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On the Metric Dimension of Corona Product of a Graph with K_1

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Abstract. For an ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of vertices and a vertex v in a connected graph G, the k-vector

$$r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$$

is called the metric representation of v with respect to W, where $d(x, y)$ is the distance between the vertices x and y. A set W is called a resolving set for G if distinct vertices of G have distinct metric representations with respect to W. The minimum cardinality of a resolving set for G is its metric dimension $\dim(G)$, and a resolving set of minimum cardinality is a basis of G. The corona product, $G \odot H$ of graphs G and H is obtained by taking one copy of G and $n(G)$ copies of H, and by joining each vertex of the ith copy of H to the ith vertex of G. In this paper, we obtain bounds for $\dim(G \odot K_1)$, characterize all graphs G with $\dim(G \odot K_1) = \dim(G)$, and prove that $\dim(G \odot K_1) = n - 1$ if and only if G is the complete graph K_n or the star graph $K_{1,n-1}$.

1. Introduction

Throughout this paper G is a connected finite simple graph of order $n(G)$. The vertex and edge sets of G are $V(G)$ and $E(G)$, respectively. For vertices u and v in a graph G, the distance of two vertices u and v, denoted by $d_G(u, v)$, is the length of a shortest path between u and v in G. We write it simply $d(u, v)$ when no confusion can arise. The diameter of G, $D(G)$, is $\max_{u,v \in V(G)} d(u, v)$. The symbol $(v_1, v_2, \ldots, v_n)$ represents a path of order n, P_n.

For an ordered set $W = \{w_1, w_2, \ldots, w_k\} \subseteq V(G)$ and a vertex v of G, the k-vector $r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$ is called the *metric representation* of v with respect to W. The set W is called a *resolving set* for G if distinct vertices of G have different metric representations. A resolving set W for G with minimum cardinality is called a *metric basis* of G, and its cardinality is the *metric*

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dimension, $\dim(G)$, of G. It is obvious that to see whether a given set W is a resolving set, it is sufficient to consider the vertices in $V(G)\backslash W$, because $w \in W$ is the unique vertex of G for which $d(w,w) = 0$.

In [12], Slater introduced the idea of a resolving set and used a *locating set* and the *location number* for a resolving set and the metric dimension, respectively. He described the usefulness of these concepts when working with U.S. Sonar and Coast Guard Loran stations. Independently, Harary and Melter [8] discovered the concept of the location number as well and called it the metric dimension. The concept of a resolving set has various applications in diverse areas including coin weighing problems [11], network discovery and verification [1], robot navigation [9], mastermind game [3], problems of pattern recognition and image processing [10], and combinatorial search and optimization [11]. For more results related to these concepts see [2, 3, 4, 6, 9].

The join of two graphs G and H, $G \vee H$ is the graph with vertex set $V(G)\cup V(H)$ and edge set $E(G)\cup E(H)$ and joining each vertex of G to all vertices of H. Also, the disjoint union of two graphs G and H, $V(G)\cap V(H) = \emptyset$, is the graph with vertex set $V(G)\cup V(H)$ and edge set $E(G)\cup E(H)$. Chartrand et al. [5] determined the metric dimension of some families of graphs such as paths, trees, and complete graphs. The following theorem gives the metric dimension of some well-known classes of graphs.

Theorem 1.1. ([5, 13]) *Let G be a graph of order $n \geq 2$.*

(a) $\dim(G) = 1$ *if and only if* $G = P_n$*.*

(b) $\dim(G) = n - 1$ *if and only if* $G = K_n$*.*

(c) For $n \geq 3$, $\dim(C_n) = 2$*.*

(d) For $n \geq 4$, $\dim(G) = n-2$ *if and only if* $G = K_{s,t}(s,t \geq 1), G = K_s \vee \overline{K}_t(s \geq 1, t \geq 2)$*, or* $G = K_s \vee (K_t \cup K_1)(s,t \geq 1)$*.*

The *corona product*, $G \odot H$ of graphs G and H is obtained by taking one copy of G and $n(G)$ copy of H, and by joining each vertex of the ith copy of H to the ith vertex of G, $1 \leq i \leq n(G)$. When $n(H) \geq 2$, Fernau et al. [7], showed that $\dim(G \odot H)$ is equal to $n(G)$ times of adjacency dimension of H. When $n(H) = 1$ the problem of finding $\dim(G \odot H)$ is more difficult and there is a few results about it.

For each integer $k \geq 2$, Yero et al. [14], defined the graph $G \odot^k H$ recursively from $G \odot H$ as $G \odot^k H = (G \odot^{k-1} H) \odot H$. They proved the following theorem for the case $n(H) = 1$.

Theorem 1.2. ([14]) *For every connected graph G of order $n \geq 2$,* $\dim(G \odot^k K_1) \leq 2^{k-1}n - 1$*.*

Buczkowski et al. [2] proved that if G' is a graph by adding a leaf to a nontrivial graph G, then

$$\dim(G) \leq \dim(G') \leq \dim(G) + 1 \tag{1.1}$$

This implies $\dim(G) \leq \dim(G \odot K_1)$. Yero et al. [14] compute $\dim(G \odot K_1)$, when G is a tree.

Theorem 1.3. ([14]) If T is a tree, then the metric dimension of $T \odot K_1$ is equal to the number of leaves of T.

In this paper, we focus on the metric dimension of $G \odot K_1$. Clearly, $G \odot K_1$ is a path if and only $G \in \{K_1, K_2\}$. Hence, for graphs G of order greater than 2 we have $2 \leq \dim(G \odot K_1) \leq n(G) - 1$. We obtain an upper bound for $\dim(G \odot K_1)$, in terms of order and diameter of G. Using this bound we characterize all graphs G with $\dim(G \odot K_1) = n(G) - 1$. Then we characterize all graphs that attain the bound in Theorem 1.2. In fact we prove $\dim(G \odot^k K_1) = 2^{k-1}n - 1$ if and only if $k = 1$ and G is complete graph K_n or G is star graph $K_{1,n-1}$. Also, we give a necessary and sufficient condition to $\dim(G) = \dim(G \odot K_1)$.

2. Main Results

In this section we consider the metric dimension of the graph $G \odot K_1$ and characterize all graphs that attain the bound in Theorem 1.2 for the case $k = 1$. Using this result, we complete the characterization for all integer numbers $k \geq 1$. By the following lemma, $G \odot K_1$ has a basis consists of leaves.

Lemma 2.1. *For each graph G, the graph $G \odot K_1$ has a basis B such that all members of B are of degree one.*

Proof. For each vertex $v \in V(G)$ let v' be the leaf is adjacent to v in $G \odot K_1$. It is clear that for every $x, y \in V(G \odot K_1)$, $d(x, y') = d(x, y) + 1$. Hence, if y resolves a pair a, b of vertices of $G \odot K_1$, then y' resolves this pair, as well. Moreover, y does not resolve y' from any vertex of $N(y) \backslash \{y'\}$, but y' resolves y from all other vertices of $G \odot K_1$. Thus, if R is a resolving set for $G \odot K_1$ such that $\{y, y'\} \subseteq R$, then $R \backslash \{y\}$ is a resolving set for $G \odot K_1$, too. Now let B be a basis for $G \odot K_1$ and $v_1, v_2, \ldots, v_t$ be all none-leaf vertices of B. Therefore $B' = (B \cup \{v'_1, v'_2, \ldots, v'_t\}) \backslash \{v_1, v_2, \ldots, v_t\}$ is a basis for $G \odot K_1$, and all vertices of B' are leaves. □

By Inequality 1.1, for each graph G we have $\dim(G) \leq \dim(G \odot K_1)$. The following theorem explain a necessary and sufficient condition to $\dim(G) = \dim(G \odot K_1)$.

Theorem 2.2. *Let G be a graph. Then $\dim(G) = \dim(G \odot K_1)$ if and only if there exists a metric basis B of G such that for each $u, v \in V(G) \setminus B$, $r(u|B) - r(v|B) \neq (1, 1, \ldots, 1)$.*

Proof. For each vertex $v \in V(G)$ let v' be the leaf is adjacent to v in $G \odot K_1$. Let $\dim(G) = \dim(G \odot K_1)$ and B' is a basis of $G \odot K_1$. Consider $B = \{v \in V(G) | v \in B' \vee v' \in B'\}$. If $r(u|B) = r(v|B)$ for some $u, v \in V(G)$, then there exists a vertex $b' \in B'$ such that $d_{G \odot K_1}(u, b') \neq d_{G \odot K_1}(v, b')$ and $d_G(u, b) = d_G(v, b)$. This is impossible, because $d_{G \odot K_1}(u, b') = d_G(u, b) + 1$ and $d_{G \odot K_1}(v, b') = d_G(v, b) + 1$. Therefore B is a basis for G. If there exist vertices $u, v \in V(G) \setminus B$ with $r(u|B) - r(v|B) = (1, 1, \ldots, 1)$, then $r(u|B') - r(v|B') = (1, 1, \ldots, 1)$. Since $v \notin B$, we have $v' \notin B'$ and $r(v'|B') - r(v|B') = (1, 1, \ldots, 1)$. That means $r(v'|B') = r(u|B')$, which is a contradiction. Therefore $r(u|B) - r(v|B) \neq (1,1,\ldots,1)$, for every $u, v \in V(G) \setminus B$.

For the converse, let B be a basis of G such that $r(u|B) - r(v|B) \neq (1,1,\ldots,1)$ for each $u, v \in V(G) \setminus B$. Set $B' = \{v' \in V(G \odot K_1) | v \in B\}$. We prove that B' is a resolving set for $G \odot K_1$. Let $x, y \in V(G \odot K_1) \setminus B'$. If both of x, y are in $V(G)$ or both of them are not in $V(G)$, then $r(x|B') \neq r(y|B')$, because B is a basis of G. Now let $x \in V(G)$ and $y \in V(G \odot K_1) \setminus V(G)$ and $r(x|B') = r(y|B')$. Hence y is the only leaf of a vertex $z \in V(G)$ and $r(y|B') - r(z|B') = (1,1,\ldots,1)$. Therefore $r(x|B') - r(z|B') = (1,1,\ldots,1)$. By definition of B', we have $r(x|B) - r(z|B) = (1,1,\ldots,1)$. Note that $x, y \notin B'$ yields $x, z \notin B$, a contradiction. Therefore B' is a resolving set for $G \odot K_1$ of size $\dim(G)$. This means $\dim(G \odot K_1) \leq \dim(G)$ and Inequality 1.1 implies $\dim(G \odot K_1) = \dim(G)$. □

The following upper bound is useful to study of metric dimension of $G \odot K_1$.

Theorem 2.3. *Let G be a graph of order n and diameter D. Then* $\dim(G \odot K_1) \leq n - D + 1$.

Proof. Assume that for each $v \in V(G)$, v' is the leaf is adjacent to v in $G \odot K_1$. Let $(v_0, v_1, \ldots, v_D)$ be a shortest path of length D in G. If $W = \{v'_0, v'_D\}$, then $r(v_i|W) = (i+1, D-i+1), 0 \leq i \leq D$ and $r(v'_i|W) = (i+2, D-i+2), 1 \leq i \leq D-1$. We prove that W resolves the set $\{v_0, v_1, \ldots, v_D, v'_0, v'_1, \ldots, v'_D\}$. It is easy to see that for $i \neq j$, $r(v_i|W) \neq r(v_j|W)$ and $r(v'_i|W) \neq r(v'_j|W)$. On the hand, if $r(v_i|W) = r(v'_j|W)$ for some i, j, then $(i+1, D-i+1) = (j+2, D-j+2)$. The equality of the first entry implies that $j = i - 1$ and the equality of the second entry implies that $i = j-1$, which is impossible. Now let $W_1 = V(G \odot K_1) \setminus (V(G) \cup \{v'_1, v'_2, \ldots, v'_{D-1}\})$. It is clear that $|W_1| = n - D + 1$. To complete the proof, it is sufficient to prove that W_1 is a resolving set for $G \odot K_1$. Note that $v_i, D + 1 \leq i \leq n - 1$, is the unique vertex in $G \odot K_1$ with distance 1 to v'_i. That means W_1 is a resolving set for $G \odot K_1$, since $W \subseteq W_1$ resolves $\{v_0, v_1, \ldots, v_D, v'_0, v'_1, \ldots, v'_D\}$. □

It is easy to see that upper bound in Theorem 2.3 is tight for $G = P_n$. The following lemma gives a property of graphs that attain the bound in Theorem 2.3.

Lemma 2.4. *Let G be a graph of order n, diameter D and* $\dim(G \odot K_1) = n - D + 1$. *If P is a shortest path of length D in G, then each vertex of $G \setminus P$ is adjacent to a vertex of P.*

Proof. Assume that for each $v \in V(G)$, v' is the leaf is adjacent to v in $G \odot K_1$. Let $P = (v_0, v_1, \ldots, v_D)$ be a shortest path of length D in G. Suppose, on the contrary, there exists a vertex $x \in V(G \setminus P)$ with no adjacent in P. Since G is connected, x has a neighbour $y \notin V(P)$. Let

$$V = \{v_1, v_2, \ldots, v_{D-1}, y\}, \qquad U = \{v'_1, v'_2, \ldots, v'_{D-1}, y'\}.$$

We prove that $W = V(G \odot K_1) \setminus (V(G) \cup U)$ is a resolving set for $G \odot K_1$. It is clear that for each $v \in V(G) \setminus V$, $v' \in W$ and v is the unique vertex of $G \odot K_1$ with $d(v, v') = 1$. Hence for each $v \in V(G) \setminus V$, $r(v|W)$ is unique. That means, W is a resolving set for G if W resolves $U \cup V$. Note that, y is the unique vertex of $U \cup V$

with distance 2 to x'. Since $x' \in W$, $r(y|W)$ is unique. On the other hand, for each $i, 1 \le i \le D-1$,

$$r(v_i|\{v'_0, v'_D\} = (i+1, D-i+1), \qquad r(v'_i|\{v'_0, v'_D\} = (i+2, D-i+2).$$

Hence for each i, j, $1 \le i \ne j \le D-1$, $r(v_i|W) \ne r(v_j|W)$ and $r(v'_i|W) \ne r(v'_j|W)$. Also, if $r(v_i|W) = r(v'_j|W)$ for some i, j, $1 \le i, j \le D-1$, then $r(v_i|\{v'_0, v'_D\} = r(v'_j|\{v'_0, v'_D\}$. Thus, $i+1 = j+2$ and $D-i+1 = D-j+1$ these imply $i = j+1$ and $j = i+1$, which is impossible. Therefore to complete the prove, we need to prove that $r(y'|W)$ is different from the metric representations of vertices in $U \cup V \setminus \{y, y'\}$. Since x, y are adjacent, $d(y', x') = 3$. If there exists a vertex $t \in U \cup V \setminus \{y, y'\}$ with $r(y'|W) = r(t|W)$ then $d(t, x') = 3$. This means t is adjacent to y. Otherwise, $d(t, x') \ge 4$, because x has no neighbour in $V(P)$. Thus $t \in V$, say $t = v_i$, for some $i, 1 \le i \le D-1$. Hence,

$$r(y'|\{v'_0, v'_D\} = r(v_i|\{v'_0, v'_D\} = (i+1, D-i+1).$$

That yields, $d(y, v'_0) = d(y', v'_0) - 1 = i$ and $d(y, v'_D) = d(y', v'_D) - 1 = D - i$. Therefore,

$$D+2 = d(v'_0, v'_D) \le d(v'_0, y) + d(y, v'_D) = D.$$

Therefore W is a resolving set for $G \odot K_1$. That is, $\dim(G \odot K_1) \le |W| = n - D$. Which is a contradiction, therefore x has a neighbour in $V(P)$. □

The next lemma characterizes all graphs that attain the bound in Theorem 1.2 for the case $k = 1$.

Lemma 2.5. *Let G be a graph of order $n \ge 2$. Then $\dim(G \odot K_1) = n-1$ if and only if $G = K_n$ or $G = K_{1,n-1}$.*

Proof. Assume that for each $v \in V(G)$, v' is the leaf is adjacent to v in $G \odot K_1$. First let $\dim(G \odot K_1) = n-1$. If the diameter of G is D, then by Theorem 2.3, $n-1 \le n-D+1$, that is $D \le 2$. $D = 1$ implies $G = K_n$. Now consider $D = 2$ we claim that $G = K_{1,n-1}$. Let $P = (v_0, v_1, v_2)$ be a shortest path of length D in G. It is enough to consider the case $P \ne G$, otherwise $G = K_{1,2}$. Assume that $x \notin V(P)$, by Lemma 2.4, x has some neighbours in P. If x is adjacent to a leaf of P, say v_2, then let $W = V(G \odot K_1) \setminus V(G) \cup \{v'_1, v'_2\}$. It is easy to see that the metric representation of every vertex in $V(G \odot K_1) \setminus \{v_1, v_2, v'_1, v'_2\}$ with respect to W is unique. Also, we have $\{v'_0, x'\} \subseteq W$ and

$$r(v_1|\{v'_0, x'\}) = (2, r), \quad r(v'_1|\{v'_0, x'\}) = (3, r+1),$$

$$r(v_2|\{v'_0, x'\}) = (3, 2), \qquad r(v'_2|\{v'_0, x'\}) = (4, 3),$$

where $r \in \{2, 3\}$. Therefore W is a resolving set for G of cardinality $n-2$, this contradiction implies that no leaf of P is adjacent to any vertex of $V(G) \setminus V(P)$ and by Lemma 2.4, v_1 is adjacent to all vertices in $V(G) \setminus V(P)$. To prove our claim, we need to prove that $V(G) \setminus V(P)$ is an independent set of vertices. If $a, b \in V(G) \setminus V(P)$ are adjacent, then let $W = V(G \odot K_1) \setminus V(G) \cup \{v'_1, b'\}$. It is easy to see that the metric representation of every vertex in $V(G\odot K_1)\setminus\{v_1, b, v'_1, b'\}$ with respect to W is unique. Also, we have $\{v'_0, a'\} \subseteq W$ and

$$r(v_1|\{v'_0, a'\}) = (2, 2), r(v'_1|\{v'_0, a'\}) = (3, 3), r(b|\{v'_0, a'\}) = (3, 2), r(b'|\{v'_0, a'\}) = (4, 3).$$

Therefore W is a resolving set for G of cardinality $n - 2$. This contradiction yields $V(G) \setminus V(P)$ is an independent set and $G = K_{1,n-1}$.

Conversely, if $G = K_n$, then by Inequality 1.1, $n - 1 \leq \dim(G \odot K_1)$ and by Theorem 1.2 $\dim(G \odot K_1) \leq n - 1$. Therefore $\dim(G \odot K_1) = n - 1$. In the case $G = K_{1,n-1}$, G is a tree with $\sigma(G) = n-1$ and by Theorem 1.3, $\dim(G\odot K_1) = n-1$. □

By Theorem 1.1, if $G = K_{s,t}(s, t \geq 1), G = K_s \vee \overline{K}_t(s \geq 1, t \geq 2)$, or $G = K_s \vee (K_t \cup K_1)(s, t \geq 1)$, then $\dim(G) = n(G) - 2$. Inequality 1.1 and Theorem 1.2 imply that for all these graphs $n(G) - 2 \leq \dim(G \odot K_1) \leq n(G) - 1$. On the other hand by Theorem 1.3 the star graph $K_{1,n-1}$ is the only graph among these with $\dim(K_{1,n-1} \odot K_1) = n - 1$. Therefore we have the following corollary.

Corollary 2.6. *If* $G = K_{s,t}(s, t > 1), G = K_s \vee \overline{K}_t(s \geq 1, t \geq 2)$, *or* $G = K_s \vee (K_t \cup K_1)(s, t \geq 1)$, *then* $\dim(G \odot K_1) = n(G) - 2$.

The following theorem completes the characterization of all graphs that attain the bound in Theorem 1.2.

Theorem 2.7. *Let G be a graph of order* $n \geq 2$. *Then* $\dim(G_k) = 2^{k-1}n - 1$ *if and only if* $k = 1$ *and* $G = K_n$ *or* $G = K_{1,n-1}$.

Proof. By Lemma 2.5, it is sufficient to prove that the equality is not hold for the case $k \geq 2$. Suppose on the contrary that equality is hold for some integer $k \geq 2$. Let $H = G \odot^{k-1} K_1$. Then $G \odot^k K_1 = H \odot K_1$ and the order of H is $2^{k-1}n$. Since equality is hold for $G\odot^k K_1$, we have $\dim(H\odot K_1) = n(H)-1$. Hence, by Lemma 2.5, H is a complete graph or a star graph. Since H has some vertices of degree 1, H is not a complete graph. Therefore H must be a star graph with $n(H) - 1$ leaves. On the other hand, H has $2^{k-2}n$ leaves. It implies that $2^{k-1}n - 1 = n(H) - 1 = 2^{k-2}n$, which is a contradiction. Therefore $\dim(G \odot^k K_1) = 2^{k-1}n - 1$ if and only if $k = 1$ and $G = K_n$ or $G = K_{1,n-1}$. □

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