Commun. Korean Math. Soc. **38** (2023), No. 2, pp. 331–340
https://doi.org/10.4134/CKMS.c220097
pISSN: 1225-1763 / eISSN: 2234-3024

MAXIMAL CHAIN OF IDEALS AND n-MAXIMAL IDEAL

Hemin A. Ahmad and Parween A. Hummadi

Abstract. In this paper, the concept of a maximal chain of ideals is introduced. Some properties of such chains are studied. We introduce some other concepts related to a maximal chain of ideals such as the n-maximal ideal, the maximal dimension of a ring S $(M.\dim(S))$, the maximal depth of an ideal K of S $(M.d(K))$ and maximal height of an ideal $K(M.d(K))$.

1. Introduction

In this paper, S is a commutative ring with identity. A chain of ideals $K_0 \subset K_1 \subset K_2 \subset \cdots$ of S is called a chain of prime ideals, if K_i is a prime ideal of S [6]. Such a chain of ideals is called maximal if there is no further a prime ideal can be inserted between K_{i-1} and K_i for each $i \in \mathbb{Z}^+$ [3]. A chain of proper ideals $K_0 \subset K_1 \subset K_2 \subset \cdots$ of S is called a prime (resp. p-maximal) ascending chain of ideals if K_{i-1} is a prime (resp. prime and maximal) ideal in K_i for each $i \in \mathbb{Z}^+$[1]. The length of a finite chain of prime ideals $K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_{h-1} \subset K_h$ of S is h. The maximum length of a chain of prime ideals is called the dimension of S and the depth (resp. height) of an ideal K of S is the maximum length over all chains of prime ideals in S with an initial (resp. a terminal) ideal K [3, 6]. These ideas motivated us to introduce and study some new concepts. Let $J \subset K$ be two proper ideals of S. The ideal J is said to be maximal in K, if there is no ideal I of S such that $J \subset I \subset K$. A chain of proper ideals $K_0 \subset K_1 \subset K_2 \subset \cdots$ of S is called the maximal chain of ideals of S if K_{t-1} is a maximal ideal in K_t for each $t \in \mathbb{Z}^+$. If $K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_h$ is a finite chain, then K_0 is said to be the initial ideal and K_h is the terminal ideal of the chain. A nonmaximal proper ideal K_0 of S is called a maximal ideal of length m with respect to the maximal chain of ideals $K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_m$, if K_m is a maximal ideal of S.

A ring S has the property FMC, if for every two proper ideals $J \subset K$ of S, there is a finite maximal chain of ideals of S with an initial ideal J and

Received April 9, 2022; Accepted May 25, 2022.
2020 *Mathematics Subject Classification.* 13E99.
Key words and phrases. Maximal chain, property FMC, Artinian ring, n-maximal ideal, maximal dimension, maximal depth, maximal height.

a terminal ideal K. This property gives a clue to give a characterization of Artinian rings. In Section 3, the concept of an n-maximal ideal is introduced via a maximal chain of ideals. Some results on such ideals are obtained. The relations between an n-maximal ideal with some other types of ideals, such as a prime ideal, a weakly prime ideal, a primary ideal, a quasi prime ideal, an almost prime ideal, an irreducible ideal and a strongly irreducible ideal are discussed. In Section 4, the concepts of the maximal dimension $M.\dim(S)$ of a ring S, the maximal depth $M.d(K)$ and the maximal height $M.h(K)$ of an ideal K of S are introduced.

2. Maximal chain of ideals

In this section, the concepts of a maximal chain of ideals of a ring and the property FMC of a ring are introduced and studied. We obtain some results and properties of a maximal chain of ideals of a ring having the property FMC.

Definition 2.1. A chain of proper ideals $K_0 \subset K_1 \subset K_2 \subset \cdots$ of a ring S is called the maximal chain of ideals of S if K_{t-1} is a maximal ideal in K_t for each $t \in \mathbb{Z}^+$. If $K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_h$ is a finite chain, then K_0 is said to be the initial ideal and K_h is the terminal ideal of the chain. A nonmaximal proper ideal K_0 of S is called a maximal ideal of length m with respect to the maximal chain of ideals $K_0 \subset K_1 \subset K_2 \subset \cdots$, if there exists $m \in \mathbb{Z}^+$ such that K_m is a maximal ideal of S. The length of K_0 is said to be ∞, if there is no such the finite maximal chain of ideals with initial ideal K_0. Moreover, the length of a maximal ideal is defined to be 0. Also the chain $J_0 \supset J_1 \supset J_2 \supset \cdots$ is said to be a maximal chain of ideals of S, if J_h is a maximal ideal in J_{h-1} for each $h \in \mathbb{Z}^+$.

Examples 2.2. 1. Consider the ring $S = \mathbb{Z}_{p^n}$, where p is a prime number and $n > 1$. Let $K_i = \langle p^{n-i} \rangle$, where $0 \leq i < n$. The chain $K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_{n-1}$ is a finite maximal chain of ideals with an initial ideal $K_0 = \langle 0 \rangle$ and a terminal ideal $K_{n-1} = \langle p \rangle$ which is the maximal ideal of S and for each $0 \leq i < n$, K_i is a maximal ideal of length $(n-1) - i$ with respect to the maximal chain of ideals $K_i \subset \cdots \subset K_{n-1}$.

2. Let $S = \prod_1^\infty \mathbb{Z}_2$ be the ring of direct product of an infinite countable copies of $\mathbb{Z}_2$. For each $i \in \mathbb{Z}^+ \cup \{0\}$, consider the ideal

$$K_i = \underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{i\text{-times}} \times \{0\} \times \{0\} \times \cdots .$$

Then for each $0 \leq i$, K_i is a maximal ideal in K_{i+1}. So that $K_i \subset K_{i+1} \subset K_{i+2} \subset \cdots$ is an infinite maximal chain of ideals of S with an initial ideal K_i. Therefore, for each $0 \leq i$, K_i is a maximal ideal of length ∞ with respect to the maximal chain of ideals $K_i \subset K_{i+1} \subset K_{i+2} \subset \cdots$. Moreover for each

$i \in \mathbb{Z}^+ \cup \{0\}$, consider the ideal

$$J_i = \underbrace{\{0\} \times \{0\} \times \cdots \times \{0\}}_{(i+1)\text{-times}} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots .$$

Then for each $0 \leq i$, J_{i+1} is a maximal ideal in J_i. So that $J_0 \supset J_1 \supset J_2 \supset \cdots$ is an infinite maximal chain of ideals of S with a terminal ideal J_0 which is a maximal ideal of S. This means that for each $1 \leq i$, J_i is a maximal ideal of length i with respect to the maximal chain of ideals $J_0 \supset J_1 \supset J_2 \supset \cdots \supset J_i$.

3. The zero ideal of $\mathbb{Z}$ is neither a maximal ideal of $\mathbb{Z}$ nor maximal in any other ideal of $\mathbb{Z}$. This means that there is no a finite maximal chain of ideals with initial ideal $\langle 0 \rangle$ and a terminal ideal which is a maximal ideal of $\mathbb{Z}$. So that $\langle 0 \rangle$ is a maximal ideal of length ∞.

Definition 2.3 ([5])**.** A proper ideal I of a ring S is called strongly irreducible if for any two ideals A and B of S, $A \cap B \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$.

Remark 2.4. Consider the ideal K of a ring S. Then

1. If K is a maximal ideal in more than one ideal of S, then K is not a strongly irreducible ideal consequently not prime, since if K is a maximal ideal in two ideals J and I of S, then clearly $K \subseteq J \cap I \subset I$. Since K is a maximal ideal in I, then $K = J \cap I$. Hence K is not a strongly irreducible, since $J, I \nsubseteq K$.

2. If K is maximal in exactly one ideal, then K need not be a strongly irreducible (resp. not prime) ideal. For example, consider the ideals $K_0 = \langle 0 \rangle$, $K_1 = \langle 2 \rangle = \{0, 2\}$, $K_2 = \langle x \rangle = \{0, x\}$, $K_3 = \langle 2 + x \rangle = \{0, 2 + x\}$ and $K_4 = \langle 2, x \rangle = \{0, 2, x, 2 + x\}$ of the ring $S = \mathbb{Z}_4[x]/\langle 2x, x^2 \rangle = \{0, 1, 2, 3, x, 1 + x, 2 + x, 3 + x\}$ such that $x^2 = 2x = 0$. Clearly each of K_1, K_2 and K_3 are maximal in exactly one ideal of S that is K_4 but they are not strongly irreducible.

Directly from Remark 2.4 we get the following result.

Corollary 2.5. *A strongly irreducible ideal of a ring S is maximal in at most one ideal of S.*

Theorem 2.6. *Let S be an integral domain. The zero ideal of S can not be maximal in any other proper ideal of S.*

Proof. Suppose $\langle 0 \rangle$ is a maximal ideal in a proper ideal K of S. Clearly K is a principal ideal say $K = \langle a \rangle$ where a is non zero non unit. Then $\langle 0 \rangle \subseteq \langle a^2 \rangle \subseteq \langle a \rangle$. Being the zero ideal maximal in K, then $\langle 0 \rangle = \langle a^2 \rangle$ or $\langle a^2 \rangle = \langle a \rangle$. If $\langle 0 \rangle = \langle a^2 \rangle$, then $a^2 = 0$ which is a contradiction with S is an integral domain. If $\langle a^2 \rangle = \langle a \rangle$, then $ta^2 = a$ for some $t \in S \setminus \{0\}$. So that $a(ta - 1) = 0$. If $ta - 1 = 0$, then a is a unit, contradiction, which completes the proof. ☐

Definition 2.7. A ring S has the property FMC, if for every two proper ideals $J \subset K$ of S, there is a finite maximal chain of ideals of S with an initial ideal J and a terminal ideal K.

From Definition 2.7 we get the following result.

Remark 2.8. If a ring S has the property FMC, then every nonmaximal ideal of S is maximal in another ideal of S.

The following two theorems are needed.

Theorem 2.9 ([2])**.** *A commutative ring with identity is Noetherian if and only if each of its ideals is finitely generated.*

Theorem 2.10 ([4])**.** *For a commutative ring S with identity the following are equivalent*:

(1) *S is Artinian.*
(2) *S is Noetherian and has Krull dimension* 0*.*
(3) *Every nonempty family of ideals of S contains a minimal element under inclusion.*

Theorem 2.11. *Let S be a ring which is not a field having the property FMC. Then*

(1) *S is not an integral domain, equivalently the zero ideal is not prime.*
(2) *If H is a nonmaximal proper ideal of S, then the quotient ring S/H has the property FMC.*
(3) *Every prime ideal of S is a maximal ideal. Equivalently* $\dim(S) = 0$.
(4) *S is a Noetherian ring.*
(5) *Every chain of ideals $J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n$ of S with an initial ideal J_0 and a terminal ideal J_n is a subsequence of a finite maximal chain of ideals of S with an initial ideal J_0 and a terminal ideal J_n.*

Proof. (1) From the assumption, the zero ideal is maximal in a nonzero proper ideal of S. By Theorem 2.6, S is not an integral domain, equivalently the zero ideal is not prime.

(2) Let H be a nonmaximal proper ideal of S. Suppose $\bar{J} \subset \bar{K}$ are two ideals of S/H. Thus there are two ideals J, K of S such that $\bar{J} = J + H$ and $\bar{K} = K + H$. By the property FMC of S, there is a maximal chain $J \subset K_1 \subset K_2 \subset \cdots \subset K_m \subset K$ with an initial ideal J and a terminal ideal K. This implies that there is a chain $\bar{K}_0 = \bar{J} \subset \bar{K}_1 \subset \bar{K}_2 \subset \cdots \subset \bar{K}_m \subset \bar{K} = \bar{K}_{m+1}$, where $\bar{K}_i = K_i + H$ for each $1 \leq i \leq m+1$. If $\bar{K}_i \neq \bar{K}_{i+1}$ and $\bar{K}_i$ is not maximal in $\bar{K}_{i+1}$, then there is an ideal $\bar{L}$ of S/H such that $\bar{K}_i \subset \bar{L} \subset \bar{K}_{i+1}$. This implies that there is an ideal L of S such that $\bar{L} = L+H$ and $K_i \subset L \subset K_{i+1}$ which is a contradiction. This means if $\bar{K}_i \neq \bar{K}_{i+1}$, then $\bar{K}_i$ is maximal in $\bar{K}_{i+1}$. By removing the equal ideals in the chain $K_0 = \bar{J} \subseteq \bar{K}_1 \subseteq \bar{K}_2 \subseteq \cdots \subseteq \bar{K}_m \subseteq \bar{K} = \bar{K}_{m+1}$ it remains a finite maximal chain of ideals $\bar{J} \subset \bar{J}_0 \subset \bar{J}_1 \subset \cdots \subset \bar{J}_n \subset \bar{J}_{n+1} = \bar{K}$ of S/H with an initial ideal $\bar{J}$ and a terminal ideal $\bar{J}$, where $\bar{K}_i = \bar{J}_j$ for some $1 \leq i \leq m+1$, which completes the proof.

(3) Consider the prime ideal Q of S. Clearly S/Q is an integral domain. By (2), S/Q has the property FMC. If we suppose that S/Q is not a field, then

by part (1), S/Q is not an integral domain, so we get a contradiction. Hence S/Q is a field. Consequently Q is a maximal ideal.

(4) Let $H \neq \langle 0 \rangle$ be an ideal of S. Then there is a finite maximal chain of ideals $\langle 0 \rangle = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_m \subset H = K_{m+1}$ of S with an initial ideal $\langle 0 \rangle$ and a terminal ideal H. Clearly for each $0 < i \leq m+1$, K_i is generated by an element $a_i \in K_i - K_{i-1}$ and K_{i-1} that is $K_i = \langle K_{i-1}, a_i \rangle$. So that H is finitely generated and $H = \langle a_1, a_2, \ldots, a_m \rangle$. Therefore, by Theorem 2.9, S is Noetherian.

(5) Let $J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n$ be a chain of ideals of S with an initial ideal J_0 and a terminal ideal J_n. If J_i is not maximal in J_{i+1}, then by assumption there is a finite maximal chain of ideals of S with an initial ideal J_i and a terminal ideal J_{i+1} of the form $J_i \subset J_{i1} \subset J_{i2} \subset \cdots \subset J_{im_i} \subset J_{i+1}$. Then clearly the chain $J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n$ with an initial ideal J_0 and a terminal ideal J_n is a subsequence of a finite maximal chain of ideals of S with an initial ideal J_0 and a terminal ideal J_n of the form $J_0 \subset J_{01} \subset J_{02} \subset \cdots \subset J_{0m_0} \subset J_1 \subset \cdots \subset J_{n-1} \subset J_{(n-1)1} \subset J_{(n-1)2} \subset \cdots \subset J_{(n-1)m_{n-1}} \subset J_n$. □

Now, we give a characterization of an Artinian ring.

Theorem 2.12. *Let S be a ring. Then S is an Artinian ring if and only if S has property FMC.*

Proof. Suppose S is an Artinian ring and let $J \subset K$ be two proper ideals of S. Let $T_1 = \{H : H \text{ is an ideal of } S \text{ and } J \subset H \subseteq K\}$. Clearly $T_1 \neq \phi$. Since S is an Artinian ring, by Theorem 2.10, T_1 has a minimal ideal say I_1. So that J is a maximal ideal in I_1. If $I_1 = K$, then $J \subset K$ is a finite maximal chain of ideals of S with an initial ideal J and a terminal ideal K. If $I_1 \neq K$, then $T_2 = \{H : H \text{ is an ideal of } S \text{ and } I_1 \subset H \subseteq K\}$ is a non empty family of ideals. Since S is Artinian, then T_2 has a minimal ideal say I_2. So that I_1 is a maximal ideal in I_2. If $I_2 = K$, then $J \subset I_i \subset K$ is a finite maximal chain of ideals of S with an initial ideal J and a terminal ideal K. Proceeding in this way, we obtain a maximal chain of ideals $J \subset I_1 \subset I_2 \subset \cdots$ of S. By Theorem 2.10, the ring S is Noetherian. Then there exists $t \in \mathbb{Z}^+$ such that $I_t = I_{t+1} = \cdots$. Clearly $I_t \subseteq K$. If $I_t \subset K$, we get a contradiction. This means $I_t = K$. So that we obtain the finite maximal chain of ideal $J \subset I_1 \subset I_2 \subset \cdots \subset I_t \subset K$ with an initial ideal J and a terminal ideal K. This means that S has the property FMC. The converse, follows from (3) and (4) of Theorem 2.11. □

From Remark 2.8 and Theorem 2.12, we get the following result.

Corollary 2.13. *If a ring S has a nonmaximal proper ideal which is not maximal in any other ideal of S, then S is not an Artinian ring.*

3. n-maximal ideals

In this section, we introduce the concept of an n-maximal ideal of a commutative ring with identity and we illustrate it by some examples. We obtain some results and properties of such ideals.

Definition 3.1. Let I_0 be a nonmaximal proper ideal of a ring S which has a finite length with respect to a maximal chain of ideals with an initial ideal I_0 and let $n = \min\{t : t$ is the length of I_0 with respect to a maximal chain of ideals of the form $I_0 \subset I_1 \subset I_2 \subset \cdots\}$. Then I_0 is called an n-maximal ideal and a maximal ideal of S is said to be a 0-maximal ideal of S.

Example 3.2. Let $K = \langle p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m} \rangle$ be a nonmaximal ideal of $\mathbb{Z}$, where p_i's are distinct primes and $m \geq 1$, and $\alpha_i \in \mathbb{Z}^+$ for $1 \leq i \leq m$ with at least one of α_i, m is greater than one. Then K is an n-maximal ideal, where $n = (\sum_1^m \alpha_i) - 1$. Furthermore, the zero ideal is not an n-maximal ideal of $\mathbb{Z}$.

Remark 3.3. Let S be a ring. Then

1. If $I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_m$ is a maximal chain of ideals of S and I_m is a maximal ideal of S, then for each $0 \leq i < m$, the ideal I_i is an n-maximal ideal for some $n \in \mathbb{Z}^+$.
2. If K is a nonmaximal proper ideal of S which is not maximal in any other ideal of S, then K is not an n-maximal ideal of S.

Proof. 1. Let $I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_m$ be a maximal chain of ideals of S and I_m is a maximal ideal of S. Clearly for each $0 \leq i < m$, the length of I_i is $m-i$ with respect to the maximal chain of ideals $I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_m$. The set $A_i = \{t : t$ is the length of I_i with respect to a maximal chain of ideals of the form $I_i \subset I_{1+i} \subset I_{i+2} \subset \cdots \subset I_m\}$ is a nonempty subset of $\mathbb{Z}^+$. Therefore, I_i is an n-maximal ideal, where n is the least element of A_i.

2. It is clear. □

Example 3.4. Consider the ring $S = \mathbb{Z}_2[x_1, x_2, \ldots]$, where x_i are indeterminates. Then for each $t \in \mathbb{Z}^+$, the principal ideal $K_t = \langle x_t \rangle$ is not a maximal ideal of S. Furthermore, for each $t \in \mathbb{Z}^+$, clearly K_t is not an n-maximal ideal of S.

Theorem 3.5. *If S is an Artinian ring, then every nonmaximal proper ideal of S is an n-maximal ideal.*

Proof. Let K be a nonmaximal proper ideal of an Artinian ring S. Then K contained in a maximal ideal M of S. By Theorem 2.12, there is a finite maximal chain of ideals $K \subset I_1 \subset I_2 \subset \cdots \subset I_m \subset M$ of S with an initial ideal K and a terminal ideal M. So that K is a maximal ideal of length $m+1$ with respect to a maximal chain of ideals. Let $B = \{t : t$ is the length of K with respect to a maximal chain of ideals of the form $K \subset J_1 \subset J_2 \subset \cdots\}$. Then B is a nonempty subset of $\mathbb{Z}^+$. Let n be the least element of B. Therefore, K is an n-maximal ideal. □

Remark 3.6. The concept of an n-maximal ideal is independent with the concepts of a prime ideal, a weakly prime ideal, a primary ideal, a quasi prime ideal, an almost prime ideal, an irreducible ideal and a strongly irreducible ideal. For example, the ideal $I = \langle 30 \rangle$ of the ring $\mathbb{Z}$ is a 2-maximal but it is not any one of a prime ideal, a weakly prime ideal, a primary ideal, a quasi prime ideal, an almost prime ideal, an irreducible ideal and a strongly irreducible ideal. On the other hand the ideal $J = \langle x \rangle$ of $S = \mathbb{Z}_2[x, y, z]$ is a prime ideal, consequently is a weakly prime ideal, a primary ideal, a quasi prime ideal, an almost prime ideal, an irreducible ideal and a strongly irreducible ideal but it is not an n-maximal ideal of $S = \mathbb{Z}_2[x, y, z]$.

Proposition 3.7. *Let I be a* 1*-maximal ideal of a ring S. Then*

(1) *Either $rad(I)$ is a maximal ideal of S or $rad(I)$ is a* 1*-maximal ideal.*
(2) *I is contained in at most two maximal ideals.*
(3) *If I is contained in exactly two maximal ideals of S say M_1 and M_2, then $I = M_1 \cap M_2$.*

Proof. (1) Since I is a 1-maximal ideal, then I is maximal in every maximal ideal of S containing it. So that $I \subseteq rad(I) \subseteq M$, where M is a maximal ideal containing I. Since I is a maximal ideal in M, then either $rad(I) = M$ or $rad(I) = I$.

(2) If I is contained in at least three distinct maximal ideals of S say M_1, M_2 and M_3, then clearly the following chains are obtained:

$$I \subseteq M_1 \cap M_2 \cap M_3 \subset \begin{cases} M_1 \cap M_2 \subset \begin{cases} M_1, \\ M_2, \end{cases} \\ M_1 \cap M_3 \subset \begin{cases} M_1, \\ M_3, \end{cases} \\ M_2 \cap M_3 \subset \begin{cases} M_2, \\ M_3. \end{cases} \end{cases}$$

This means that I is not a maximal ideal in each of M_1, M_2 and M_3, contradiction.

(3) Let I be contained in exactly two maximal ideals of S say M_1 and M_2. Then clearly

$$I \subseteq M_1 \cap M_2 \subset \begin{cases} M_1, \\ M_2. \end{cases}$$

So that $I = M_1 \cap M_2$, since I is a maximal ideal in each of M_1 and M_2. □

Remark 3.8. If I is a 1-maximal ideal of S, then one of the following statements must hold:

1. $rad(I) = M_1 \cap M_2$, where M_1, M_2 are the only two distinct maximal ideals of S containing I.
2. Either $rad(I) = I$ or $rad(I) = M$, where M is a maximal ideal of S.

Proof. Let I be a 1-maximal ideal of S. By Proposition 3.7, I is contained in at most two maximal ideals.

1. If I is contained in two maximal ideals M_1 and M of S, then

$$I \subseteq M_1 \cap M_2 \subset \begin{cases} M_1, \\ M_2. \end{cases}$$

This means $I = M_1 \cap M_2$, consequently $rad(I) = M_1 \cap M_2$.

2. If I is contained in one maximal ideal of S which is M, then clearly $I \subseteq rad(I) \subseteq M$. Then either $rad(I) = I$or $rad(I) = M$. □

Theorem 3.9. *Let I be an n-maximal ideal of a ring S. Then I is contained in at most $n+1$ maximal ideals of S.*

Proof. If $n = 1$, then by Proposition 3.7, I is contained in at most two maximal ideals of S. Suppose the statement is true for $n = k$ and let I be a $(k+1)$-maximal ideal. Then there is an ideal J of S such that I is maximal in J and J is a k-maximal ideal. So that J is contained in at most $k+1$ maximal ideals of S. Suppose that it is contained in exactly r maximal ideals say $M_1, M_2, \ldots, M_r$ for some $1 \leq r \leq k+1$. If I is contained in at least $k+3$ maximal ideals say $M_1, M_2, \ldots, M_{k+2}, M_{k+3}$, then $I \subseteq J \cap M_{k+2} \cap M_{k+3} \subset J \cap M_{k+2} \subset J$. This means that I is not maximal in J, contradiction. Therefore, I is contained in at most $k+2$ maximal ideals of S. Hence for each positive integer n, every n-maximal ideal is contained in at most $n+1$ maximal ideals of S. □

4. Maximal dimensions

This section is devoted to introducing some new concepts and studying their properties such as the maximal dimension of a ring, the maximal depth and the maximal height of an ideal of a ring.

Definition 4.1. Let S be a ring. The length of a finite maximal chain of ideals of S of the form $I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_k$ is k, where $k \in \mathbb{Z}^+$ and the length of an infinite maximal chain of ideals of S of one of the forms $I_0 \subset I_1 \subset I_2 \subset \cdots$ or $J_0 \supset J_1 \supset J_2 \supset \cdots$ is ∞. The maximal dimension of S denoted by $M.\dim(S)$ is the maximum possible length of a maximal chain of ideals. Moreover, if S has at least two proper ideals $I \subset J$ such that there is no a finite maximal chain of ideals with an initial ideal I and a terminal ideal J, we say $M.\dim(S) = \infty$.

Example 4.2. Consider the ring $S = \mathbb{Z}_n$, where $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \in \mathbb{Z}^+$ and p_i's are distinct primes and $\alpha_i \in \mathbb{Z}^+$ for $1 \leq i \leq k$. Then $M.\dim(S) = \left(\sum_i^k \alpha_i\right) - 1$.

Remark 4.3. Let S be a ring. Then

1. If S is a field, then $M.\dim(S) = 0$.
2. If S is not an Artinan ring equivalently S is not a Noetherian ring or $\dim(S) > 0$, then $M.\dim(S) = \infty$.

Proof. 1. It is obvious.

2. Let S be a non Artinian ring. By Theorem 2.12, S has at least two proper ideals $I \subset J$ such that there is no a finite chain of ideals with an initial ideal I and a terminal ideal J. Therefore, $M.\dim(S) = \infty$. □

Definition 4.4. Let J be a proper ideal of a ring S. The maximal depth of J denoted by $M.d(J)$ is the maximum length over all maximal chains of ideals of S with an initial ideal J. If there is an ideal K of S such that $J \subset K$ but there is no a finite maximal chain of ideals with an initial ideal J and a terminal ideal K, then $M.d(J) = \infty$. The maximal height of J denoted by $M.h(J)$ is the maximum length over all maximal chains of ideals S with a terminal ideal J. If there is an ideal I of S such that $I \subset J$ but there is no a finite maximal chain of ideals with an initial ideal I and a terminal ideal J, then $M.h(J) = \infty$.

The following remark is obvious.

Remark 4.5. Let S be a ring. Then

1. K is a maximal ideal of S if and only if $M.d(K) = 0$.
2. J is the zero ideal of S if and only if $M.h(J) = 0$.
3. S is a field if and only if $M.d(\langle 0 \rangle) = M.h(\langle 0 \rangle) = 0$.

TABLE 1. The multiplication table of the ring $S = \mathbb{Z}_2[x, y]/\langle x^3, xy, y^2 \rangle$.

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	$\times$	0	1	x	x^2	$1+x$	$1+x^2$	$x+x^2$	$1+x+x^2$	y	$1+y$	$x+y$	x^2+y	$1+x+y$	$1+x^2+y$	$x+x^2+y$	$1+x+x^2+y$
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	0	1	x	x^2	$1+x$	$1+x^2$	$x+x^2$	$1+x+x^2$	y	$1+y$	$x+y$	x^2+y	$1+x+y$	$1+x^2+y$	$x+x^2+y$	$1+x+x^2+y$
3	x	0	x	x^2	0	$x+x^2$	x	x^2	$x+x^2$	0	x	x^2	0	$x+x^2$	x	x^2	$x+x^2$
4	x^2	0	x^2	0	0	x^2	x^2	0	x^2	0	x^2	0	0	x^2	x^2	0	x^2
5	$1+x$	0	$1+x$	$x+x^2$	x^2	$1+x^2$	$1+x+x^2$	x	1	y	$1+x+y$	$x+x^2+y$	x^2+y	$1+x^2+y$	$1+x+x^2+y$	$x+y$	$1+y$
6	$1+x^2$	0	$1+x^2$	x	x^2	$1+x+x^2$	1	$x+x^2$	$1+x$	y	$1+x^2+y$	$x+y$	x^2+y	$1+x+x^2+y$	$1+y$	$x+x^2+y$	$1+x+y$
7	$x+x^2$	0	$x+x^2$	x^2	0	x	$x+x^2$	x^2	x	0	$x+x^2$	x^2	0	x	$x+x^2$	x^2	x
8	$1+x+x^2$	0	$1+x+x^2$	$x+x^2$	x^2	1	$1+x$	x	$1+x^2$	y	$1+x+x^2+y$	$x+x^2+y$	x^2+y	$1+y$	$1+x+y$	$x+y$	$1+x^2+y$
9	y	0	y	0	0	y	y	0	y	0	y	0	0	y	y	0	y
10	$1+y$	0	$1+y$	x	x^2	$1+x+y$	$1+x^2+y$	$x+x^2$	$1+x+x^2+y$	y	1	$x+y$	x^2+y	$1+x$	$1+x^2$	$x+x^2+y$	$1+x+x^2$
11	$x+y$	0	$x+y$	x^2	0	$x+x^2+y$	$x+y$	x^2	$x+x^2+y$	0	$x+y$	x^2	0	$x+x^2+y$	$x+y$	x^2	$x+x^2+y$
12	x^2+y	0	x^2+y	0	0	x^2+y	x^2+y	0	x^2+y	0	x^2+y	0	0	x^2+y	x^2+y	0	x^2+y
13	$1+x+y$	0	$1+x+y$	$x+x^2$	x^2	$1+x^2+y$	$1+x+x^2+y$	x	$1+y$	y	$1+x$	$x+x^2+y$	x^2+y	$1+x^2$	$1+x+x^2$	$x+y$	1
14	$1+x^2+y$	0	$1+x^2+y$	x	x^2	$1+x+x^2+y$	$1+y$	$x+x^2$	$1+x+y$	y	$1+x^2$	$x+y$	x^2+y	$1+x+x^2$	1	$x+x^2+y$	$1+x$
15	$x+x^2+y$	0	$x+x^2+y$	x^2	0	$x+y$	$x+x^2+y$	x^2	$x+y$	0	$x+x^2+y$	x^2	0	$x+y$	$x+x^2+y$	x^2	$x+y$
16	$1+x+x^2+y$	0	$1+x+x^2+y$	$x+x^2$	x^2	$1+y$	$1+x+y$	x	$1+x^2+y$	y	$1+x+x^2$	$x+x^2+y$	x^2+y	1	$1+x$	$x+y$	1

Example 4.6. 1. Consider the ring $\mathbb{Z}_{\langle 2 \rangle}$, the localization of $\mathbb{Z}$ at the prime ideal $\langle 2 \rangle$. Since $\mathbb{Z}_{\langle 2 \rangle}$ is not an Artinian ring, then by Remark 4.3, $M.\dim(\mathbb{Z}_{\langle 2 \rangle}) = \infty$. The non zero proper ideals of $\mathbb{Z}_{\langle 2 \rangle}$ are of the form $I_k = \langle 2^k \rangle$ where $k \in \mathbb{Z}^+$. Then for each $k \in \mathbb{Z}^+$, $M.d(I_k) = k - 1$ and $M.h(I_k) = \infty$, since there is no a finite maximal chain of ideals with an initial ideal $\langle 0 \rangle$ and a terminal ideal I_k. Furthermore, $M.h(\langle 0 \rangle) = 0$ and $M.d(\langle 0 \rangle) = \infty$, since there is no a finite maximal chain of ideals with an initial ideal $\langle 0 \rangle$ and a terminal ideal $\langle 2 \rangle$.

2. Consider the ring $S = \mathbb{Z}_2[x, y]/\langle x^3, xy, y^2 \rangle$=$\{0, 1, x, x^2, 1 + x, 1 + x^2, x + x^2, 1+x+x^2, y, 1+y, x+y, x^2+y, 1+x+y, 1+x^2+y, x+x^2+y, 1+x+x^2+y\}$ such that $x^3 = xy = y^2 = 0$. The proper ideals of S are $I_0 = \langle 0 \rangle, I_1 = \langle x^2 \rangle = \{0, x^2\}, I_2 = \langle x \rangle = \{0, x, x^2, x + x^2\}, I_3 = \langle y \rangle = \{0, y\}, I_4 = \langle x + y \rangle = \{0, x + y, x^2, x + y + x^2\}, I_5 = \langle x^2 + y \rangle = \{0, x^2 + y\}, I_6 = \langle x^2, y \rangle = \{0, x^2, y, x^2 + y\}$ and $I_7 = \langle x, y \rangle = \{0, x, x^2, x + x^2, y, y + x, y + x^2, y + x + x^2\}$. Therefore, $M.\dim(S) = 3$, $M.d(I_0) = 3$, $M.d(I_1) = M.d(I_3) = M.d(I_5) = 2$, $M.d(I_2) =$

$M.d(I_4) = M.d(I_6) = 1$, $M.d(I_7) = 0, M.h(I_0) = 0$, $M.h(I_1) = M.h(I_3) = M.h(I_5) = 1$, $M.h(I_2) = M.h(I_4) = M.h(I_6) = 2$ and $M.h(I_7) = 3$.

The following diagram illustrates the maximal chains of ideals of the ring $S = \mathbb{Z}_2[x, y]/\langle x^3, xy, y^2\rangle$:

$$I_0 \subset \begin{cases} I_1 \subset \begin{cases} I_2 \subset I_7, \\ I_4 \subset I_7, \\ I_6, \subset I_7, \end{cases} \\ I_3 \subset I_6 \subset I_7, \\ I_5 \subset I_6 \subset I_7. \end{cases}$$

References

[1] H. A. Ahmad and P. A. Hummadi, *N sequence prime ideals*, Iraqi J. Sci. **62** (2021), no. 10, 3672–3678. https://doi.org/10.24996/ijs.2021.62.10.25

[2] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley Publishing Co., Reading, MA, 1969.

[3] I. S. Cohen, *Lengths of prime ideal chains*, Amer. J. Math. **76** (1954), 654–668. https://doi.org/10.2307/2372708

[4] D. S. Dummit and R. M. Foote, *Abstract Algebra*, third edition, John Wiley & Sons, Inc., Hoboken, NJ, 2004.

[5] P. A. Hummadi and R. S. Rasheed, *Toplogies on modules*, Zanko. **5** (1992), no. 4, 59–66.

[6] H. Matsumura, *Commutative ring theory*, translated from the Japanese by M. Reid, second edition, Cambridge Studies in Advanced Mathematics, 8, Cambridge University Press, Cambridge, 1989.

Hemin A. Ahmad
Department of Mathematics
College of Education
Salahaddin University-Erbil
Erbil 4401, Iraq
Email address: hemin.ahmad@su.edu.krd

Parween A. Hummadi
Department of Mathematics
College of Education
Salahaddin University-Erbil
Erbil 4401, Iraq
Email address: Parween.Hummadi@su.edu.krd