Let $k > 1$ be a positive integer. Given a $2k$-sided polygon $\mathscr{P}_{2k} = A_1A_2 \cdots A_{2k}$, we call k-median point of $\mathscr{P}_{2k}$ every median point $M_{k,i}$ of a k-subpolygon $A_iA_{i+1} \cdots A_{i+k-1}$ of $\mathscr{P}_{2k}$. We call the $2k$-sided polygon $\mathscr{V}(\mathscr{P}_{2k}) = M_{k,1}M_{k,2} \cdots M_{k,2k}$ the Varignon polygon of $\mathscr{P}_{2k}$.

For example, the Varignon parallelogram $\mathscr{V}(\mathscr{P}_4)$ of a given quadrilateral $\mathscr{P}_4$ is also denoted by $\mathscr{V}(\mathscr{P}_4)$.

We can now extend Theorem 3.1 proving, in a synthetic way, the following result.

Theorem 3.3. *Let i and k be positive integers, with $i < 2k$. Let $\mathscr{P}_{2k} = A_1A_2 \cdots A_{2k}$ be a $2k$-sided polygon and let $\mathscr{V}(\mathscr{P}_{2k}) = M_{k,1}M_{k,2} \cdots M_{k,2k}$ the corresponding Varignon's polygon. Then,*

a) *each side $M_{k,i}M_{k,i+1}$ of $\mathscr{V}(\mathscr{P}_{2k})$ is parallel to the k-diagonal A_iA_{i+k} of $\mathscr{P}_{2k}$; moreover $M_{k,i}M_{k,i+1} = \frac{1}{k}A_iA_{i+k}$,*
b) *the opposite sides of $\mathscr{V}(\mathscr{P}_{2k})$ are equal and parallel in pairs,*
c) *the perimeter of $\mathscr{V}(\mathscr{P}_{2k})$ is $\frac{2}{k}$-times the sum of the k-diagonals of $\mathscr{P}_{2k}$.*

Proof. The point a) can be obtained by applying to the $(k+1)$-sided polygon $A_iA_{i+1} \cdots A_{i+k}$ the thesis a) of Theorem 2.2. Indeed, $M_{k,i}$ and $M_{k,i+1}$ are, respectively, the median points opposite to the vertices A_{i+k} and A_i of $A_iA_{i+1} \cdots A_{i+k}$. Then $M_{k,i}M_{k,i+1} \| A_iA_{i+k}$ and $M_{k,i}M_{k,i+1} = \frac{1}{k+1-1}A_iA_{i+k}$ (see Figure 7).

Thus, by substituting i with $i+k$ in the above relationships, we have $M_{k,i+k}M_{k,i+k+1} \| A_{i+k}A_{i+k+k} = A_{i+k}A_i$ and $M_{k,i+k}M_{k,i+k+1} = \frac{1}{k}A_{i+k}A_{i+k+k}$

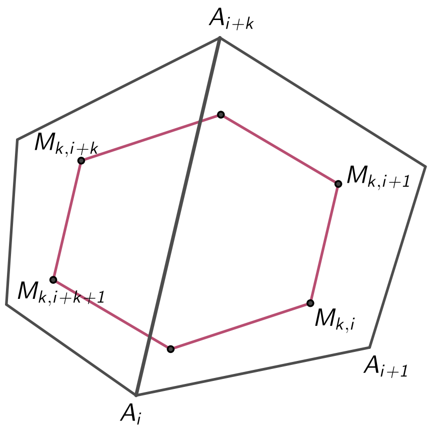

FIGURE 7. $M_{k,i}M_{k,i+1} \parallel A_iA_{i+k} \parallel M_{k,i+k}M_{k,i+k+1}$, $M_{k,i}M_{k,i+1} = \frac{1}{k}A_iA_{i+k} = M_{k,i+k}M_{k,i+k+1}$. The perimeter of $\mathscr{V}(\mathscr{P}_{2k})$ is $\frac{2}{k}$ times the sum of the k-diagonals of $\mathscr{P}_{2k}$.

$= \frac{1}{k}A_{i+k}A_i$. Then $M_{k,i}M_{k,i+1}$ and $M_{k,i+k}M_{k,i+k+1}$ are both parallel to A_iA_{i+k} and equal to $\frac{1}{k}A_iA_{i+k}$. Therefore, point b) is also proved.

Since $A_{i+k}A_{i+k+k} = A_iA_{i+k}$, the assertion c) follows from $\sum_{i=1}^{2k} M_{k,i}M_{k,i+1} = \frac{1}{k}\sum_{i=1}^{2k} A_iA_{i+k} = \frac{1}{k}\sum_{i=1}^{k} A_iA_{i+k} + \frac{1}{k}\sum_{i=1}^{k} A_{i+k}A_{i+k+k} = \frac{2}{k}\sum_{i=1}^{k} A_iA_{i+k}$. □

Note that Theorem 3.3 integrates the generalization given by Lord in [9], establishing a relationship between the perimeters of $\mathscr{P}_{2k}$ and its Varignon polygon.

4. Further generalizations of the medial triangle and Varignon's theorem

To provide further generalizations of the medial triangle theorem and Varignon's theorem, we need to extend the median point concept. Therefore we adopt the following definition.

Definition 3. Let k and n be positive integers, with $k < n$. Given an n-sided polygon $\mathscr{P}_n = A_1A_2\cdots A_n$, we call k-median point of $\mathscr{P}_n$ every median point $M_{k,i}$ of the k-subpolygon $A_iA_{i+1}\cdots A_{i+k-1}$ of $\mathscr{P}_n$. Moreover, we call the n-sided polygon $\mathscr{V}_k(\mathscr{P}_n) = M_{k,1}M_{k,2}\cdots M_{k,n}$ the k-medial polygon of $\mathscr{P}_n$.

Note that Definition 3 naturally extends the notation used in Proposition 3.2 and Definition 2. In particular, the medial triangle of a given triangle $\mathscr{P}_3$ is the 2-medial polygon $\mathscr{V}_2(\mathscr{P}_3)$, while the Varignon parallelogram of a quadrilateral $\mathscr{P}_4$ is the 2-medial polygon $\mathscr{V}_2(\mathscr{P}_4) = \mathscr{V}(\mathscr{P}_4)$. More generally,

$\mathscr{V}_2(\mathscr{P}_n)$ is the polygon whose vertex are the median points of the sides of $\mathscr{P}_n$, while $\mathscr{V}_1(\mathscr{P}_n) = \mathscr{P}_n$. Finally, we can denote with $\mathscr{V}_n(\mathscr{P}_n)$ the n-gon that degenerate on the median point of $\mathscr{P}_n$.

At this point, we can prove a general theorem that provides a simultaneous extension of the median triangle theorem and Varignon's theorem.

Theorem 4.1. *Let i, k and n be positive integers, with $k, i < n$. Let $\mathscr{P}_n = A_1A_2\cdots A_n$ be an n-sided polygon and let $\mathscr{V}_k(\mathscr{P}_n) = M_{k,1}M_{k,2}\cdots M_{k,n}$ be a k-median polygons of $\mathscr{P}_n$. Then,*

a) *each side $M_{k,i}M_{k,i+1}$ of $\mathscr{V}_k(\mathscr{P}_n)$ is parallel to the k-diagonal A_iA_{i+k} of $\mathscr{P}_n$, moreover $M_{k,i}M_{k,i+1} = \frac{1}{k}A_iA_{i+k}$,*
b) *each side $M_{n-k,i+k}M_{n-k,i+k+1}$ of $\mathscr{V}_{n-k}(\mathscr{P}_n)$ is parallel to the side $M_{k,i}M_{k,i+1}$ of $\mathscr{V}_k(\mathscr{P}_n)$, moreover $M_{n-k,i+k}M_{n-k,i+k+1} = \frac{k}{n-k}M_{k,i}M_{k,i+1}$,*
c) *the perimeter of $\mathscr{V}_k(\mathscr{P}_n)$ is equal to $\frac{1}{k}\sum_{i=1}^{n} A_iA_{i+k}$.*

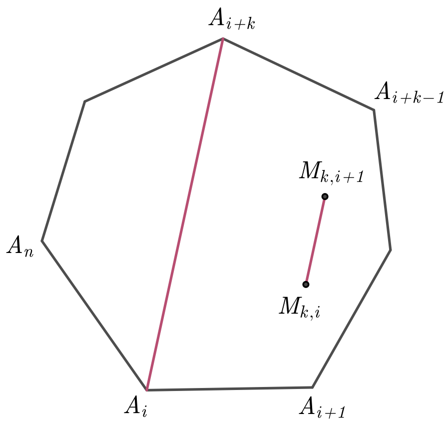

FIGURE 8. $M_{k,i}M_{k,i+1} \parallel A_iA_{i+k}$ and $M_{k,i}M_{k,i+1} = \frac{1}{k}A_iA_{i+1}$.

Proof. The thesis a) can be obtained by applying to the $(k+1)$-sided polygon $A_iA_{i+1}\ldots A_{i+k}$ the thesis a) of Theorem 2.2 (see Figure 8). In fact, the points $M_{k,i}$ and $M_{k,i+1}$ are, respectively, the median points opposite to the vertices A_{i+k} and A_i of $A_iA_{i+1}\cdots A_{i+k}$. Then $M_{k,i}\, M_{k,i+1} \parallel A_iA_{i+k}$ and $M_{k,i}M_{k,i+1} = \frac{1}{k+1-1}A_iA_{i+k}$.

Moreover, substituting k by $n-k$ and i by $i+k$ in the last two relations, we have

$$M_{n-k,i+k}M_{n-k,i+k+1} \parallel A_{i+k}A_{i+k+n-k} = A_{i+k}A_i, \text{ and}$$
$$M_{n-k,i+k}M_{n-k,i+k+1} = \frac{1}{n-k}A_{i+k}A_{i+k+n-k} = \frac{1}{n-k}A_{i+k}A_i.$$

Therefore, statement b) is also proved. The point c) follows from a). □

From Theorem 4.1, as the angles of $\mathscr{V}_k(\mathscr{P}_n)$ and $\mathscr{V}_{n-k}(\mathscr{P}_n)$ are equal in pairs and all the ratios of the pairs of the corresponding sides are equal, the corollary given below follows (see [4, Definition 1, p. 156]). A similar result was obtained for a Cartesian coordinates system in [10, Corollary 2, p. 123].

Corollary 4.2. *Let $\mathscr{P}_n$ be an n-sided polygon. Then, for any integer k, with $1 \leq k < n$ the k-medial polygon $\mathscr{V}_k(\mathscr{P}_n)$ is similar to the $(n-k)$-medial polygon $\mathscr{V}_{n-k}(\mathscr{P}_n)$, with ratio $\frac{n-k}{k}$.*

Furthermore, as particular cases of Theorem 4.1, we obtain the following natural generalizations of the statements $\boldsymbol{\mathcal{M}}$ and $\boldsymbol{\mathcal{V}}$ given in the introduction. In Figures 9 and 10 we show some examples.

Theorem 4.3. *The k-median points of an arbitrary $2k$-gon form a polygon for which the opposite sides are equal and parallel in pairs and the perimeter is $2/k$ times the sum of its k-diagonals.*

Theorem 4.4. *The $(k+1)$-median points of an arbitrary $(2k+1)$-gon form a polygon with sides that are $\frac{k}{k+1}$-times and parallel to the sides of the polygon formed by the k-median points of the initial $(2k+1)$-gon.*

Theorem 4.5. *The $(k+1)$-median points of an arbitrary $(2k+1)$-gon form a polygon with sides that are $\frac{1}{k+1}$-times and parallel to the $(k+1)$-diagonals of the initial $(2k+1)$-gon.*

Note that Varignon's theorem and its immediate consequences can be obtained from Theorem 4.3 for $k = 2$. Similarly, the medial triangle theorem can be obtained from Theorems 4.4 and 4.5 for $k = 1$.

5. Conclusions

As is well known, a simple consequence of Varignon's theorem is that the area of the Varignon parallelogram is half that of the generating quadrilateral (see, for example [2, Theorem 3.11, p. 53]. [12, p. 407] and [14]). In the previous pages we have seen that the concept of Varignon polygon of a quadrilateral can be extended to generic n-gons, by defining the k-medial polygons $\mathscr{V}_k(\mathscr{P}_n)$. It has also been shown that the perimeters of $\mathscr{V}_k(\mathscr{P}_n)$ are related to the sum of the k-diagonals of $\mathscr{P}_n$. However, other than specifying the relationship between the areas of similar polygons, previous results provide no information on possible relationships between the areas of $\mathscr{P}_n$ and the areas of its k-medial polygons. We are interested, in particular, in this relationship for Varignon sub-polygons $\mathscr{V}_k(\mathscr{P}_{2k})$ here. Although such a relationship does not appear to be immediately evident, it is reasonable to expect the existence of a well defined range within which the ratio between the aforementioned areas can vary. This unresolved problem could be the subject of subsequent investigations.

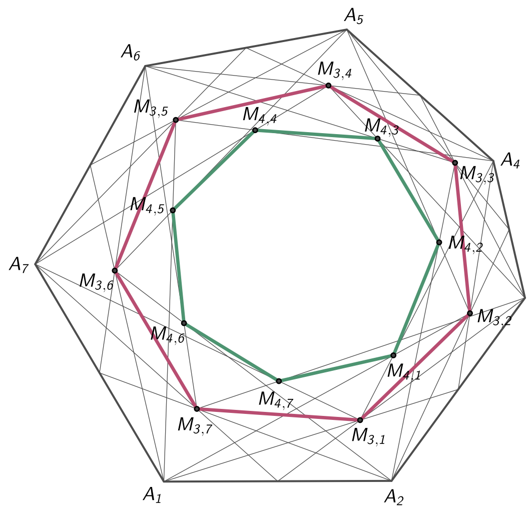

FIGURE 9. Each side of $\mathscr{V}_4(\mathscr{P}_7)$ is parallel to a side of $\mathscr{V}_3(\mathscr{P}_7)$, precisely, $M_{4,i}M_{4,i+1} \parallel M_{3,i+4}M_{3,i+5}$, moreover their ratio is $\frac{3}{4}$.

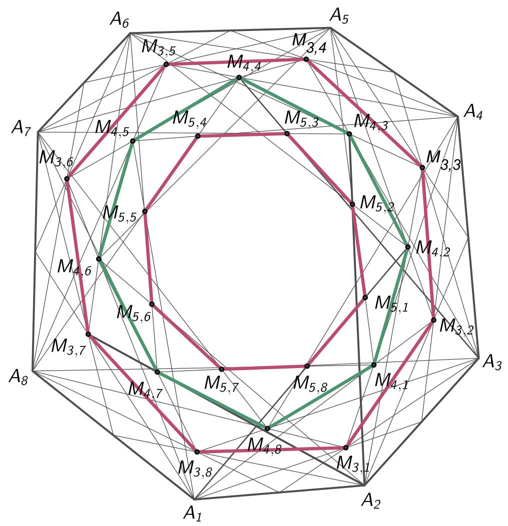

FIGURE 10. The sides of $\mathscr{V}_4(\mathscr{P}_8)$ are parallel and equal in pairs, precisely, $M_{4,i}M_{4,i+1} \parallel M_{4,i+4}M_{4,i+5}$. Each side of $\mathscr{V}_5(\mathscr{P}_8)$ is parallel to a side of $\mathscr{V}_3(\mathscr{P}_8)$, precisely, $M_{5,i}M_{5,i+1} \parallel M_{3,i+5}M_{3,i+6}$, moreover their ratio is $\frac{3}{5}$.

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