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ON GRADED J-IDEALS OVER GRADED RINGS

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Abstract. The goal of this article is to present the graded J-ideals of G-graded rings which are extensions of J-ideals of commutative rings. A graded ideal P of a G-graded ring R is a graded J-ideal if whenever $x, y \in h(R)$, if $xy \in P$ and $x \notin J(R)$, then $y \in P$, where $h(R)$ and $J(R)$ denote the set of all homogeneous elements and the Jacobson radical of R, respectively. Several characterizations and properties with supporting examples of the concept of graded J-ideals of graded rings are investigated.

1. Introduction

The theory of graded commutative rings is the focus of this paper. Rings with gradings are used to describe several subjects in algebraic geometry. We assume throughout this article that all rings are commutative with identity. Let R be a ring and P be a proper ideal of R. Recently, Khashan in [10], defined the concept of a J-ideal. A proper ideal P of R is said to be a J-ideal of R if whenever $x, y \in R$ with $xy \in P$ and $x \notin J(R)$, then $y \in P$, where $J(R)$ is the Jacobson radical of R. We are dealing with graded J-ideals in a G-graded ring in this case.

Let G be an abelian group with identity e and R be a commutative ring with $1 \neq 0$. Then R is called a G-graded ring if $R = \bigoplus_{g \in G} R_g$ with the property $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$, where R_g is an additive subgroup of R for all $g \in G$. The elements of R_g are called homogeneous of degree g. If $x \in R$, then x can be written uniquely as $\sum_{g \in G} x_g$, where x_g is the component of x in R_g. The set of all homogeneous elements of R is $h(R) = \bigcup_{g \in G} R_g$. Let P be an ideal of a G-graded ring R. Then P is called a graded ideal if $P = \bigoplus_{g \in G} P_g$, i.e., for $x \in P$ and $x = \sum_{g \in G} x_g$, where $x_g \in P_g$ for all $g \in G$. An ideal of a G-graded ring is not necessary a graded ideal (see [1]). In [9], the Jacobson radical of a G-graded ring is defined as $J(R) = \bigoplus_{g \in G} J(R) \cap R_g$. Let P be a proper graded ideal of R. Then the graded radical of P is denoted by $Grad(P)$

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and it is defined as written below:

$$Grad(P) = \Big\{x = \sum_{g \in G} x_g \in R : \text{for all } g \in G, \text{ there exists } n_g \in \mathbf{N} \text{ such that } {x_g}^{n_g} \in P\Big\}.$$

Note that $Grad(P)$ is always a graded ideal of R (see [15]). If P is a graded ideal of R, then the quotient ring R/P is a G-graded ring. Indeed,

$$R/P = \bigoplus_{g \in G} (R/P)_g,$$

where $(R/P)_g = \{x + P : x \in R_g\}$. A G-graded ring R is called a G-graded integral domain if whenever $x, y \in h(R)$ with $xy = 0$, then either $x = 0$ or $y = 0$. Many studies have been conducted since a graded algebra plays such an important role (see for example [2–4]).

Ashby first proposed the concept of a graded principal ideal in [6]. A graded ideal P of R is said to be a graded principal ideal of R if $P = \langle x \rangle$ for some $x \in h(R)$. The G-graded ring R is said to be a G-graded principal ring if every graded ideal of R is a graded principal ideal of R. Let R be a G-graded ring. A proper graded ideal P of R is said to be graded prime if whenever $x, y \in h(R)$ with $xy \in P$, then either $x \in P$ or $y \in P$ (see [15]). In [5], P is said to be a graded n-ideal of R if whenever $x, y \in h(R)$ with $xy \in P$ and $x \notin Grad(0)$, then $y \in P$. In Section 2, we introduce and study the concept of a graded J-ideal. We call a proper graded ideal P of a graded ring R a graded J-ideal if whenever $x, y \in h(R)$ with $xy \in P$ and $x \notin J(R)$, then $y \in P$.

Among many results in this paper, first we investigate the location of a graded J-ideal between the concept of a graded ideal in the literature such as graded prime, a graded n-ideal, a graded r-ideal by illustrating some examples. For instance, it is clear from the definitions that every graded n-ideal is a graded J-ideal. But the converse of this implication is not true in general (see Examples 2.3 and 2.4). In general, Proposition 2.13 is a useful result to built some examples for graded J-ideals which are not graded n-ideals. On the other hand, a graded prime ideal needs not to be a graded J-ideal and vice versa (see Example 2.5). Then, we give the relationship between these two concepts (see Theorems 2.6 and 2.9). Also, the connection among graded J-ideals of a G-graded ring, J-submodules of the R_e-module M and J-ideals of R_e is justified (see Proposition 2.15). A characterization of graded J-ideals in a graded ring R is given (Theorem 2.10). Moreover, the graded rings of which every graded ideal is a graded J-ideal is exactly characterized (see Theorem 2.12).

In Section 3, we discuss the behaviour of graded J-ideals over a graded epimorphism (Proposition 3.5), over graded factor rings (Corollary 3.6), over localizations (Proposition 3.9) and over direct product of graded rings (Remark 3.10). Finally, in Theorem 3.8, we prove that if P is a graded ideal of a graded ring R and S is a graded J-multiplicatively closed subset of R disjoint with P, then there exists a graded J-ideal I of R that is disjoint with S containing P.

2. Graded J-ideals in graded rings

In this section, we introduce the concept of graded J-ideals and investigate some of their properties and characterizations. Moreover, we clarify their relation with some other types of graded ideals such as graded n-ideals, graded prime, and graded r-ideals.

Definition 2.1. Let R be a G-graded ring. A proper graded ideal P of R is said to be a graded J-ideal if whenever $x, y \in h(R)$ with $xy \in P$ and $x \notin J(R)$, then $y \in P$.

Proposition 2.2. *Let P be a graded ideal of a G-graded ring R.*

(1) *Any graded n-ideal is a graded J-ideal.*

(2) *If P is a proper graded J-ideal of R, then $P \subseteq J(R)$.*

Proof. (1) The implication is clear as $N(R) \subseteq J(R)$.

(2) Suppose that P is a graded J-ideal of R and $P \nsubseteq J(R)$. Then we have $x \in P$ such that $x \notin J(R)$. Now, $x \cdot 1 \in P$ and $x \notin J(R)$ implies that $1 \in P$, a contradiction. Thus, $P \subseteq J(R)$. □

However, the following examples are given to show that the converses of these implications above do not hold in general.

Example 2.3. Let $R = \{\frac{x}{y} : x, y \in h(\mathbf{Z}), 2 \nmid y\}$ be a quasi-local G-graded ring. By Theorem 2.9 any ideal of R is J-ideal. Consider the graded J-ideal $P = \{\frac{x}{y} : x \in \langle 8 \rangle, 2 \nmid y\}$ of R which is not a graded n-ideal of R. Since $\frac{2}{7}, \frac{4}{3} \in h(R)$ and $\frac{2}{7} \cdot \frac{4}{3} = \frac{8}{21} \in P$ but $\frac{2}{7} \notin N(R)$ and $\frac{4}{3} \notin P$.

Example 2.4. Let $R = \mathbf{Z}_{36}$ be a G-graded ring with $R_e = \mathbf{Z}_{36}$ and $R_g = \{0\}$, where $g \in G - \{e\}$ and $p = \langle 12 \rangle$ be a graded ideal of R. Then $J(R) = \langle 6 \rangle$ and then $P \subseteq J(R)$. But P is not a graded J-ideal of R since $3 \cdot 4 \in P$ with $3 \notin J(R)$ and $4 \notin P$.

Note that graded J-ideals are not comparable with graded prime ideals in general. A graded J-ideal needs not to be graded prime. Indeed, let R and P be as in Example 2.3. P is a graded J-ideal that is not graded prime as $\frac{2}{7} \cdot \frac{4}{3} = \frac{8}{21} \in P$ but $\frac{2}{7}, \frac{4}{3} \notin P$. Conversely, we present the following example to show that a graded prime ideal needs not to be a graded J-ideal.

Example 2.5. Consider $G = \mathbf{Z}_2$ and a graded ring $R = \mathbf{Z}[i]$, where $R_0 = \mathbf{Z}$ and $R_1 = i\mathbf{Z}$. Then $I = p\mathbf{Z}$, where p is a prime integer, is a graded prime ideal. However, it is not a graded J-ideal as $p \cdot 1 \in I$ but neither $p \in J(R)$ nor $1 \in I$.

Theorem 2.6. *Let R be a G-graded ring and P be a graded prime ideal of R. Then P is a graded J-ideal of R if and only if $P = J(R)$. In particular, $J(R)$ is a graded J-ideal of R if and only if $J(R)$ is a graded prime ideal of R.*

Proof. Suppose that P is a graded prime ideal of R. It is easy to show that $J(R) \subseteq P$. Now, if P is a graded J-ideal of R, then by Proposition 2.2 we have $P \subseteq J(R)$. Therefore, $P = J(R)$. Conversely, assume that $P = J(R)$ and P is a graded prime ideal of R. Let $x, y \in h(R)$ with $xy \in P$ and $x \notin J(R) = P$ but P is a graded prime ideal of R so $y \in P$. Therefore, P is a graded J-ideal of R.

For the "particular" part, suppose that $J(R)$ is a graded J-ideal of R. Let $x, y \in h(R)$ such that $xy \in J(R)$ and $x \notin J(R)$. Then $y \in J(R)$ as $J(R)$ is a graded J-ideal of R. Therefore, $J(R)$ is a graded prime ideal of R. The converse is the special case of the situation above. $\square$

Let I and K be two graded ideals of a G-graded ring R. Then the ideal $\{x \in R : xK \subseteq I\}$, denoted by $(I : K)$, is a graded ideal [14, Proposition 1.13]. Also, we need to state that for $a \in h(R)$ and graded ideals I, J of R, aR, $I \cap J$ and $I + J$ and are graded ideals (see [12, Lemma 2.1]).

Lemma 2.7. *If I and P are two graded J-ideals of R with $I \nsubseteq P$, then so is $(P : I)$. In particular, if P is a graded J-ideal of R and $x \in h(R) - P$, then $(P : x)$ is a graded J-ideal of R.*

Proof. If $(P : I) = R$, then $1 \in (P : I)$ and $I \subseteq P$, which is a contradiction. Hence, $(P : I)$ is a proper graded ideal of R. Suppose that $xy \in (P : I)$ and $x \notin J(R)$ for some $x, y \in h(R)$. Then $xyi \in P$ for every $i \in I$. Since P is a graded J-ideal of R, it follows $yi \in P$; and so $y \in (P : I)$. Hence $(P : I)$ is a graded J-ideal of R. For the particular case, put $I = xR$. $\square$

Definition 2.8. Let R be a G-graded ring. A proper graded J-ideal P of R is called a graded maximal J-ideal if there is no a graded J-ideal which contains P properly.

Now, we are ready to give another the relationship between graded J-ideals and graded prime ideals of a graded ring R.

Theorem 2.9. *Every graded maximal J-ideal of a graded ring R is graded prime.*

Proof. Let P be a graded maximal J-ideal of R. Suppose that $xy \in P$ and $x \notin P$ for some $x, y \in h(R)$. Then $(P : x)$ is a graded J-ideal of R by Lemma 2.7. However, since P is maximal, we conclude the equality $P = (P : x)$; and thus $y \in P$. Therefore, P is graded prime. $\square$

In the following, we give a characterization for graded J-ideals in a graded ring R.

Theorem 2.10. *Let R be a G-graded ring and P be a graded ideal of R. Then the following are equivalent:*

(1) $P = \bigoplus_{g \in G} P_g$ *is a graded J-ideal of R.*
(2) $P = (P : x)$ *for every $x \in h(R) - J(R)$.*

(3) *For each $g, h \in G$, $IJ \subseteq P$, where I and J are R_e-submodules of R_g and R_h, respectively, we have either $I \subseteq J(R)$ or $J \subseteq P$.*

Proof. (1) $\Rightarrow$ (2) Suppose that P is a graded J-ideal of R. Let $x \in h(R) - J(R)$. It is clearly $P \subseteq (P : x)$. Now, let $y \in (P : x)$, where $y \in h(R)$. Then P is a graded J-ideal of R and $x \notin J(R)$ implies that $y \in P$. Thus $(P : x) \subseteq P$. Thus, $P = (P : x)$.

(2) $\Rightarrow$ (3) Suppose that $IJ \subseteq P$ and $I \nsubseteq J(R)$, where I and J are submodules of R_g and R_h, respectively. If $I \nsubseteq J(R)$, then there exists $x \in I$ such that $x \notin J(R)$. Thus $xJ \subseteq P$, and then $J \subseteq (P : x) = P$.

(3) $\Rightarrow$ (1) Let $x, y \in h(R)$ such that $xy \in P$ and $x \notin J(R)$. Then there exist $g, h \in G$ such that $x \in R_g$ and $y \in R_h$. Take two graded submodules $I := R_e x$ and $J := R_e y$ of R_g and R_h, respectively. Then $IJ \subseteq P$. By our assumption, $x \in I \subseteq J(R)$ or $y \in J \subseteq P$. Therefore, P is a graded J-ideal of R. $\square$

For graded J-ideals I and J of R, if $IP = JP$, then we have the following cancellation whenever $P \not\subset J(R)$:

Corollary 2.11. *Let R be a G-graded ring and P be a graded ideal of R with $P \not\subset J(R)$. Then the following hold.*

(1) *If I and J are graded J-ideals of R with $IP = JP$, then $I = J$.*

(2) *If I is a graded ideal and IP is a graded J-ideal of R, then $IP = I$.*

Proof. (1) Assume that $IP = JP$. Since $JP \subseteq I$ and I is a graded J-ideal of R, by Theorem 2.10 we have $J \subseteq I$. Similarly, since J is a graded J-ideal of R, we have $I \subseteq J$. Thus $I = J$.

(2) It is similar to (1). $\square$

It is known from [15] that I is a graded maximal ideal if I is proper and there exists no a graded ideal K of R such that $I \subset K \subset R$. Next, we determine the graded rings of which every graded ideal is a graded J-ideal.

Theorem 2.12. *Let R be a G-graded ring. Then the following assertions are equivalent*:

(1) *R is a quasi-local G-graded ring.*

(2) *Every graded ideal of R is a graded J-ideal of R.*

(3) *Every graded principal ideal of R, then P is a graded J-ideal of R.*

Proof. (1) $\Rightarrow$ (2) Let R be a quasi-local G-graded ring with the unique graded maximal ideal M of R and $x, y \in h(R)$ with $xy \in P$ and $x \notin J(R) = M$. Then $x \in U(R)$ and it follows that $x^{-1}(xy) \in P$. Therefore, $y \in P$ and P is a graded J-ideal of R.

(2) $\Rightarrow$ (3) It is immediate.

(3) $\Rightarrow$ (1) Assume that P is a graded principal ideal of R and P is a graded J-ideal of R. Let M be a graded maximal ideal of R and $x \in M$. Now, let $P = \langle x \rangle$ and $1 \cdot x \in P$. If $x \notin J(R)$, then $1 \in P$ this is a contradiction. Thus $x \in J(R)$ and $J(R) = M$ is the unique graded maximal ideal of R. Therefore, R is a quasi-local G-graded ring. $\square$

In the following, we state a result which enables to built some examples for graded J-ideals which are not graded n-ideals.

Proposition 2.13. *Let R be a quasi-local G-graded ring for which $N(R) \neq J(R)$. Then $J(R)$ is a graded J-ideal of R which is not a graded n-ideal of R.*

Proof. Since R is a quasi-local G-graded ring, $J(R)$ is a graded J-ideal of R by Theorem 2.12. Take $x \in J(R) - N(R)$. Then $x = x \cdot 1 \in J(R)$ but neither $x \in N(R)$ nor $1 \in J(R)$. Therefore, $J(R)$ is not a graded n-ideal of R. ☐

We call a graded ring R as gr-presimplifiable if $Z(R) \cap h(R) \subseteq J(R) \cap h(R)$. According to this definition, we conclude the following connection between graded J-ideals and graded r-ideals.

Proposition 2.14. *Any graded r-ideal of a gr-presimplifiable ring R is a graded J-ideal.*

Proof. Let R be a gr-presimplifiable ring and I be a graded r-ideal. Suppose that $xy \in I$ and $x \notin J(R)$ for some $x, y \in h(R)$. Then $x \notin Z(R)$ by our assumption and since I is a graded r-ideal, it follows that $y \in I$, as required. ☐

A proper submodule N of an R-module M is called a J-submodule of M if for $a \in R$ and $m \in M$, whenever $am \in N$ and $a \notin (J(R)M : M)$, then $m \in N$ [10]. We end this section by giving a relationship among graded J-ideals of a G-graded ring, J-submodules of the R_e-module M and J-ideals of R_e.

Proposition 2.15. *Let $I = \oplus_{g \in G} I_g$ be a graded ideal of a G-graded ring R. Then the following statements hold*:

(1) *Let I be a graded J-ideal of R. Then I_g is a J-submodule of the R_e-module R_g for each $g \in G$. Additionally, $(I_g :_{R_e} R_g)$ is a J-ideal of R_e for all $g \in G$ provided that R_g is a multiplication R_e-module.*

(2) *If R is strongly graded and I_e is a J-ideal of R_e, then I_g is a J-submodule of the R_e-module R_g; and also I is a graded J-ideal of R.*

Proof. (1) Let $x \in R_e$ and $m \in R_g$ with $am \in I_g$. Then it implies that $x \in J(R)$ or $m \in I$. Hence, we have $x \in J(R) \cap R_e \subseteq (J(R_e)R_g : R_g)$ or $m \in I \cap R_g = I_g$, and thus I_g is a J-submodule of R_e-module R_g. Now, if R_g is a multiplication R_e-module, then $(I_g :_{R_e} R_g)$ is a J-ideal of R_e by [10, Proposition 3.3].

(2) Let $g \in G$. If $I_g = R_g$, then clearly $I_e = R_e$ which contradicts with our assumption that I_e is a J-ideal of R_e. Hence, I_g is proper. Let $x \in R_e$ and $m \in R_g$ such that $xm \in I_g$. Choose a unit element $u \in R_{g^{-1}}$. Then $x(um) \in I_e$ which implies either $x \in J(R_e)$ or $um \in I_e$ as I_e is a J-ideal. Thus, $x \in J(R_e) \subseteq (J(R_e)R_g : R_g)$ or $m \in I_g$. Thus I_g is a J-submodule of the R_e-module R_g. Now, we show that I is a graded J-ideal of R. Let $x, y \in h(R)$ such that $xy \in I$. Say, $x \in R_g$ and $y \in R_h$ for some $g \in, h \in G$. Choose a unit $v \in R_g$. Then $(xv)y \in R_h \cap I = I_h$ and being I_h a J-submodule of an R_e-module R_h implies that either $xv \in (J(R_e)R_h : R_h)$ or $y \in I_h$. Therefore, we conclude $x \in J(R)$ or $y \in I$, so we are done. ☐

3. Behavior of graded J-ideals

This section is devoted to investigate the behavior of graded J-ideals over a graded epimorphism, over graded factor rings, over localizations and over direct product of graded rings. Let R and R' be G-graded rings. A ring homomorphism $f : R \to R'$ is said to be a graded ring homomorphism if $f(R_g) \subseteq R'_g$ for every $g \in G$ (see [11]). Let $f : R \to R'$ be a graded ring homomorphism and I, J are two graded ideals of R, R', respectively. Then $f^{-1}(J)$ and $f(I)$ are graded rings of R, R', respectively (see [13]). Analogue of [10, Proposition 2.23], we have the following result.

Definition 3.1. Let R be a G-graded ring and M be a graded ideal of R. We say that M is a graded maximal ideal of R if $M \neq R$ and M is not properly contained in a proper graded ideal of R.

Theorem 3.2. *Let R be a G-graded ring and M be a graded ideal of R. Then M is a graded maximal ideal of R if and only if R/M is a graded field.*

Note that, R is a G-graded field if $h(R) \subseteq U(R) \cup \{0\}$.

Lemma 3.3. *Let R_1 and R_2 be G-graded rings and $f : R_1 \to R_2$ be a graded ring homomorphism. If f is a graded epimorphism and M is a graded maximal ideal of R_2, then $f^{-1}(M)$ is a graded maximal ideal of R_1 with $R_1/f^{-1}(M) \approx R_2/M$.*

Proof. Let $\phi : R_1 \to R_2/M$ be a map define by $\phi(a) = f(a) + M$ for all $a \in h(R_1)$, since f is a graded epimorphism so ϕ is a graded homomorphism with $Ker(\phi) = \{a \in h(R_1) : f(a) \in M\} = f^{-1}(M)$. By first isomorphism theorem for graded rings we have that $R_1/f^{-1}(M) \approx R_2/M$. But R_2/M is a graded field, then $R_1/f^{-1}(M)$ is a graded field. Therefore, $f^{-1}(M)$ is a graded maximal ideal of R_1 with $f(f^{-1}(M)) = M$. $\square$

We denoted the set of all graded maximal ideal of R by $GSpecm(R)$.

Lemma 3.4. *Let R_1 and R_2 be G-graded rings and $f : R_1 \to R_2$ be a graded ring homomorphism. If f is a graded epimorphism, then $f(J(R_1)) \subseteq J(R_2)$.*

Proof. Let

$$J(R_1) = \bigcap_{M^* \in GSpecm(R_1)} M^* \subseteq \bigcap_{M \in GSpecm(R_2)} f^{-1}(M)$$

by Lemma 3.3.

$$\begin{aligned} f(J(R_1)) &= f(\bigcap_{M^* \in GSpecm(R_1)} M^*) \\ &\subseteq f(\bigcap_{M \in GSpecm(R_2)} f^{-1}(M)) \\ &\subseteq \bigcap_{M \in GSpecm(R_2)} f(f^{-1}(M)) \end{aligned}$$

$$= \bigcap_{M \in GSpecm(R_2)} M = J(R_2).$$

□

Proposition 3.5. *Let R_1 and R_2 be G-graded rings and $f : R_1 \to R_2$ be a graded ring homomorphism. Then the following hold*:

(1) *If P is a graded J-ideal of R_1 and f is a graded epimorphism with $Ker(f) \subseteq P$, then $f(P)$ is a graded J-ideal of R_2.*

(2) *If I is a graded J-ideal of R_2 and f is a graded monomorphism, then $f^{-1}(I)$ is a graded J-ideal of R_1.*

Proof. (1) Let $x, y \in h(R_2)$ such that $xy \in f(P)$ and $x \notin J(R_2)$. Since f is a graded epimorphism, we can take $a, b \in h(R_1)$ such that $x = f(a)$ and $y = f(B)$, then $xy = f(ab) \in f(P)$. But $Ker(f) \subseteq P$, we have $ab \in P$. Also, $a \notin J(R_1)$, if $a \in J(R_1)$, then $f(a) = x \in J(R_2)$ by Lemma 3.4 which is a contradiction. But P is a graded J-ideal of R_1 we have $b \in P$ and then $y = f(b) \in f(P)$. Therefore, $f(P)$ is a graded J-ideal of R_2.

(2) Let $x, y \in h(R_1)$ with $xy \in f^{-1}(I)$ and $x \notin J(R_1)$. Then we have $f(x)f(y) = f(xy) \in I$. We need to show that $f(x) \notin J(R_2)$. Suppose that $f(x) \in J(R_2)$ and let M be a graded maximal ideal of R_1. Then $f(M)$ is a graded maximal ideal of R_2 since $Ker(f) \subseteq J(R_1) \subseteq M$. Hence $f(x) \in f(M)$, and then $x \in M$. Thus, $x \in J(R_1)$ which is a contradiction. Since I is a graded J-ideal of R_2 we have $f(y) \in I$ and so $y \in f^{-1}(I)$. Therefore, $f^{-1}(I)$ is a graded J-ideal of R_1. □

Let R be a G-graded ring and P be an ideal and I be a graded ideal of R with $I \subseteq P$. Then R/I is G-graded by $(R/I)_g = (R_g + I)/I$ for all $g \in G$ [11], and also, P is a graded ideal of R if and only if P/I is a graded ideal of R/I [16, Lemma 3.2].

Corollary 3.6. *Let R be a G-graded ring and P, I be graded ideals of R with $I \subseteq P$. Then the following are hold*:

(1) *If P is a graded J-ideal of R, then P/I is a graded J-ideal of R/I.*

(2) *If P/I is a graded J-ideal of R/I and $I \subseteq J(R)$, then P is a graded J-ideal of R.*

(3) *If P/I is a graded J-ideal of R/I and I is a graded J-ideal of R, then P is a graded J-ideal of R.*

Proof. (1) Suppose that P is a graded J-ideal of R with $I \subseteq P$. Let $f : R \to R/I$ be a graded epimorphism defined by $f(x) = x + I$, where $x \in h(R)$. Note that $Ker(f) = I \subseteq P$. Then by Proposition 3.5(1) and then $f(P) = P/I$. Therefore, P/I is a graded J-ideal of R/I.

(2) Let $f : R \to R/I$ be a graded epimorphism defined by $f(x) = x+I$, where $x \in h(R)$. Since $I \subseteq J(R)$ by Proposition 3.5(2). Therefore, $P = f^{-1}(P/I)$ is a graded J-ideal of R.

(3) It is similar to (2) and Proposition 3.5. □

The converse of (1) in Corollary 3.6 is not true in general. For example $\langle 0\rangle \simeq \langle 5\rangle/\langle 5\rangle$ is a graded J-ideal of the G-graded field $\mathbf{Z}/\langle 5\rangle \simeq \mathbf{Z}_5$ but $\langle 5\rangle$ is not a graded J-ideal of $\mathbf{Z}$ as $\langle 5\rangle \nsubseteq J(\mathbf{Z}) = \langle 0\rangle$.

In a G-graded ring R, if I and J are graded ideals of R, then $I+J$ is a graded ideal of R (see [8]). Moreover, if $Grad(I) + Grad(J) = R$, then $I + J = R$ by [14]. For the next result, first we need the following lemma: It is well-known that if P and Q are graded ideals of a graded ring R, then $P \cap Q$ is a graded ideal of R. Then we have a following result for graded J-ideals of R.

Lemma 3.7. *Let R be a G-graded ring and P_i for all $i = \{1, 2, \ldots, n\}$ be a graded ideal of R. Then we have the following*:

(1) *If P_i is a graded J-ideal of R for each i, then $P = \bigcap_{i=1}^{n} P_i$ is a graded J-ideal of R.*

(2) *If P_i's are pairwise non-comparable prime graded ideals and $P = \bigcap_{i=1}^{n} P_i$ is a graded J-ideal of R, then P_i is a graded J-ideal of R for each i.*

Proof. (1) Let $x, y \in h(R)$ with $xy \in P$ and $x \notin J(R)$. Hence $xy \in P_i$ for all $i = \{1, 2, \ldots, n\}$ but P_i is a graded J-ideal of R for each i with $x \notin J(R)$ follows $y \in P_i$ for each i. Thus $y \in P = \bigcap_{i=1}^{n} P_i$. Therefore, P is a graded J-ideal of R.

(2) Let $n = 2$. Suppose that $xy \in P_1$ and $x \notin J(R)$ for some $x, y \in h(R)$. Since P_1 and P_2 are non-comparable, we may choose $z \in P_2 \backslash P_1$. Hence, $xyz \in P_1 \cap P_2 = P$ and since P is a graded J-ideal, it follows that $yz \in P \subseteq P_1$. Now, since P_1 is prime and $z \notin P_1$, we conclude $y \in P_1$, as needed. By a symmetric way, P_2 is a graded J-ideal of R. For $n \geq 3$, it is easy to prove that the claim holds by using mathematical induction. □

Theorem 3.8. *Let R be a G-graded ring and P, I be graded ideals of R. If P and I are graded J-ideals of R, then $P + I$ is a graded J-ideal of R.*

Proof. Suppose that P and I are graded J-ideals of R. Then $P + I$ is a graded ideal of R. Since $P \cap I$ is a graded J-ideal of R by (1) in Lemma 3.7, then $P/(P \cap I)$ is a graded J-ideal of $R/(P \cap I)$ by Corollary 3.6(1). By the isomorphism $P/(P \cap I) \cong (P + I)/I$, we have $(P + I)/I$ is a graded J-ideal of R/I. Therefore, $P + I$ is a graded J-ideal of R by Corollary 3.6. □

Note that, $S^{-1}R$ is a G-graded ring with $S^{-1}R = \{\frac{r}{s} : r \in R, s \in S\}$. If P is a graded ideal of R and $P \cap S = \emptyset$, then $S^{-1}P \neq S^{-1}R$ and $S^{-1}P$ is a graded ideal of $S^{-1}R$ (see [7] and [11]). We will use the notation $G - Z_I(R) = \{x \in h(R) : xs \in I \text{ for some } s \in h(R)\backslash h(I)\}$ below.

Proposition 3.9. *Let R be a G-graded ring and $S \subseteq h(R)$ be a multiplicative set of R such that $J(S^{-1}R) = S^{-1}J(R)$.*

(1) *If P is a graded J-ideal of R with $P \cap S = \emptyset$, then $S^{-1}P$ is a graded J-ideal of $S^{-1}R$.*

(2) *If $S^{-1}P$ is a graded J-ideal of $S^{-1}R$ and $S \cap G - Z_I(R) = S \cap G - Z_{J(R)}(R) = \emptyset$, then P is a graded J-ideal of R.*

Proof. (1) Let $x, y \in h(R)$ and $t, s \in S$ such that $\frac{x}{s}.\frac{y}{t} \in S^{-1}P$ and $\frac{x}{s} \notin J(S^{-1}R)$, then $\frac{x}{s} \notin S^{-1}J(R)$ and then $x \notin J(R)$. Now $\frac{x}{s}.\frac{y}{t} = \frac{z}{w}$ for some $z \in P$ and $w \in S$ and so there exists $s_1 \in S$ such that $s_1xy \in P$. But P is a graded J-ideal of R we have $s_1y \in P$ and then $\frac{y}{t} = \frac{s_1y}{s_1t} \in S^{-1}P$. Therefore, $S^{-1}P$ is a graded J-ideal of $S^{-1}R$.

(2) Suppose that $xy \in P$ for some $x, y \in h(R)$. Then $\frac{x}{1}\frac{y}{1} \in S^{-1}P$ which implies that either $\frac{x}{1}\frac{y}{1} \in J(S^{-1}R) = S^{-1}J(R)$ or $\frac{y}{t} \in S^{-1}P$. Hence, there exist some $u, v \in S$ such that $ux \in J(R)$ or $vy \in P$. From our assumption, we conclude that $x \in J(R)$ or $y \in P$, as needed. □

Let R_1 and R_2 be G-graded rings. Then $R = R_1 \times R_2$ is a G-graded ring with $R_g = (R_1)_g \times (R_2)_g$ for all $g \in G$ (see [11]).

Remark 3.10. Let R_1 and R_2 be G-graded rings. Then $R = R_1 \times R_2$ has no a graded J-ideal. Indeed, suppose that $P = P_1 \times P_2$ is a graded J-ideal of $R = R_1 \times R_2$, where P_1 and P_2 are graded ideals of R_1 and R_2, respectively. Since $(0,1)(1,0) \in P$ but $(0,1),(1,0) \notin J(R) = J(R_1 \times R_2)$, then we have $(0,1),(1,0) \in P$ and then $P = R_1 \times R_2$ which is a contradiction. Therefore, $R = R_1 \times R_2$ has no a graded J-ideal.

Definition 3.11. Let $S \neq \emptyset$ be a subset of a G-graded ring R with $h(R) - J(R) \subseteq S$. Then S is called a graded J-multiplicatively closed subset of R if $ts \in S$ for all $t \in h(R) - J(R)$ and $s \in S$.

Proposition 3.12. *Let R be a G-graded ring and P be a graded ideal of R. Then the following are equivalent*:

(1) *P is a graded J-ideal of R.*

(2) *$h(R) - P$ is a graded J-multiplicatively closed subset of R.*

Proof. (1) ⇒ (2) Let P be a graded J-ideal of R. Assume that $t \in h(R) - J(R)$ and $s \in h(R) - P$. Since $t \notin J(R)$, $s \notin P$ and P is a graded J-ideal of R, then we have $ts \notin P$. Therefore, $st \in h(R) - P$. Since P is a graded J-ideal of R we have $P \subseteq J(R)$ by Proposition 2.2. Then $h(R) - J(R) \subseteq h(R) - I$.

(2) ⇒ (1) Suppose that $t, s \in h(R)$ with $rs \in P$ and $r \notin J(R)$. If $s \in h(R) - P$, then by assumption (2), we have $ts \in h(R) - P$ which is a contradiction. Thus $s \in I$. Therefore, P is a graded J-ideal of R. □

Theorem 3.13. *Let R be a G-graded ring, P be a graded ideal of R and S be a graded J-multiplicatively closed subset of R with $P \cap S = \emptyset$. Then there exists a graded J-ideal I of R such that $P \subseteq I$ and $I \cap S = \emptyset$.*

Proof. Let $T = \{J : J$ is a graded ideal of R with $P \subseteq J$ and $J \cap S = \emptyset\}$. Note that $T \neq \emptyset$ since $P \in T$. Suppose that $J_1 \subseteq J_2 \subseteq \cdots$ is a chain in T. Suppose that $xy \in \bigcup_{i=1}^{n} J_i$ and $x \notin J(R)$, where $x, y \in h(R)$. Thus $xy \in J_i$ for some $i = 1, \ldots, n$. Since J_i is a graded J-ideal of R and $x \notin J(R)$, then $y \in J_i \subseteq \bigcup_{i=1}^{n} J_i$. Therefore, $\bigcup_{i=1}^{n} J_i$ is a graded J-ideal of R. Since $P \subseteq \bigcup_{i=1}^{n} J_i$ and $(\bigcup_{i=1}^{n} J_i) \cap S = \bigcup_{i=1}^{n}(J_i \cap S) = \emptyset$, we have $\bigcup_{i=1}^{n} J_i$ is the upper bound of this chain. By using Zorn's Lemma, there is a maximal element I of T. Now, we show that I is a graded J-ideal of R. Suppose not, then we have $xy \in I$ and $x \notin J(R)$ and $y \notin I$. Thus we have $y \in (I : x)$ and $I \subset (K : x)$. But I is a maximal we have $(I : x) \cap S \neq \emptyset$ and so there exists $s \in S$ such that $s \in (I : x)$. Then $xs \in I$. If $x \in J(R)$, then we are done. So assume that $x \notin J(R)$. Since S is a graded J-multiplicatively closed subset of R, we conclude that $xs \in S$. Thus $xs \in S \cap I$ which is a contradiction. Therefore, I is a graded J-ideal of R. □

References

[1] R. Abu-Dawwas and M. Bataineh, *Graded r-ideals*, Iran. J. Math. Sci. Inform. **14** (2019), no. 2, 1–8.

[2] T. Al-Shorman and M. Bataineh, *On graded classical S-primary submodules*, arXiv preprint arXiv:2204.07578 (2022).

[3] T. Al-Shorman, M. Bataineh, and R. Abu-Dawwas, *Generalizations of graded S-primary ideals*, Proyecciones **41** (2022), 1353–1376.

[4] K. Al-Zoubi and M. Al-Azaizeh, *On graded classical 2-absorbing second submodules of graded modules over graded commutative rings*, Afr. Mat. **33** (2022), no. 2, Paper No. 43, 10 pp. https://doi.org/10.1007/s13370-022-00982-1

[5] K. Al-Zoubi, F. Al-Turman, and E. Y. Celikel, *gr-n-ideals in graded commutative rings*, Acta Univ. Sapientiae Math. **11** (2019), no. 1, 18–28. https://doi.org/10.2478/ausm-2019-0002

[6] W. T. Ashby, *On graded principal ideal domains*, JP J. Algebra Number Theory Appl. **24** (2012), no. 2, 159–171.

[7] M. Atiyah, *Introduction to Commutative Algebra*, CRC Press, 2018.

[8] F. Farzalipour and P. Ghiasvand, *On the union of graded prime submodules*, Thai J. Math. **9** (2011), no. 1, 49–55.

[9] A. V. Kelarev, *On the Jacobson radical of graded rings*, Comment. Math. Univ. Carolin. **33** (1992), no. 1, 21–24.

[10] H. A. Khashan and A. B. Bani-Ata, *J-ideals of commutative rings*, Int. Electron. J. Algebra **29** (2021), 148–164.

[11] C. Năstăsescu and F. Van Oystaeyen, *Methods of graded rings*, Lecture Notes in Mathematics, 1836, Springer-Verlag, Berlin, 2004. https://doi.org/10.1007/b94904

[12] K. H. Oral, Tekir, and A. G. Ağargün, *On graded prime and primary submodules*, Turkish J. Math. **35** (2011), no. 2, 159–167.

[13] M. Refai and R. Abu-Dawwas, *On generalizations of graded second submodules*, Proyecciones **39** (2020), no. 6, 1537–1554.

[14] M. Refai and K. Al-Zoubi, *On graded primary ideals*, Turkish J. Math. **28** (2004), no. 3, 217–229.

[15] M. Refai, M. Hailat, and S. Obiedat, *Graded radicals and graded prime spectra*, Far East J. Math. Sci. (FJMS) **2000**, Special Volume, Part I, 59–73.

[16] H. Saber, T. Alraqad, and R. Abu-Dawwas, *On graded s-prime submodules*, AIMS Math. **6** (2021), no. 3, 2510–2524. https://doi.org/10.3934/math.2021152

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