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THE MEASURABILITY OF HEWITT-STROMBERG MEASURES AND DIMENSIONS

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Abstract. The aim of this paper is to study the descriptive set-theoretic complexity of the Hewitt-Stromberg measure and dimension maps.

1. Introduction and statement of the results

The main motivation for this paper is the articles by Falconer and Mauldin in [10], Mattila and Mauldin in [20], and Olsen in [21], where the following question is considered: what can be said about the study of the measurability and the Baire's classes of multifractal and fractal measures and dimensions? The aim of this work is to study the smoothness of the Hewitt-Stromberg measures and dimensions.

The notion of dimension is fundamental in the study of fractals. Various definitions of dimension have been proposed, such as the Hausdorff dimension, the packing dimension and the modified lower and upper box dimensions etc. Unlike the Hausdorff and packing dimensions, the modified lower and upper box dimensions are not defined in terms of measures. Hewitt-Stromberg measures were introduced by Hewitt and Stromberg in [13, Exercise (10.51)]. Since then, they have been investigated by several authors, highlighting their importance in the study of local properties of fractals and products of fractals. One can cite, for example [1-6, 11, 12, 15, 18, 22, 26-31]. In particular, Edgar's textbook [7] provides an excellent and systematic introduction to these measures, which also appears explicitly, for example, in Pesin's monograph [23] and implicitly in Mattila's text [19].

A function $g : (0, +\infty) \to (0, +\infty)$ is called a dimension function if g is increasing, right continuous and $\lim_{r\to 0} g(r) = 0$. The Hausdorff measure associated with a dimension function g is defined as follows. Let X be a metric

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space and $E \subseteq X$. For $\varepsilon > 0$, we write

$$\mathscr{H}^g_\varepsilon(E) = \inf\left\{ \sum_i g\Big(\operatorname{diam}(E_i)\Big) \;\Big|\; E \subseteq \bigcup_i E_i,\ \ \operatorname{diam}(E_i) < \varepsilon \right\}.$$

This allows to define the g-dimensional Hausdorff measure $\mathscr{H}^g(E)$ of E by

$$\mathscr{H}^g(E) = \sup_{\varepsilon > 0} \mathscr{H}^g_\varepsilon(E).$$

The packing measure with a dimension function g is defined, for $\varepsilon > 0$, as follows:

$$\overline{\mathscr{P}}^g_\varepsilon(E) = \sup\left\{ \sum_i g\Big(2r_i\Big) \right\},$$

where the supremum is taken over all closed balls $\Big(B(x_i, r_i)\Big)_i$ such that $r_i \leq \varepsilon$ and with $x_i \in E$ and $B(x_i, r_i) \cap B(x_j, r_j) = \emptyset$ for $i \neq j$. The g-dimensional packing pre-measure $\overline{\mathscr{P}}^g(E)$ of E is now defined by

$$\overline{\mathscr{P}}^g(E) = \inf_{\varepsilon > 0} \overline{\mathscr{P}}^g_\varepsilon(E).$$

This makes us able to define the g-dimensional packing measure $\mathscr{P}^g(E)$ of E as

$$\mathscr{P}^g(E) = \inf\left\{ \sum_i \overline{\mathscr{P}}^g(E_i) \;\Big|\; E \subseteq \bigcup_i E_i \right\}.$$

While Hausdorff and packing measures are defined using coverings and packings by families of sets (balls) with diameters less than a given positive number δ, say, the Hewitt-Stromberg measures are defined using packings and coverings of balls with the same diameter δ. Let E be a (totally) bounded subset of X. The Hewitt-Stromberg pre-measures are defined as follows:

$$\overline{\mathscr{U}}^g_\delta(E) = \inf_{0 < r \leq \delta} M_r(E) g(2r) \quad \text{and} \quad \overline{\mathscr{U}}^g(E) = \lim_{\delta \to 0} \overline{\mathscr{U}}^g_\delta(E)$$

and

$$\overline{\mathscr{V}}^g_\delta(E) = \sup_{0 < r \leq \delta} N_r(E) g(2r) \quad \text{and} \quad \overline{\mathscr{V}}^g(E) = \lim_{\delta \to 0} \overline{\mathscr{V}}^g_\delta(E),$$

where the packing number $M_r(E)$ of E is given by

$$M_r(E) = \sup\left\{ \sharp\{I\} \;\Big|\; \Big(B(x_i, r)\Big)_{i \in I} \text{ is a family of closed balls with } x_i \in E \right.$$
$$\left. \text{and } B(x_i, r) \cap B(x_j, r) = \emptyset \text{ for } i \neq j \right\}$$

and the covering number $N_r(E)$ of E is given by

$$N_r(E) = \inf\left\{ \sharp\{I\} \;\Big|\; \Big(B(x_i,r)\Big)_{i\in I} \text{ is a family of closed balls with } x_i \in X \text{ and } E \subseteq \bigcup_i B(x_i,r) \right\}.$$

Now, we define the lower and upper g-dimensional Hewitt-Stromberg measures, which we denote respectively by $\mathscr{U}^g(E)$ and $\mathscr{V}^g(E)$, as follows:

$$\mathscr{U}^g(E) = \inf\left\{ \sum_i \overline{\mathscr{U}}^g(E_i) \;\Big|\; E \subseteq \bigcup_i E_i,\ \ E_i \text{ is bounded in } X \right\}$$

and

$$\mathscr{V}^g(E) = \inf\left\{ \sum_i \overline{\mathscr{V}}^g(E_i) \;\Big|\; E \subseteq \bigcup_i E_i,\ \ E_i \text{ is bounded in } X \right\}.$$

It is easy to see that N_r and M_r are closely related, more precisely, we have

$$M_{2r}(E) \le N_r(E) \le M_r(E). \tag{1.1}$$

Because of this relation, we will use M_r most of the time.

We recall in the following the basic inequalities satisfied by the Hewitt-Stromberg, the Hausdorff and the packing measures. It follows from (1.1) that there exists a constant $c > 0$ such that for all E

$$\overline{\mathscr{U}}^g(E) \le c\ \overline{\mathscr{V}}^g(E) \le c\ \overline{\mathscr{P}}^g(E)$$

and

$$\mathscr{H}^g(E) \le \mathscr{U}^g(E) \le c\ \mathscr{V}^g(E) \le c\ \mathscr{P}^g(E)$$

provided g satisfies the doubling condition.

Note that the definition of the Hewitt-Stromberg measures is differing slightly from those introduced in [7,15]. Our main reason for modifying the definition is to allow us to prove results without having to assume some extra conditions. The reader is referred to Edgar's book [7, p. 32] (see also [15, 22, Proposition 2.1]) for a systematic introduction to the Hewitt-Stromberg, Hausdorff and packing measures.

As above, we note that if $t > 0$ and g_t denotes the dimension function defined by $g_t(r) = r^t$, then we will follow the traditional convention and write

$$\mathscr{H}^{g_t}(E) = \mathscr{H}^t(E), \quad \mathscr{P}^{g_t}(E) = \mathscr{P}^t(E),$$

$$\overline{\mathscr{U}}^{g_t}(E) = \overline{\mathscr{U}}^t(E), \quad \overline{\mathscr{V}}^{g_t}(E) = \overline{\mathscr{V}}^t(E),$$

and

$$\mathscr{U}^{g_t}(E) = \mathscr{U}^t(E), \quad \mathscr{V}^{g_t}(E) = \mathscr{V}^t(E).$$

The lower and upper Hewitt-Stromberg dimension $\underline{\dim}_{MB}(E)$ and $\overline{\dim}_{MB}(E)$ are defined by

$$\underline{\dim}_{MB}(E) = \inf\left\{t \geq 0 \;\middle|\; \mathscr{U}^t(E) = 0\right\} = \sup\left\{t \geq 0 \;\middle|\; \mathscr{U}^t(E) = +\infty\right\}$$

and

$$\overline{\dim}_{MB}(E) = \inf\left\{t \geq 0 \;\middle|\; \mathscr{V}^t(E) = 0\right\} = \sup\left\{t \geq 0 \;\middle|\; \mathscr{V}^t(E) = +\infty\right\}.$$

The lower and upper box dimensions, denoted by $\underline{\dim}_B(E)$ and $\overline{\dim}_B(E)$, respectively, are now defined by

$$\underline{\dim}_B(E) = \liminf_{r\to 0} \frac{\log N_r(E)}{-\log r} = \liminf_{r\to 0} \frac{\log M_r(E)}{-\log r}$$

and

$$\overline{\dim}_B(E) = \limsup_{r\to 0} \frac{\log N_r(E)}{-\log r} = \limsup_{r\to 0} \frac{\log M_r(E)}{-\log r}.$$

There are similar formulas for the lower and upper box dimensions as follows:

$$\underline{\dim}_B(E) = \inf\left\{t \geq 0 \;\middle|\; \overline{\mathscr{U}}^t(E) = 0\right\} = \sup\left\{t \geq 0 \;\middle|\; \overline{\mathscr{U}}^t(E) = +\infty\right\} \tag{1.2}$$

and

$$\overline{\dim}_B(E) = \inf\left\{t \geq 0 \;\middle|\; \overline{\mathscr{V}}^t(E) = 0\right\} = \sup\left\{t \geq 0 \;\middle|\; \overline{\mathscr{V}}^t(E) = +\infty\right\}. \tag{1.3}$$

These dimensions satisfy the following inequalities:

$$\dim_H(E) \leq \underline{\dim}_{MB}(E) \leq \overline{\dim}_{MB}(E) \leq \dim_P(E),$$

$$\dim_H(E) \leq \dim_P(E) \leq \overline{\dim}_B(E) \;\text{ and }\; \dim_H(E) \leq \underline{\dim}_B(E) \leq \overline{\dim}_B(E).$$

In particular, we have (see [9])

$$\underline{\dim}_{MB}(E) = \inf\left\{\sup_i \underline{\dim}_B(E_i) \;\middle|\; E \subseteq \bigcup_i E_i,\;\; E_i \text{ is bounded in } X\right\} \tag{1.4}$$

and

$$\overline{\dim}_{MB}(E) = \inf\left\{\sup_i \overline{\dim}_B(E_i) \;\middle|\; E \subseteq \bigcup_i E_i,\;\; E_i \text{ is bounded in } X\right\}. \tag{1.5}$$

The reader is referred to [7, 9] for an excellent discussion of the Hausdorff dimension, the packing dimension, lower and upper Hewitt-Stromberg dimension and the box dimensions.

Remark 1.1. It follows from the standard Method I construction [24, Theorem 4] that $\mathscr{U}^g$ and $\mathscr{V}^g$ are outer measures and thus they are measures on the Carathéodory-measurable algebra. The function $\mathscr{U}^g$ is a metric outer measure and thus it is a measure on the Borel algebra. Unfortunately, the function $\mathscr{V}^g$ is not a metric outer measure (see [8]) which implies in particular that $\mathscr{V}^g$ is not a Borel measure (see Theorem 1.7 in [19]).

In this work, we study the descriptive set theoretic complexity of the following maps:

$$\begin{aligned}
\mathscr{K}(X) &\longrightarrow [0,\ +\infty] \ : \quad K \mapsto \mathscr{U}^g(K),\\
\mathscr{K}(X) &\longrightarrow [0,\ +\infty] \ : \quad K \mapsto \mathscr{V}^g(K),\\
\mathscr{K}(X) &\longrightarrow [0,\ +\infty] \ : \quad K \mapsto \underline{\dim}_{MB}(K),\\
\mathscr{K}(X) &\longrightarrow [0,\ +\infty] \ : \quad K \mapsto \overline{\dim}_{MB}(K),\\
\mathscr{K}(X) &\longrightarrow [0,\ +\infty] \ : \quad K \mapsto \underline{\dim}_{B}(K),\\
\mathscr{K}(X) &\longrightarrow [0,\ +\infty] \ : \quad K \mapsto \overline{\dim}_{B}(K),
\end{aligned}$$

here $\mathscr{K}(X)$ denotes the family of non-empty compact subsets of X equipped with the Hausdorff metric. More precisely, we will prove that the upper and lower box dimension maps are of Baire class 2 and not of Baire class 1 (note that this result is proved by Mattila and Mauldin [20], the purpose of this paper is to give another proof by using the smoothness of the Hewitt-Stromberg pre-measures). We show also that the upper and lower Hewitt-Stromberg dimension maps are measurable with respect to the σ-algebra generated by the analytic sets of $\mathscr{K}(X)$ and, in general, they are not Borel measurable. Finally, we prove that the Hewitt-Stromberg measure maps (if we assume that the "*subset of positive and finite measure*" property is satisfied) are measurable with respect to the σ-algebra generated by the analytic sets of $\mathscr{K}(X)$ and they need not be Borel measurable. As an application, by composing suitable functions, we can use the results to deduce the measurability of various functions, i.e., we apply the measurability results to study the measurability of the Hewitt-Stromberg measures and dimensions of sections. Remark that some of these results can be viewed as the analogues of the descriptive set-theoretic complexity of Hausdorff and packing measures and dimensions by Olsen [21] when $X = \mathbb{R}^n$, and Mattila and Mauldin [20] in the case where X is a Polish space.

Throughout this paper (X, d) will be a Polish space, that is, a complete separable metric space. We equip the space $\mathscr{K}(X)$ of non-empty compact subsets of X with the Hausdorff distance ρ denoted by

$$\rho(K, L) = \max\left\{\sup_{x\in K}\ d(x, L),\ \sup_{y\in L}\ d(y, K)\right\}.$$

Then $(\mathscr{K}(X), \rho)$ is a complete separable metric space. For $A \subset X$, the closure of A will be $\bar{A}$ and the interior of A will be A°. The σ-algebra generated by analytic sets will be denoted by $\mathscr{B}(\mathscr{A})$. In a product space, π stands for the projection onto the first factor.

The first result gives that the upper and lower box dimension maps are of Baire class 2 and not of Baire class 1. Let us mention that this result has been proved by Mattila and Mauldin [20], the main purpose in the following is to give another proof by using the smoothness (Borel measurability) of the Hewitt-Stromberg pre-measures.

Theorem 1.1. *The functions* $\underline{\dim}_B$, $\overline{\dim}_B : \mathscr{K}(X) \longrightarrow [0,+\infty]$ *are of Baire class* 2 *and not of Baire class* 1.

By using Theorem 1.1 and the characterizations (1.4) and (1.5) of the Hewitt-Stromberg dimensions, we show that these dimension functions are $\mathscr{B}(\mathscr{A})$-measurable, i.e., are measurable with respect to the σ-algebra generated by the analytic sets of $\mathscr{K}(X)$.

Theorem 1.2. *The maps* $\underline{\dim}_{MB}$, $\overline{\dim}_{MB} : \mathscr{K}(X) \longrightarrow [0,+\infty]$ *are* $\mathscr{B}(\mathscr{A})$*-measurable, where* $\mathscr{B}(\mathscr{A})$ *denotes the* σ*-algebra generated by the family* $\mathscr{A}$ *of analytic subsets of* $\mathscr{K}(X)$.

In the following, we present one of our main results which shows that the Hewitt-Stromberg measure functions are measurable with respect to the σ-algebra generated by the analytic sets. In order to obtain optimal results, we occasionally will have to use that the Hewitt-Stromberg measures satisfy the "*subset of positive and finite measure*" property as which is given in [14]. We say that the measure μ has the "*subset of positive and finite measure*" property, if for any compact set E of X with $\mu(E) > 0$ there exists a compact set $F \subseteq E$ such that $0 < \mu(F) < \infty$.

Theorem 1.3. *If the Hewitt-Stromberg measures have the "subset of positive and finite measure" property, then the functions* $\mathscr{U}^g : \mathscr{K}(X) \longrightarrow [0,+\infty]$ *and* $\mathscr{V}^g : \mathscr{K}(X) \longrightarrow [0,+\infty]$ *are measurable with respect to the* σ*-algebra generated by the analytic sets of* $\mathscr{K}(X)$.

As an application, we apply the measurability results established in Theorems 1.2 and 1.3 to study the measurability of sections. Let Y be a Polish space. If $E \subset Y \times X$, and $y \in Y$, then we denote $E_y = \{x \in X \mid (y,x) \in E\}$. Assume that each E_y is compact, it follows from [20, Section 6] that the map $y \mapsto E_y$ is Borel measurable from Y into $\mathscr{K}(X)$. We shall now study the measurability of the Hewitt-Stromberg dimensions and measures of the sections. By applying Theorems 1.2 and 1.3 we can use exactly the above argument to prove the following result.

Theorem 1.4. *Let* $E \subset Y \times X$ *be a Borel set such that all the sections* E_y *are compact for all* $y \in Y$. *One has*

(1) *the maps* $y \mapsto \underline{\dim}_{MB}(E_y)$ *and* $y \mapsto \overline{\dim}_{MB}(E_y)$ *are* $\mathscr{B}(\mathscr{A})$*-measurable.*
(2) *If the Hewitt-Stromberg measures have the "subset of positive and finite measure" propriety, then the maps* $y \mapsto \mathscr{U}^g(E_y)$ *and* $y \mapsto \mathscr{V}^g(E_y)$ *are* $\mathscr{B}(\mathscr{A})$*-measurable, where* $\mathscr{B}(\mathscr{A})$ *denotes the* σ*-algebra generated by the family* $\mathscr{A}$ *of analytic subsets of* $\mathscr{K}(X)$.

In following result we show that the Hewitt-Stromberg measures and dimensions functions need not be Borel measurable.

Theorem 1.5. *The sets*

$$E_1 := \Big\{ K \in \mathscr{K}([0.1]) \;\big|\; \underline{\dim}_{MB}(K) > 0 \Big\}$$

and

$$E_2 := \Big\{ K \in \mathscr{K}([0.1]) \;\big|\; \overline{\dim}_{MB}(K) > 0 \Big\}$$

are an analytic non-Borel sets. In particular, the maps $\underline{\dim}_{MB}$*,* $\overline{\dim}_{MB}$*,* $\mathscr{U}^g$*,* $\mathscr{V}^g : \mathscr{K}(X) \longrightarrow [0, +\infty]$ *are, in general, not Borel measurable.*

Remark 1.2. (1) Here we will give an indication that the subset of positive and finite measure propriety is satisfied for the Hewitt-Stromberg measures. By the way, there are many special cases for which the above-mentioned property is satisfied, for instance, we give a simple case in which it holds for the lower Hewitt-Stromberg measure (the symmetrical result is true as well for the upper Hewitt-Stromberg measure). For $X = \mathbb{R}$, let $E = [0, 1]$ and $g_t(r) = r^t$ with $t = \frac{\log 2}{\log 3}$. It follows from $\underline{\dim}_{MB}(E) = 1$ that $\mathscr{U}^t(E) > 0$. Now, if we take F the middle-$\frac{1}{3}$-Cantor set and by using the fact that the lower Hewitt-Stromberg measure is a natural interpolation between the original Hausdorff and packing measures, then we obtain $0 < \mathscr{U}^t(F) < +\infty$.

(2) Theorem 1.3 (in particular, Theorem 1.4) shows that, if the Hewitt-Stromberg measures have the "*subset of positive and finite measure*" propriety, then these measures are measurable with respect to the σ-algebra generated by the analytic sets. It is natural to ask if this condition can be omitted. We, therefore, pose the following question:

Is it true that the Hewitt-Stromberg measures (even if the gauge function satisfies the doubling condition) have the "*subset of positive and finite measure*" propriety?

Let us mention that we have not been able to give a positive answer to this problem.

(3) The definitions for the box-counting dimensions and Hewitt-Stromberg dimensions can also be given in terms of the diameters of balls instead of their radii as in [20, Section 5] and we obtain the same results.

(4) The study of the descriptive set-theoretic complexity of the multifractal Hewitt-Stromberg measure and dimension maps introduced in [2, 27] will be achieved in further works.

2. Proof of the main results

We present the tools, as well as the intermediate results and some notations, which will be used in the proof of our main results. We will now briefly describe the Borel Hierarchy used in the classification of the smoothness of the Hewitt-Stromberg measure and dimension maps. For an ordinal γ with $1 < \gamma < \omega_1$ (where ω_1 is the first uncountable cardinal) we define the Baire classes Σ^0_γ and Π^0_γ as follows:

$$\Sigma^0_1 = \Sigma^0_1(X) = \Big\{ \mathscr{O} \subseteq X \;\Big|\; \mathscr{O} \text{ is open} \Big\},$$

$$\Pi^0_1 = \Pi^0_1(X) = \Big\{ \mathscr{C} \subseteq X \;\Big|\; \mathscr{C} \text{ is closed} \Big\},$$

$$\Sigma^0_\gamma = \Sigma^0_\gamma(X) = \left\{ \bigcup_{n=1}^{+\infty} \mathscr{W}_n \;\middle|\; \mathscr{W}_n \in \bigcup_{\eta<\gamma} \Pi^0_\eta \right\}$$

and

$$\Pi^0_\gamma = \Pi^0_\gamma(X) = \left\{ \bigcap_{n=1}^{+\infty} \mathscr{W}_n \;\middle|\; \mathscr{W}_n \in \bigcup_{\eta<\gamma} \Sigma^0_\eta \right\}.$$

It is known that

$$\bigcup_{\eta<\omega_1} \Sigma^0_\eta = \bigcup_{\eta<\omega_1} \Pi^0_\eta = \mathscr{B}(X),$$

where $\mathscr{B}(X)$ denotes the Borel σ-algebra on X. The Borel hierarchy, therefore, gives a ramification of the Borel sets in ω_1 levels. Sometimes we will use the traditional notation, $\mathscr{G}(X) = \mathscr{G}$ for the family of open subsets of X, and the traditional notation, $\mathscr{F}(X) = \mathscr{F}$ for the family of closed subsets of X. We have

$$\begin{array}{ll} \Sigma^0_1 = \mathscr{G}, & \Pi^0_1 = \mathscr{F}, \\ \Sigma^0_2 = \mathscr{F}_\sigma, & \Pi^0_2 = \mathscr{G}_\delta, \\ \Sigma^0_3 = \mathscr{G}_{\delta\sigma}, & \Pi^0_3 = \mathscr{F}_{\sigma\delta}, \\ \Sigma^0_4 = \mathscr{F}_{\sigma\delta\sigma}, & \Pi^0_4 = \mathscr{G}_{\delta\sigma\delta}, \\ \vdots & \vdots \end{array}$$

Let X and Y be two metric spaces and $n \in \mathbb{N}$. A function $f : X \to Y$ is said to be of Baire class n if f is Σ^0_{n+1}-measurable. The functions of Baire class 0 are continuous, The functions of Baire class 1 are 1 *step away from being continuous*, etc. It is known that a function f is of Baire class n if and only if the function f is the pointwise limit of a sequence of functions of Baire class $n-1$.

Finally we recall the definition of an analytic set. Let A be a subset of a Polish space is X. We say that A is analytic if it is the continuous image of a Polish space, i.e., if there exist a Polish space Y and a continuous map $f : Y \to X$ such that $f(Y) = A$. Let X be a separable metric space, we denote the family of analytic subsets of X by $\mathscr{A}(X)$. It is known that every Borel set is analytic, i.e.,

$$\mathscr{B}(X) \subseteq \mathscr{A} := \mathscr{A}(X).$$

In particular, we see that every Borel measurable map is $\mathscr{B}(\mathscr{A})$-measurable, where $\mathscr{B}(\mathscr{A})$ denotes the σ-algebra generated by the family $\mathscr{A}$ of analytic subsets of $\mathscr{K}(X)$.

The reader is referred to [16] for an excellent discussion of the Borel Hierarchy, Baire's functions and analytic sets.

2.1. Proof of Theorem 1.1

The next result is essentially Lemma 9 in [25] (see also [15, Lemma 3.1] and [22, Lemma 4.1]).

Proposition 2.1. *For all $r > 0$, the functions $N_r : \mathscr{K}(X) \longrightarrow [0, +\infty]$ and $M_r : \mathscr{K}(X) \longrightarrow [0, +\infty]$ are respectively lower and upper semi-continuous.*

As a consequence of Proposition 2.1 we have the next result.

Proposition 2.2. *Let $\delta > 0$.*

(1) *The function $\overline{\mathscr{U}}^g_\delta : \mathscr{K}(X) \longrightarrow [0, +\infty]$ is upper semi-continuous, in particular of Baire class 1.*
(2) *The function $\overline{\mathscr{V}}^g_\delta : \mathscr{K}(X) \longrightarrow [0, +\infty]$ is lower semi-continuous, in particular of Baire class 1.*
(3) *The functions $\overline{\mathscr{U}}^g$, $\overline{\mathscr{V}}^g : \mathscr{K}(X) \longrightarrow [0, +\infty]$ are of Baire class 2 and not of Baire class 1.*

Proof. (1) It follows from Proposition 2.1 that the function $K \longmapsto M_r(K)$ is upper semi-continues, which implies that the map

$$K \longmapsto \overline{\mathscr{U}}^g_\delta(K) = \inf_{0<r\leq\delta} M_r(K)g(2r)$$

is upper semi-continues and of Baire class 1.

(2) Proposition 2.1 implies that the function $K \longmapsto N_r(K)$ is lower semi-continues, which implies that the map

$$K \longmapsto \overline{\mathscr{V}}^g_\delta(K) = \sup_{0<r\leq\delta} N_r(K)g(2r)$$

is lower semi-continues and of Baire class 1.

(3) It follows from the previous assertions that the functions $\overline{\mathscr{U}}^g$, $\overline{\mathscr{V}}^g$: $\mathscr{K}(X) \longrightarrow [0, +\infty]$ are of Baire class 2, since

$$\overline{\mathscr{U}}^g(K) = \lim_{n\to+\infty} \overline{\mathscr{U}}^g_{\frac{1}{n}}(K) \quad \text{and} \quad \overline{\mathscr{V}}^g(K) = \lim_{n\to+\infty} \overline{\mathscr{V}}^g_{\frac{1}{n}}(K)$$

for all $K \in \mathscr{K}(X)$.

We will now show that the function $K \mapsto \overline{\mathscr{U}}^g(K)$ is not of Baire class 1. The proof that the map $K \mapsto \overline{\mathscr{V}}^g(K)$ is not of Baire class 1 is very similar and is therefore omitted. If $t > 0$ and $g(r) = r^t$, then we will follow the traditional convention and write

$$\overline{\mathscr{U}}^g(E) = \overline{\mathscr{U}}^t(E),$$

it suffices to show that $\overline{\mathscr{U}}^{\frac{1}{3}} : \mathscr{K}(X) \longrightarrow [0, +\infty]$ is not of Baire class 1. Let

$$A = \Big\{ E \in \mathscr{K}(\mathbb{R}) \;\Big|\; E \text{ is finite} \Big\}$$

and

$$B = \Big\{ E \cup F \;\Big|\; E \in \mathscr{K}(\mathbb{R}) \text{ is finite and } F \text{ is a compact line segment in } \mathbb{R} \text{ of positive length} \Big\}.$$

Then we have

$$\overline{\mathscr{U}}^{\frac{1}{3}}(G) = \begin{cases} 0 & \text{if} \quad G \in A, \\ +\infty & \text{if} \quad G \in B. \end{cases}$$

Which proves that the map $K \mapsto \overline{\mathscr{U}}^g(K)$ is everywhere discontinuous and consequently is not of Baire class 1. $\square$

The following lemma gives some elementary properties of the box dimensions which are verified in [9, Section 3.2].

Lemma 2.1. *One has*

1. *a smooth m-dimensional sub-manifold F of $\mathbb{R}^n$ has $\dim_B(F) = m$.*
2. *The functions $\underline{\dim}_B$ and $\overline{\dim}_B$ are monotonic.*
3. *$\overline{\dim}_B$ is finitely stable, i.e.,*
$$\overline{\dim}_B(E \cup F) = \max\left(\overline{\dim}_B(E), \overline{\dim}_B(F)\right),\ \forall E, F \subseteq \mathbb{R}^n,$$
the corresponding identity does not hold for $\underline{\dim}_B$.
4. *$\underline{\dim}_B$ is finitely sub-stable, i.e.,*
$$\underline{\dim}_B(E \cup F) \geq \max\left(\underline{\dim}_B(E), \underline{\dim}_B(F)\right),\ \forall E, F \subseteq \mathbb{R}^n.$$

Now, let us prove that the function $\underline{\dim}_B : \mathscr{K}(X) \longrightarrow [0, +\infty]$ is of Baire class 2 and not of Baire class 1. Let s and t be two real numbers. It follows from (1.2) and Proposition 2.2 that

$$\begin{aligned} &\left\{K \in \mathscr{K}(X) \,\middle|\, s < \underline{\dim}_B(K) < t\right\} \\ = &\left(\bigcup_{n\geq 1} \left\{K \in \mathscr{K}(X) \,\middle|\, \overline{\mathscr{U}}^{s+\frac{1}{n}}(K) > 1\right\}\right) \\ &\bigcap \left(\bigcup_{n\geq 1} \left\{K \in \mathscr{K}(X) \,\middle|\, \overline{\mathscr{U}}^{t-\frac{1}{n}}(K) < 1\right\}\right) \in \mathscr{G}_{\delta\sigma}. \end{aligned}$$

Therefore the map $K \mapsto \underline{\dim}_B(K)$ is of Baire class 2. We will now show that the map $K \mapsto \underline{\dim}_B(K)$ is not of Baire class 1. Consider the following sets

$$A = \left\{E \in \mathscr{K}(\mathbb{R}) \,\middle|\, E \text{ is finite}\right\}$$

and

$$B = \left\{E \cup F \,\middle|\, E \in \mathscr{K}(\mathbb{R}) \text{ is finite and } F \text{ is a compact line segment in } \mathbb{R} \text{ of positive length}\right\}.$$

The sets A and B are dense in $\mathscr{K}(\mathbb{R})$. Moreover, if $G \in A$, $\underline{\dim}_B(G) = 0$ and if $G \in B$, then $G = E \cup F$ with E is finite and F is a compact line segment in $\mathbb{R}$ of positive length, which leads by using Lemma 2.1 to

$$\begin{aligned} 1 = \max\left(\underline{\dim}_B(E), \underline{\dim}_B(F)\right) &\leq \underline{\dim}_B(G) \\ &\leq \overline{\dim}_B(G) \\ &= \max\left(\overline{\dim}_B(E), \overline{\dim}_B(F)\right) = 1. \end{aligned}$$

Therefore, $\underline{\dim}_B(G) = 1$. Which proves that the map $K \mapsto \underline{\dim}_B(K)$ is everywhere discontinuous and consequently is not of Baire class 1.

By using similar techniques the other statement follows immediately from (1.3), Proposition 2.2 and Lemma 2.1.

2.2. Proof of Theorem 1.2

The characterizations (1.4) and (1.5) of the Hewitt-Stromberg dimensions lead to the following proposition.

Proposition 2.3.

(1) (a) *Suppose $c \in \mathbb{R}$, $K \in \mathscr{K}(X)$ and $\underline{\dim}_{MB}(K) > c$. Then there is a non-empty compact set $L \subset K$ such that $\underline{\dim}_{MB}(L \cap V) \geq c$ for all open sets V with $L \cap V \neq \emptyset$.*

(b) *Let $c \in \mathbb{R}$ and $K \in \mathscr{K}(X)$. Then $\underline{\dim}_{MB}(K) \geq c$ if and only if for every $d < c$ there is a non-empty compact set $L \subset K$ such that $\underline{\dim}_B(L \cap \overline{V}) \geq d$ for all open sets V with $L \cap V \neq \emptyset$.*

(2) (a) *Suppose $c \in \mathbb{R}$, $K \in \mathscr{K}(X)$ and $\overline{\dim}_{MB}(K) > c$. Then there is a non-empty compact set $L \subset K$ such that $\overline{\dim}_{MB}(L \cap V) \geq c$ for all open sets V with $L \cap V \neq \emptyset$.*

(b) *Let $c \in \mathbb{R}$ and $K \in \mathscr{K}(X)$. Then $\overline{\dim}_{MB}(K) \geq c$ if and only if for every $d < c$ there is a non-empty compact set $L \subset K$ such that $\overline{\dim}_B(L \cap \overline{V}) \geq d$ for all open sets V with $L \cap V \neq \emptyset$.*

Proof. (1) (a) Consider $L_0 = K$ and define a transfinite sequence of compact subsets of K by iteration for each ordinal α, as follows:

$$L_{\alpha+1} = \Big\{ x \in L_\alpha \;\Big|\; \underline{\dim}_{MB}\left(L_\alpha \cap V\right) \geq c \text{ for all neighborhood } V \text{ of } x \Big\}.$$

For each limit ordinal λ, we let $L_\lambda = \bigcap_{\beta<\lambda} L_\beta$. As $(L_\alpha)_\alpha$ is a descending transfinite sequence of compact sets, there exists a countable ordinal γ such that $L_\gamma = L_{\gamma+1}$. Note that for each ordinal α we have $\underline{\dim}_{MB}\left(L_\alpha \backslash L_{\alpha+1}\right) \leq c$. We assume that for each countable ordinal α we have $\underline{\dim}_{MB}(L_\alpha) > c$. This is certainly true for $\alpha = 0$. Now, suppose this claim holds for all $\beta < \alpha$. Let $\alpha = \tau + 1$, it follows from $L_\tau = L_{\tau+1} \cup (L_\tau \backslash L_{\tau+1})$ that

$$\underline{\dim}_{MB}(L_{\tau+1}) = \underline{\dim}_{MB}(L_\alpha) > c.$$

Now suppose that α is a countable limit ordinal, it follows from

$$L_0 = \bigcup_{\beta<\alpha} \left(L_\beta \backslash L_{\beta+1}\right) \cup L_\alpha$$

that $\underline{\dim}_{MB}(L_\alpha) > c$. Finally, put $L = L_\gamma$ which implies that $L \neq \emptyset$. Suppose there were some open set V such that

$$\underline{\dim}_{MB}(L \cap V) < c \quad \text{and} \quad L \cap V \neq \emptyset,$$

then $L_{\gamma+1} \neq L_\gamma$ which is a contradiction.

(b) "$\Longrightarrow$" It follows immediately from the first assertion.

"$\Longleftarrow$" We assume that the stated condition holds and

$$\underline{\dim}_{MB}(K) < c.$$

Then we can choose $d < c$ for which $\underline{\dim}_{MB}(K) < d$. It follows from (1.4) that there are compact sets $K_1, K_2, \dots$ such that

$$K \subset \bigcup_{i=1}^{\infty} K_i \quad \text{and} \quad \underline{\dim}_B K_i < d \quad \text{for all} \ \ i.$$

Now, we let $L \subset K$ be non-empty, compact such that $\underline{\dim}_B(L \cap \overline{V}) \geq d$ for all open sets V with $L \cap V \neq \emptyset$. Since $L = \bigcup_{i=1}^{\infty} (L \cap K_i)$, it follows from Baire's category theorem that $L \cap K_i$ has non-empty interior relative to L, for some i. Then, there is an open set V with

$$\emptyset \neq L \cap V \subset L \cap \overline{V} \subset L \cap K_i$$

which implies that

$$\underline{\dim}_B(L \cap \overline{V}) \leq \underline{\dim}_B (L \cap K_i) < d$$

which is a contradiction.

(2) The proof of assertion 2 is very similar to the proof of assertion 1 and is therefore omitted. $\square$

Proof of Theorem 1.2. Let us prove that $\underline{\dim}_{MB} : \mathscr{K}(X) \longrightarrow [0, \infty]$ is $\mathscr{B}(\mathscr{A})$-measurable. It is sufficient to prove that, for any $c \in \mathbb{R}$, the set

$$A = \Big\{ K \in \mathscr{K}(X) \ \Big| \ \underline{\dim}_{MB}(K) \geq c \Big\}$$

is analytic. Let $(x_i)_i$ be a countable dense subset of X and r be a positive rational. For positive integers i and m we consider the following sets

$$F = \Big\{ (K, L) \in \mathscr{K}(X) \times \mathscr{K}(X) \ \Big| \ L \subset K \Big\},$$

$$B_i(r) = \Big\{ (K, L) \in \mathscr{K}(X) \times \mathscr{K}(X) \ \Big| \ B^\circ(x_i, r) \cap L = \emptyset \Big\},$$

and

$$C_{m,i}(r) = \left\{ (K, L) \in \mathscr{K}(X) \times \mathscr{K}(X) \ \middle| \ \underline{\dim}_B \left(B(x_i, r) \cap L \right) \geq c - \frac{1}{m} \right\}.$$

It follows from Proposition 2.3(1) that

$$\begin{aligned} &\Big\{ K \in \mathscr{K}(X) \ \Big| \ \underline{\dim}_{MB}(K) \geq c \Big\} \\ = &\bigcap_m \Big\{ K \in \mathscr{K}(X) \ \Big| \ \text{there exists } L \subset K \text{ such that } L \text{ is non-empty compact} \\ &\qquad \text{and if } U \subset X \text{ is open and } L \cap U \neq \emptyset, \text{ then } \underline{\dim}_B(L \cap \overline{U}) \geq c - \frac{1}{m} \Big\} \\ = &\bigcap_m \pi\Big(\Big\{ (K, L) \in \mathscr{K}(X) \times \mathscr{K}(X) \ \Big| \ L \subseteq K \Big\} \end{aligned}$$

$$\bigcap\Big\{(K,L)\in\mathscr{K}(X)\times\mathscr{K}(X)\;\Big|\;\text{if } i\in\mathbb{N},\;\; r\in\mathbb{Q}_+^* \text{ with}$$
$$B^\circ(x_i,r)\cap L\neq\emptyset,\text{ then } \underline{\dim}_B(L\cap B(x_i,r))\geq c-\frac{1}{m}\Big\}\Big)$$
$$=\bigcap_m\Big(\pi\Big(F\cap\bigcap_i\bigcap_{r\in\mathbb{Q}_+^*}\big(B_i(r)\cup C_{m,i}(r)\big)\Big)\Big).$$

The sets F and $B_i(r)$ are clearly closed, and it follows from Theorem 1.1 that the set $C_{m,i}(r)$ is Borel. Consequently, A is an analytic subset of $\mathscr{K}(X)$. $\square$

2.3. Proof of Theorem 1.3

First, let us prove that the map $\mathscr{U}^g:\mathscr{K}(X)\to[0,+\infty]$ is $\mathscr{B}(\mathscr{A})$-measurable. It is sufficient to prove that, for all $c\in\mathbb{R}$, the set

$$\Big\{K\in\mathscr{K}(X)\;\Big|\;\mathscr{U}^g(K)\geq c\Big\}$$

is analytic. We may assume $c>0$ and let

$$\begin{aligned}\pi:\mathscr{K}(X)\times\mathscr{P}(X)\quad&\longrightarrow\quad\mathscr{K}(X)\\ (K,\mu)\quad&\longmapsto\quad K,\end{aligned}$$

here $\mathscr{P}(X)$ denotes the family of Borel probability measures on X equipped with the weak topology. It follows from the "*subset of positive and finite measure*" propriety that, for any compact K checked $\mathscr{U}^g(K)\geq c$, there exists a compact set $L\subseteq K$ such that $c\leq\mathscr{U}^g(L)<+\infty$. Now, we define a Borel probability measure

$$\mu(E)=\frac{\mathscr{U}^g(E\cap L)}{\mathscr{U}^g(L)}\qquad\text{for all Borel subset } E \text{ of } X.$$

Observe that $\operatorname{supp}\mu\subset K$ and $c\;\mu(M)\leq\mathscr{U}^g(M)$ for all $M\in\mathscr{K}(X)$, where $\operatorname{supp}\mu$ denotes the topological support of μ. Inversely, if there exists $\mu\in\mathscr{P}(X)$ such that $\operatorname{supp}\mu\subset K$ and $c\;\mu(M)\leq\mathscr{U}^g(M)$ for all $M\in\mathscr{K}(X)$, then $\mathscr{U}^g(K)\geq c$. Which means that

$$\Big\{K\in\mathscr{K}(X)\;\Big|\;\mathscr{U}^g(K)\geq c\Big\}=\pi(A),$$

where

$$A=\Big\{(K,\mu)\in\mathscr{K}(X)\times\mathscr{P}(X)\;\Big|\;\operatorname{supp}\mu\subset K\text{ and}$$
$$c\;\mu(M)\leq\mathscr{U}^g(M),\;\forall M\in\mathscr{K}(X)\Big\}.$$

Then it is sufficient to prove that A is a Borel set. One has

$$A=B\bigcap\left(\bigcap_{i=1}^{\infty}A_i\right),$$

where

$$B=\Big\{(K,\mu)\in\mathscr{K}(X)\times\mathscr{P}(X)\;\Big|\;\operatorname{supp}\mu\subset K\Big\}$$

and

$$A_i=\Big\{(K,\mu)\in\mathscr{K}(X)\times\mathscr{P}(X)\;\Big|\;c\,\mu(M)\le\overline{\mathscr{U}}^g_{\frac{1}{i}}(M)\text{ for compact sets }M\subset K\Big\}.$$

The set B is closed and $(k,\mu)\in A_i$ if and only if for every $k\in\mathbb{N}$, every compact $M\subseteq K$ and $0<r<\frac{1}{i}$ there exists a packing $\big(B(x_j,r)\big)_{j=1}^n$ of M such that

$$\left(c-\frac{1}{k}\right)\frac{\mu(M)}{n}<g(2r).$$

For a fixed k, this set is open, which implies that the set

$$A_i=\bigcap_{k=1}^{+\infty}\Big\{(K,\mu)\in\mathscr{K}(X)\times\mathscr{P}(X)\;\Big|\;\left(c-\frac{1}{k}\right)\mu(M)<\overline{\mathscr{U}}^g_{\frac{1}{i}}(M)\\ \text{for all compact }M\subset K\Big\}$$

is a $\mathscr{G}_\delta$-set. Thus, we can conclude that A is a Borel set.

The proof of the measurability of the measure $\mathscr{V}^g$ is very similar and is therefore omitted. Which achieve the proof of Theorem 1.3.

2.4. Proof of Theorem 1.5

The proof of Theorem 1.5 is similar to the original argument by Mattila and Mauldin in [20, Theorem 7.5]. For the reader's convenience, we give short proofs of forms suitable for our purposes. We will prove that E_1 is an analytic non-Borel set. The proof for the set E_2 is very similar and is therefore omitted.

It is well-known that each $x\in\mathbb{R}$ possesses a unique continued fraction expansion of the form

$$x=\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{a_4+\cfrac{1}{a_5+\ddots}}}}},$$

where $a_k\in\mathbb{N}:=\{1,2,3,\dots\}$ is the k-th partial quotient of x. This expansion is usually denoted by $x=[a_1,a_2,a_3,\dots]$. $x\in\mathbb{Q}$ if and only if the sequence $(a_k)_k$ is finite. Let $\mathbb{Q}^c$ denote the set of irrational number and

$$x=[a_1,a_2,\dots]\in[0,1]\cap\mathbb{Q}^c.$$

We define

$$\Gamma(x)=\Big\{[a_1,k_1,a_2,k_2,\dots]\;\big|\;k_i\in\mathbb{N}\;\text{ for }i\in\mathbb{N}\Big\}.$$

By using almost identical to the original argument by Mauldin and Mattila in [20, Section 7] there is $c\in(0,\frac{1}{2}]$ such that, for all $x\in[0,1]\cap\mathbb{Q}^c$, we have

$$c\le\underline{\dim}_{MB}(\Gamma(x))\le\overline{\dim}_{MB}(\Gamma(x)).\tag{2.1}$$

It follows from the proof of Theorem 1.2 that E_1 is an analytic set. To prove that E_1 is a non-Borel set (using the completeness method in [16,17]), it suffices to show that, for an analytic subset A of a Polish space X, we can find a Borel measurable function f of X into $\mathscr{K}([0,1])$ such that $f^{-1}(E_1) = A$, i.e., E_1 is a Σ_1^1-complete set. For this, let A be an analytic subset of a Polish space X and g be a continuous function of $\mathscr{Y} = \mathbb{N}^{\mathbb{N}}$ onto A. For $x = (a_1, a_2, a_3, \ldots) \in \mathscr{Y}$, we define the function

$$\phi(x) = (a_1, a_3, a_5, \ldots) .$$

It is clear that ϕ is a continuous function of $\mathscr{Y}$ onto $\mathscr{Y}$. Let $\psi = g \circ \phi : \mathscr{Y} \to A$, which is also continuous. Also, for all $t \in A$ with $g\,(a_1, a_2, a_3, \ldots) = t$, we have

$$\Gamma(x) = \Gamma\,(a_1, a_2, a_3, \ldots) \subseteq \psi^{-1}\{t\}.$$

It follows from (2.1) that

$$c \leq \underline{\dim}_{MB}(\psi^{-1}\{t\}) \leq \overline{\dim}_{MB}(\psi^{-1}\{t\}) \text{ for all } t \in A.$$

Let K be the closure of $\Big\{(\psi(\tau), \tau) \;\Big|\; \tau \in \mathscr{Y}\Big\}$, which is a closed subset of $X \times [0,1]$. Let $t \in A$. Then

$$\psi^{-1}\{t\} \subseteq \mathrm{K}_t := \Big\{y \in [0,1] \;\Big|\; (t,y) \in \mathrm{K}\Big\},$$

which implies that

$$c \leq \underline{\dim}_{MB}(\mathrm{K}_t) \leq \overline{\dim}_{MB}(\mathrm{K}_t).$$

Now, let $t \in X$ and $y \in \mathrm{K}_t$ then we can choose a sequence $\tau_i \in \mathscr{Y}$ with $(\psi\,(\tau_i)\,, \tau_i) \to (t, y)$ as $i \to +\infty$. If $y \in \mathscr{Y}$ is irrational, then $t = \lim_{i \to +\infty} \psi\,(\tau_i)$ $= \psi(y) \in A$ which gives that, for all $t \in X \backslash A$,

$$\mathrm{K}_t \subseteq \mathbb{Q} \quad \text{and} \quad \underline{\dim}_{MB}(\mathrm{K}_t) = \overline{\dim}_{MB}(\mathrm{K}_t) = 0.$$

Finally, we have proven that there is a function $f : X \to \mathscr{K}([0,1])$, $t \mapsto f(t) = K_t$ such that $f^{-1}(E_1) = A$. Which achieve the proof of Theorem 1.5.

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