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A SIMPLE PROOF FOR A RESULT ON n-JORDAN HOMOMORPHISMS

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Abstract. In this short note, we give a simple proof of the main theorem of [5] which states that every n-Jordan homomorphism $h : A \longrightarrow B$ between two commutative algebras A and B is an n-homomorphism.

1. Introduction and preliminaries

The study of additive mappings from one ring $\mathcal{R}$ into another ring $\mathcal{R}'$ which preserve squares was initiated by Ancochea [2] in connection with problems arising in projective geometry. Among others, Kaplansky [11], Jacobson and Rickart [10] and Herstein [9] then proceeded to carry out an extensive study of such functions.

Let $n \geq 2$ be an integer and $\mathcal{R}$ be an associative ring. Following [9], we say that $\mathcal{R}$ is of characteristic not n if $na = 0$ implies $a = 0$ for every $a \in \mathcal{R}$, and $\mathcal{R}$ is of characteristic greater than n if $n!a = 0$ implies $a = 0$ for every $a \in \mathcal{R}$.

The additive mapping $\varphi : \mathcal{R} \longrightarrow \mathcal{R}'$ between two rings is called an n-*homomorphism* if for all $a_1, a_2, \ldots, a_n \in \mathcal{R}$,

$$\varphi(a_1 a_2 \cdots a_n) = \varphi(a_1)\varphi(a_2)\cdots\varphi(a_n),$$

and it is called an n-Jordan homomorphism if $\varphi(a^n) = \varphi(a)^n$ for all $a \in \mathcal{R}$.

The concept of an n-homomorphism was studied for complex algebras in [8], and the notion of n-Jordan homomorphism was dealt with firstly by Herstein [9].

If $n = 2$, then this concepts coincides the classical definitions of homomorphism and Jordan homomorphism, respectively.

It is clear that every n-homomorphism is an n-Jordan homomorphism, but in general the converse is false. There are plenty of known examples of n-Jordan homomorphisms which are not n-homomorphisms. For $n = 2$, it is proved in [10] that some Jordan homomorphism on the polynomial rings cannot be homomorphism.

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For characterization of n-Jordan homomorphisms on rings and Banach algebras we refer the reader to [1, 9, 13–17], and a list of references.

It is shown in [6] that each n-Jordan homomorphism between two commutative algebras is an n-ring homomorphism for $n \in \{3, 4\}$, and this result extended to $n < 8$, in [4]. Note that for $n = 2$, the proof is simple and routine.

In [4], the author asked the following:

Is every n-Jordan homomorphism between commutative algebras also a n-ring homomorphism when $n \in \mathbb{N}$?

In 2013, Lee [12] and in 2014, Gselmann in [7] independently answered this problem in the affirmative without knowing the above question. The following general result is due to Gselmann.

Theorem 1.1 ([7, Theorem 2.1]). *Let $n \in \mathbb{N}$, $\mathcal{R}$ and $\mathcal{R}'$ be two commutative rings such that char$(\mathcal{R}') > n$ and suppose that $\varphi : \mathcal{R} \longrightarrow \mathcal{R}'$ is an n-Jordan homomorphism. Then φ is an n-homomorphism.*

Moreover, if $\mathcal{R}$ is unital, then $\varphi(e) = \varphi(e)^n$ and the map $\psi : \mathcal{R} \longrightarrow \mathcal{R}'$ defined by $\psi(x) = \varphi(e)^{n-2}\varphi(x)$ is a homomorphism.

In 2018, Bodaghi and İnceboz solved this problem in [3, Theorem 2.2] based on the property of the Vandermonde matrix, which is different from the methods that are used in [7] and [12].

In 2020 the following result was appeared in [5], by Cheshmavar et al.

Theorem 1.2 ([5, Theorem 2.3]). *Let A and B be two commutative algebras, $n \geq 3$ and let $h : A \longrightarrow B$ be an n-Jordan homomorphism. Then h is an n-homomorphism.*

We need the following lemma for a simple proof of Theorem 1.2.

Lemma 1.3 ([9, Lemma 1]). *Let $\varphi(x_1, x_2, \ldots, x_n)$ be multilinear from one ring $\mathcal{R}$ into another ring $\mathcal{R}'$, and suppose that $\varphi(x, x, \ldots, x) = 0$ for all $x \in \mathcal{R}$. Then*

$$\sum_{\sigma \in S_n} \varphi(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) = 0,$$

where S_n denotes the symmetric group of $\{1, 2, \ldots, n\}$.

Next we give a simple proof of Theorem 1.2.

Theorem 1.4. *Let $\mathcal{R}$ and $\mathcal{R}'$ be two rings and $\varphi : \mathcal{R} \longrightarrow \mathcal{R}'$ be an n-Jordan homomorphism. Then*

$$\sum_{\sigma \in S_n} \big(\varphi(x_{\sigma(i)} x_{\sigma(2)} \cdots x_{\sigma(n)}) - \varphi(x_{\sigma(1)})\varphi(x_{\sigma(2)}) \cdots \varphi(x_{\sigma(n)})\big) = 0, \tag{1}$$

where S_n denotes the symmetric group of $\{1, 2, \ldots, n\}$.

Moreover, if $\mathcal{R}$ and $\mathcal{R}'$ are commutative and char$(\mathcal{R}') > n$, then φ is an n-ring homomorphism.

Proof. Define the map $\psi : \mathcal{R}^n \longrightarrow \mathcal{R}'$ by

$$\psi(x_1, x_2, \ldots, x_n) = \varphi(x_1 x_2 \cdots x_n) - \varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)$$

for all $x_1, x_2, \ldots, x_n \in \mathcal{R}$. Then ψ is multilinear and $\psi(x, x, \ldots, x) = 0$ for all $x \in \mathcal{R}$. It follows from Lemma 1.3, that

$$\sum_{\sigma \in S_n} \psi(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) = 0,$$

and hence the equation (1) holds true. If $\mathcal{R}$ and $\mathcal{R}'$ are commutative, then the equation (1) yields that

$$\varphi(n! x_1 x_2 \cdots x_n) = n!\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)$$

for all $x_1, x_2, \ldots, x_n \in \mathcal{R}$. Since char($\mathcal{R}'$)$> n$, this forces

$$\varphi(x_1 x_2 \cdots x_n) = \varphi(x_1)\varphi(x_2)\cdots\varphi(x_n).$$

Therefore, φ is an n-ring homomorphism. □

It should be point out that if B is an algebra, then char$(B) > n$ and so Theorem 1.4 is stronger than of Theorem 1.2.

The next example provided that the commutativity of $\mathcal{R}$ and $\mathcal{R}'$ in the above theorem is essential.

Example 1.5. Let

$$\mathcal{R} = \left\{ \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} : M, N \in M_2(\mathbb{C}) \right\}.$$

Then under the usual matrix operations, $\mathcal{R}$ is a unital ring which is not commutative. Define an additive map $\varphi : \mathcal{R} \longrightarrow \mathcal{R}$ by

$$\varphi\left(\begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} \right) = \begin{bmatrix} M & 0 \\ 0 & N^T \end{bmatrix},$$

where N^T is the transpose of matrix N. Then, for all $X \in \mathcal{R}$, we have $\varphi(X^n) = \varphi(X)^n$. Therefore, φ is an n-Jordan homomorphism, but φ is not an n-ring homomorphism.

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