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Evaluation Subgroups of Mapping Spaces over Grassmann Manifolds

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Abstract. Let $V_{k,n}(\mathbb{C})$ denote the complex Steifel and $Gr_{k,n}(\mathbb{C})$ the Grassmann manifolds for $1 \leq k < n$. In this paper, we compute, in terms of the Sullivan minimal models, the evaluation subgroups and, more generally, the relative evaluation subgroups of the fibration $p : V_{k,k+n}(\mathbb{C}) \to Gr_{k,k+n}(\mathbb{C})$. In particular, we prove that $G_*(Gr_{k,k+n}(\mathbb{C}), V_{k,k+n}(\mathbb{C}); p)$ is isomorphic to $G_*^{\mathrm{rel}}(Gr_{k,k+n}(\mathbb{C}), V_{k,k+n}(\mathbb{C}); p) \oplus G_*(V_{k,k+n}(\mathbb{C}))$.

1. Introduction

A basic object of study in homotopy theory is the Gottlieb groups. They are very interesting homotopy invariants but their calculations in general are difficult. As is well known, rational homotopy theory provides a natural framework to study these groups, where topological spaces are replaced by commutative differential graded algebras and topological fibrations replaced by algebraic fibrations.

For a CW-complex X, an element $\alpha \in \pi_n(X)$ is a Gottlieb element if there is a continuous map $H : \mathbb{S}^n \times X \to X$ such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{S}^n \vee X \arrow[r, "h \vee Id_X"] \arrow[d, "inc"] & X \vee X \arrow[d, "\nabla"] \\ \mathbb{S}^n \times X \arrow[r, dashed, "H"] & X \end{tikzcd}$$

where $h : \mathbb{S}^n \to X$ is a representative map of α and ∇ is the folding map. Moreover, the set of all Gottlieb elements in $\pi_n(X)$ is a subgroup of $\pi_n(X)$ denoted by $G_n(X)$ and is called the n-th Gottlieb group of X or the n-th evaluation subgroup of $\pi_n(X)$. Alternately, $G_n(X)$ is the image of the map induced on homotopy groups by the

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evaluation map, ev : $\mathrm{aut}_1(X) \to X$, where $\mathrm{aut}_1(X)$ denotes the monoid of self-homotopy equivalences of X [3].

Similarly, if $p : X \to Y$ is a based map of simply connected CW-complexes and $\mathrm{map}(X, Y; p)$, the space of maps from X to Y which are homotopic to p, then the n-th evaluation subgroup of p, also called the n-th generalized evaluation subgroup, is defined in [9] by:

$$G_n(Y, X; p) = Im(ev_\sharp : \pi_n(map(X, Y; p)) \to \pi_n(Y)).$$

The n-th evaluation subgroup $G_n(X)$ occurs as the special case in which $X = Y$ and $p = \mathrm{Id}_X$. The generalized evaluation subgroups play a well-known role in fixed point theory. In order to apply the generalized evaluation subgroups to fixed point theory, we need the computation of $G_n(Y, X; p)$ which are proper subgroups of $\pi_n(Y)$ and contain $G_n(Y)$ properly. Unfortunately, there are not many explicit computations of $G_n(Y, X; p)$ in literature.

In [10], K.Y. Lee and M.H. Woo introduced the n-th relative evaluation subgroup $G_n^{rel}(Y, X; p)$, also called the n-th relative Gottlieb group, and showed that they fit in a sequence, called G-sequence, which is not necessarily exact:

$$(1.1) \qquad ... \to G_{n+1}^{rel}(Y, X; p) \to G_n(X) \to G_n(Y, X; p) \to G_n^{rel}(Y, X; p) \to ...$$

The exactness of the G-sequence plays an important role in computing homotopy groups.

The complex Steifel and Grassmann form a very well-studied and interesting class of manifolds. They appear abundantly in geometry and topology. Here, we recall that, for $1 \leq k < n$,

$$V_{k,n}(\mathbb{C}) = \frac{U(n)}{U(n-k)} \text{ and } Gr_{k,n}(\mathbb{C}) = \frac{U(n)}{U(k) \times U(n-k)}$$

where $U(n)$ is the unitary group [[2], Example 1.83 and Example 1.84]. There is a fibration $U(k) \overset{i}{\to} V_{k,k+n}(\mathbb{C}) \overset{p}{\to} Gr_{k,k+n}(\mathbb{C})$ for $1 \leq k < n$.

In this paper, we use the notion of Sullivan minimal model and derivation to determine the rational evaluation subgroups and the rational relative evaluation subgroups of the fibration $p : V_{k,k+n}(\mathbb{C}) \to Gr_{k,k+n}(\mathbb{C})$.

2. Preliminaries in Rational Homotopy Theory

This section cannot provide and is not intended to give an introduction to the theory. We expect the reader to have gained a certain familiarity with necessary concepts for example from [1] or [2]. We merely recall some tools and aspects which play a larger role in the paper. All our spaces will be simply connected with the homotopy type of CW-complex with rational cohomology of finite type.

Definition 2.1. A commutative differential graded algebra (cdga) is a graded algebra $A = \oplus_{i \geq 0} A^i$ with a differential $d : A^i \to A^{i+1}$ such that $d^2 = 0$, $xy = (-1)^{ij} yx$, and $d(xy) = d(x)y + (-1)^i xd(y)$ for all $x \in A^i$ and $y \in A^j$. A morphism $f : (A,d) \to (B,d)$ of cdga's is called a quasi-isomorphism if $H^*(f)$ is an isomorphism. a cdga (A,d) is called simply connected if $H^0(A) = \mathbb{Q}$ and $H^1(A) = 0$.

A commutative graded algebra A is free if it is of the form

$$\Lambda V = S(V^{even}) \otimes E(V^{odd}),$$

where $V^{even} = \oplus_{i\geq 1} V^{2i}$ and $V^{odd} = \oplus_{i\geq 0} V^{2i+1}$. A Sullivan algebra is a cgda $(\Lambda V, d)$, where $V = \oplus_{i\geq 1} V^i$ admits a homogeneous basis $\{x_i\}_{i\in I}$ indexed by a well ordered set I such $dx_i \in \Lambda(\{x_i\})_{i<j}$. A Sullivan algebra is called minimal if $dV \subset \Lambda^{\geq 2}V$. If there is a quasi-isomorphism $f : (\Lambda V, d) \to (A,d)$, where $(\Lambda V, d)$ is a minimal Sullivan algebra, then we say that $(\Lambda V, d)$ is a minimal Sullivan algebra of (A,d).

To a simply connected topological space X of finite type, Sullivan associates in a functorial way a cdga $A_{PL}(X)$ of piecewise linear forms on X such that $H^*(A_{PL}(X)) \cong H^*(X;\mathbb{Q})$ [8]. A Sullivan minimal model of X is a Sullivan minimal model of $A_{PL}(X)$. Moreover, the rational homotopy type of X is completely determined by its Sullivan minimal model $(\Lambda V, d)$. In particular, there are isomorphisms

$$H^*(\Lambda V, d) \cong H^*(X;\mathbb{Q}) \text{ as commutative graded algebras,}$$
$$V \cong \pi_*(X) \otimes \mathbb{Q} \text{ as graded vector spaces.}$$

A fibration $p : X \to Y$ of simply connected CW-complexes with fiber F has a Sullivan model which is an inclusion: $(\Lambda W, d) \to (\Lambda W \otimes \Lambda V, D)$ of cdga in which $(\Lambda W, d)$ is a Sullivan minimal model of Y and $(\Lambda W \otimes \Lambda V, D)$ is a Sullivan model (not necessarily minimal) of the total space X.

Definition 2.2. Let $\phi : (\Lambda W, d_W) \to (\Lambda V, d_V)$ be a morphism of cdga's. Define a ϕ-derivation θ of degree n to be a linear map $\theta : \Lambda W \to \Lambda V$ that reduces degree by n such that $\theta(xy) = \theta(x)\phi(y) + (-1)^{n|x|}\phi(x)\theta(y)$. When $n = 1$ we require additionally that $d_V \circ \theta = -\theta \circ d_W$. Let $\mathrm{Der}_n(\Lambda W, \Lambda V; \phi)$ denote the vector space of ϕ-derivations of degree n for $n > 0$. Define a linear map $\partial : Der_n(\Lambda W, \Lambda V; \phi) \to Der_{n-1}(\Lambda W, \Lambda V; \phi)$ by $\partial(\theta) = d_V \circ \theta - (-1)^{|\theta|}\theta \circ d_W$.

Note that $(Der_*(\Lambda W, \Lambda V; \phi), \partial)$ is a chain complex, where $\mathrm{Der}_*(\Lambda W, \Lambda V; \phi) = \oplus_n Der_n(\Lambda W, \Lambda V; \phi)$. In case $\Lambda W \cong \Lambda V$ and $\phi = Id_{\Lambda V}$, the chain complex of derivations $\mathrm{Der}_*(\Lambda V, \Lambda V; Id_{\Lambda V})$ is just the usual complex of derivations on the cdga ΛV which we denoted by $\mathrm{Der}_*(\Lambda V)$. There is an isomorphism of graded vector spaces

$$Der_*(\Lambda W, \Lambda V; \phi) \cong Hom_*(W, \Lambda V).$$

The detailed discussion of the following are in [4]. The post-composition with the augmentation $\varepsilon : \Lambda V \to \mathbb{Q}$ gives a chain complex map

$$\varepsilon_* : Der_*(\Lambda W, \Lambda V; \phi) \to Der_*(\Lambda W, \mathbb{Q}; \varepsilon).$$

The n-th evaluation subgroup of ϕ is defined as follows:

$$G_n(\Lambda W, \Lambda V; \phi) = Im\{H_n(\varepsilon_*) : H_n(Der(\Lambda W, \Lambda V; \phi)) \to Hom_n(W, \mathbb{Q})\} \text{ for } n \geq 2.$$

Then $w^* \in Hom_n(W, \mathbb{Q})$ (w^* is the dual of the basis element w of W^n) is in $G_n(\Lambda W, \Lambda V; \phi)$ if and only if w^* extends to a derivation θ of $Der_n(\Lambda W, \Lambda V; \phi)$ such that $\partial(\theta) = 0$.

In case $\Lambda W \cong \Lambda V$ and $\phi = Id_{\Lambda V}$, we get the Gottlieb group of $(\Lambda V, d_V)$, defined as follows:

$$G_n(\Lambda V) = Im\{H_n(\varepsilon_*) : H_n(Der(\Lambda V)) \to Hom_n(V, \mathbb{Q})\} \text{ for } n \geq 2.$$

In particular, if X is a finite CW-complex, then from [4] and [7] we have

$$G_n(Y, X; p) \otimes \mathbb{Q} \cong G_n(Y_{\mathbb{Q}}, X_{\mathbb{Q}}; p_{\mathbb{Q}}) \cong G_n(\Lambda W, \Lambda V; \phi) \text{ for } n \geq 2.$$

We now recall the definition of the mapping cone of a chain map $\phi : A \to B$.

Definition 2.3. Let $\phi : (A, d_A) \to (B, d_B)$ be a map of differential graded vector spaces. The mapping cone of ϕ denoted by $\mathrm{Rel}_*(\phi)$ is defined as follows: $\mathrm{Rel}_n(\phi) = A_{n-1} \oplus B_n$ with the differential $\delta(a, b) = (-d_A(a), \phi(a) + d_B(b))$.
Further, define inclusion and projection maps $J : B_n \to Rel_n(\phi)$ by $J(b) = (0, b)$ and $P : Rel_n(\phi) \to A_{n-1}$ by $P(a, b) = a$. These yield a short exact sequence of chain complexes

$$0 \to B_* \xrightarrow{J} Rel_*(\phi) \xrightarrow{P} A_{*-1} \to 0.$$

This definition can be applied to the Sullivan model $\phi : (\Lambda W, d_W) \to (\Lambda V, d_V)$ of the fibration $p : X \to Y$.

Note that the pre-composition with ϕ give maps

$$\phi^* : Der_*(\Lambda V) \to Der_*(\Lambda W, \Lambda V; \phi) \text{ and } \widehat{\phi^*} : Der_*(\Lambda V, \mathbb{Q}; \varepsilon) \to Der_*(\Lambda W, \mathbb{Q}; \varepsilon)$$

where ε is the augmentation of either ΛV or ΛW. Following G. Lupton and S.B. Smith [4] (see also [11]), we consider the commutative diagram

$$\begin{tikzcd}
Der_*(\Lambda V) \arrow[r, "\phi^*"] \arrow[d, "\varepsilon_*"] & Der_*(\Lambda W, \Lambda V; \phi) \arrow[r, "J"] \arrow[d, "\varepsilon_*"] & Rel_*(\phi^*) \arrow[d, "(\varepsilon_*, \varepsilon_*)"] \\
Der_*(\Lambda V, \mathbb{Q}; \varepsilon) \arrow[r, "\widehat{\phi^*}"] & Der_*(\Lambda W, \mathbb{Q}; \varepsilon) \arrow[r, "\widehat{J}"] & Rel_*(\widehat{\phi^*})
\end{tikzcd}$$

On passing to homology and using the naturality of the mapping cone construction, we obtain the following homology ladder for $n \geq 2$.

$$\begin{tikzcd}
\ldots \arrow[r, "H(J)"] & H_{n+1}(Rel(\phi^*)) \arrow[r, "H(P)"] \arrow[d, "H(\varepsilon_*, \varepsilon_*)"] & H_n(Der(\Lambda V)) \arrow[r, "H(\phi^*)"] \arrow[d, "H(\varepsilon_*)"] & H_n(Der(\Lambda W, \Lambda V; \phi)) \arrow[r, "H(J)"] \arrow[d, "H(\varepsilon_*)"] & \ldots \\
\ldots \arrow[r, "H(\widehat{J})"] & H_{n+1}(Rel(\widehat{\phi^*})) \arrow[r, "H(\widehat{P})"] & H_n(Der(\Lambda V, \mathbb{Q}; \varepsilon)) \arrow[r, "H(\widehat{\phi^*})"] & H_n(Der(\Lambda W, \mathbb{Q}; \varepsilon)) \arrow[r, "H(\widehat{J})"] & \ldots
\end{tikzcd}$$

The following definition is very useful to compute the relative evaluation subgroups of a map.

Definition 2.4. Suppose $\phi : (\Lambda W, d_W) \to (\Lambda V, d_V)$ is a map of cdga's. We define the n-th relative evaluation subgroup of ϕ by:

$$G_n^{rel}(\Lambda W, \Lambda V; \phi) = \mathrm{Im}\left\{H(\varepsilon_*, \varepsilon_*) : H_n(Rel(\phi^*)) \to H_n\left(Rel\left(\widehat{\phi^*}\right)\right)\right\} \text{ for } n \geq 2.$$

We end this section by an overriding hypothesis. In general, we assume that all spaces appearing in the sequel are *rational simply connected CW-complex and are of finite type.*

3. Evaluation Subgroups of Mapping Spaces over Grassmann Manifolds

Let $Gr_{k,n}(\mathbb{C})$ be the complex Grassmann manifold and $V_{k,n}(\mathbb{C})$ the complex Steifel manifold for $1 \leq k < n$. There is a fibration $U(k) \overset{i}{\to} V_{k,k+n}(\mathbb{C}) \overset{p}{\to} Gr_{k,k+n}(\mathbb{C})$. Hence for $1 \leq k < n$, a Sullivan minimal model of p is given by

$$\phi : \left(\Lambda\left(x_2, ..., x_{2k}, y_{2n+1}, ..., y_{2(n+k)-1}\right), d\right) \to \left(\Lambda\left(z_{2n+1}, ..., z_{2(n+k)-1}\right), 0\right)$$

where $\phi(x_{2i}) = 0$ for $i \in \{1, ..., k\}$ and $\phi(y_{2j+1}) = z_{2j+1}$ for $j \in \{n, ..., n+k-1\}$. In this section, we use Sullivan minimal models to compute the evaluation subgroups and the relative evaluation subgroups of the fibration $p : V_{k,k+n}(\mathbb{C}) \to Gr_{k,k+n}(\mathbb{C})$.

We begin by the following which we will use in the sequel.

Theorem 3.1. $G_*(V_{k,n}(\mathbb{C})) = \pi_*(V_{k,n}(\mathbb{C}))$ *for* $1 \leq k < n$.

Proof. First, we recall that an H-space X is a space with a multiplication $\mu : X \times X \to X$ that is associative up to homotopy and admits a unit up to homotopy. Secondly, Since the complex Steifel manifold has the rational homotopy type of an H-space, then the multiplication μ provides a composition $\mathbb{S}^n \times X \to X \times X \to X$ giving $G_*(V_{k,n}(\mathbb{C})) = \pi_*(V_{k,n}(\mathbb{C}))$, as nedeed. □

We note that Theorem 3.1 can also be proved from the Sullivan minimal model for $V_{k,n}(\mathbb{C})$. We turn now to the evaluation subgroups of the fibration p.

Theorem 3.2. $G_*(Gr_{k,k+n}(\mathbb{C}), V_{k,k+n}(\mathbb{C}); p) = \pi_*(Gr_{k,k+n}(\mathbb{C}))$ *for* $1 \leq k < n$.

Proof. Write the Sullivan minimal model for $V_{k,k+n}(\mathbb{C})$ [[2], Example 2.40] as

$$(\Lambda V, d_V) = \left(\Lambda\left(z_{2n+1}, ..., z_{2(n+k)-1}\right), 0\right)$$

and the Sullivan minimal model for $Gr_{k,k+n}(\mathbb{C})$ [[6], Lemma 1] as

$$(\Lambda W, d_W) = \left(\Lambda\left(x_2, ..., x_{2k}, y_{2n+1}, ..., y_{2(n+k)-1}\right), d_W\right)$$

where $d_W(x_{2i}) = 0$ for $i \in \{1, ..., k\}$ and $d(y_{2j+1}) \in \Lambda^{\geq 2}(x_2, ..., x_{2k})$ for $j \in \{n, ..., n+k-1\}$. Here, in both Sullivan minimal models subscripts denoting degrees. Let us calculate $G_*(Gr_{k,k+n}(\mathbb{C}), V_{k,k+n}(\mathbb{C}); p)$ as follows.

Let $(x_{2i},1)$ denote the derivation α_i in $\mathrm{Der}_{2i}(\Lambda W,\Lambda V;\phi)$ for $i\in\{1,...,k\}$ such that $\alpha_i(x_{2i})=1$ and zero on other generators. Further, let $\beta_j=(y_{2j+1},1)$ in $Der_{2j+1}(\Lambda W,\Lambda V;\phi)$ for $j\in\{n,...,n+k-1\}$. Since $d(y_{2j+1})\in\Lambda^{\geq 2}(x_2,...,x_{2k})$, a direct computation shows that β_j is a ∂-cycle in $Der_{2j+1}(\Lambda W,\Lambda V;\phi)$. Furthermore, a simple analysis on the differential prove that β_j cannot bound. This means that, for $j\in\{n,...,n+k-1\}$

$$[\beta_j]\neq 0 \text{ in } H_{2j+1}(Der(\Lambda W,\Lambda V;\phi)).$$

Next, consider the derivation α_i for $i\in\{1,...,k\}$. We see that

$$\begin{aligned}\partial(\alpha_i(e)) &= d_V(\alpha_i(e))-\alpha_i(d_W(e)) \text{ for } e\in W\\ &= 0-\alpha_i(d_W(e))\end{aligned}$$

So, by the minimality of $(\Lambda W,d_W)$ and $\phi(x_{2i})=0$, we get $\alpha_i(d_W(e))=0$ and further $\partial(\alpha_i)=0$. Otherwise, it is easy to check that α_i cannot bound for $i\in\{1,...,k\}$. We omit this detail for clarity. Suppose that there is an odd derivation $\alpha_i^{'}$ in $Der(\Lambda W,\Lambda V;\phi)$ such that $\partial(\alpha_i^{'})=\alpha_i$. This means that in particular $(\partial(\alpha_i^{'}))(x_{2i})=\alpha_i(x_{2i})$ and hence $0=1$. So, this is a contradiction. Hence, we have proved that $[\alpha_i]$ are non zero homology class in $H_{2i}(Der(\Lambda W,\Lambda V;\phi))$ for $i\in\{1,...,k\}$. Furthermore, since

$$\varepsilon_*(\alpha_i)=x_{2i}^* \text{ for } i\in\{1,...,k\} \text{ and } \varepsilon_*(\beta_j)=y_{2j+1}^* \text{ for } j\in\{n,...,n+k-1\}$$

We deduce that

$$\begin{aligned}G_*(\Lambda W,\Lambda V;\phi) &= \mathrm{Hom}(W,\mathbb{Q})\\ &= \pi_*(Gr_{k,k+n}(\mathbb{C})).\end{aligned}$$

□

We now continue with the main result. We prove the following:

Theorem 3.3. *Consider the fibration $V_{k,k+n}(\mathbb{C})\xrightarrow{p}Gr_{k,k+n}(\mathbb{C})$ for $1\leq k<n$ and $\varphi:(\Lambda W,d_W)\to(\Lambda V,d_V)$ its Sullivan minimal model. Then*

$$G_*^{rel}(\Lambda W,\Lambda V;\phi)=\langle[(0,x_{2i}^*)] \text{ for } i\in\{1,...,k\}\rangle.$$

Proof. First of all, recall that

$$\phi:\left(\Lambda\left(x_2,...,x_{2k},y_{2n+1},...,y_{2(n+k)-1}\right),d\right)\to\left(\Lambda\left(z_{2n+1},...,z_{2(n+k)-1}\right),0\right)$$

where $\phi(x_{2i})=0$ for $i\in\{1,...,k\}$ and $\phi(y_{2j+1})=z_{2j+1}$ for $j\in\{n,...,n+k-1\}$. As in the Proof of Theorem 3.3, we denote by $\alpha_i=(x_{2i},1)$ in $Der_{2i}(\Lambda W,\Lambda V;\phi)$ for $i\in\{1,...,k\}$, $\beta_j=(y_{2j+1},1)$ in $Der_{2j+1}(\Lambda W,\Lambda V;\phi)$ and $\theta_j=(z_{2j+1},1)$

in $Der_{2j+1}(\Lambda V)$ for $j \in \{n,...,n+k-1\}$. Thus, the map $\phi^* : Der_*(\Lambda V) \to Der_*(\Lambda W, \Lambda V; \phi)$ is given on generators by

$$\phi^*(\theta_j) = \beta_j \text{ for } j \in \{n,...,n+k-1\}.$$

Further, in $\mathrm{Rel}_*(\phi^*)$, one gets

$$\delta(\theta_j, 0) = (0, \beta_j) \text{ for } j \in \{n,...,n+k-1\} \text{ and } \delta(0, \alpha_i) = 0 \text{ for } i \in \{1,...,k\}.$$

Therefore, by contradiction it is easy to show that $(0, \alpha_i)$ are non-bounding δ-cycles. Hence, we have for $i \in \{1,...,k\}$

$$[(0, \alpha_i)] \neq 0 \text{ in } H_*(Rel(\phi^*)).$$

On other hand, we see that

$$Rel_*\left(\widehat{\phi^*}\right) = Der_{*-1}(\Lambda V, \mathbb{Q}; \varepsilon) \oplus Der_*(\Lambda W, \mathbb{Q}; \varepsilon).$$

Moreover for degree reason, it is spanned by

$$\left\{(0, x_{2i}^*), \left(0, y_{2j+1}^*\right), \left(z_{2j+1}^*, 0\right) \text{ for } i \in \{1,...,k\} \text{ and } j \in \{n,...,n+k-1\}\right\}.$$

However, in $\mathrm{Rel}_*\left(\widehat{\phi^*}\right)$, one gets

$$\widehat{\delta}\left(z_{2j+1}^*, 0\right) = \left(0, y_{2j+1}^*\right) \text{ and } \widehat{\delta}(0, x_{2i}^*) = 0.$$

Therefore, $(0, x_{2i}^*)$ are cycles, which are not boundaries. Combining all the above, we obtain for $i \in \{1,...,k\}$

$$H(\varepsilon_*, \varepsilon_*)([(0, \alpha_i)]) = [(0, x_{2i}^*)].$$

It follows that

$$G_*^{rel}(\Lambda W, \Lambda V; \phi) = \langle [(0, x_{2i}^*)] \text{ for } i \in \{1,...,k\} \rangle.$$

□

By Theorem 3.1, Theorem 3.2, Theorem 3.3 and the sequence (1.1), we have the exactness of the G-sequence for $j \in \{n,...,n+k-1\}$

$$0 \to G_{2j+1}(\Lambda V) \overset{\cong}{\to} G_{2j+1}(\Lambda W, \Lambda V; \phi) \to 0$$

and for $i \in \{1,...,k\}$

$$0 \to G_{2i}(\Lambda W, \Lambda V; \phi) \overset{\cong}{\to} G_{2i}^{rel}(\Lambda W, \Lambda V; \phi) \to 0.$$

Remark 3.4. Since $V_{1,n}(\mathbb{C}) = \mathbb{S}^{2n-1}$ and $Gr_{1,n}(\mathbb{C}) = \mathbb{C}P^{2n-1}$ [2]. Then, our results motivated us to extend (up to changing degree) the O. Maphane results [9].

To will illustrate Theorem 3.3, we propose the following example.

Example 3.5. Consider the fibration $V_{2,5}(\mathbb{C}) \overset{p}{\to} Gr_{2,5}(\mathbb{C})$. The Sullivan minimal model of $Gr_{2,5}(\mathbb{C})$ is given by $(\Lambda(x_2, x_4, y_7, y_9), d)$ where $dx_2 = dx_4 = 0, dy_7 = x_4^2 - 3x_2^2 x_4 + x_2^4$ and $dy_9 = 4x_2^3 x_4 - 3x_2 x_4^2 - x_2^5$. Hence a Sullivan minimal model of p, which we denote by

$$\phi : (\Lambda(x_2, x_4, y_7, y_9), d) \to (\Lambda(z_7, z_9), 0)$$

is given on generators by $\phi(x_2) = 0 = \phi(x_4), \phi(y_7) = z_7$ and $\phi(y_9) = z_9$.
Let $\alpha_2 = (x_2, 1), \alpha_4 = (x_4, 1), \beta_7 = (y_7, 1), \beta_9 = (y_9, 1)$ in $\mathrm{Der}(\Lambda W, \Lambda V; \phi)$, and $\theta_7 = (z_7, 1)$ and $\theta_9 = (z_9, 1)$ in $\mathrm{Der}(\Lambda V)$. Hence, we have

$$\phi^*(\theta_7) = \beta_7 \text{ and } \phi^*(\theta_9) = \beta_9.$$

Then, a short computation shows that $\delta(0, \alpha_2) = 0 = \delta(0, \alpha_4)$, $\delta(\theta_7, 0) = (0, \beta_7)$ and $\delta(\theta_9, 0) = (0, \beta_9)$. It follows that $[(0, \alpha_2)]$ and $[(0, \alpha_4)]$ are non zero homology classes in $H_*(\mathrm{Rel}(\phi^*))$. Moreover, since

$$(\varepsilon_*, \varepsilon_*)(0, \alpha_2) = (0, x_2^*) \text{ and } (\varepsilon_*, \varepsilon_*)(0, \alpha_4) = (0, x_4^*)$$

we conclude that

$$G_*^{rel}(Gr_{2,5}(\mathbb{C}), V_{2,5}(\mathbb{C}); p) = \mathbb{Q}\langle [(0, x_2^*)], [(0, x_4^*)] \rangle.$$

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