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Almost Kenmotsu Metrics with Quasi Yamabe Soliton

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Abstract. In the present paper, we characterize, for a class of almost Kenmotsu manifolds, those that admit quasi Yamabe solitons. We show that if a $(k,\mu)'$-almost Kenmotsu manifold admits a quasi Yamabe soliton (g, V, λ, α) where V is pointwise collinear with ξ, then (1) V is a constant multiple of ξ, (2) V is a strict infinitesimal contact transformation, and (3) $(\pounds_V h')X = 0$ holds for any vector field X. We present an illustrative example to support the result.

1. Introduction

In 1989, Hamilton [7] proposed the concept of Yamabe flow defined as

$$\frac{\partial g}{\partial t} = -rg,$$

where r is the scalar curvature of the manifold. A Riemannian manifold (M, g) is said to admit a Yamabe soliton (g, V, λ) if

$$\frac{1}{2}\pounds_V g = (r - \lambda)g, \tag{1.1}$$

where $\pounds_V$ denotes the Lie derivative along V and λ is a constant. A Yamabe soliton is said to be shrinking, steady or expanding according as $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$ respectively. If V is the gradient Df of some smooth function $f : M \to \mathbb{R}$, then the Yamabe soliton is said to be a gradient Yamabe soliton, and (1.1) reduces to

$$\nabla^2 f = (r - \lambda)g, \tag{1.2}$$

where $\nabla^2 f$ is the Hessian of f.

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Extending the notion of a Yamabe soliton, Chen and Deshmukh [2] introduced the notion of a quasi Yamabe soliton which can be defined on a Riemannian manifold as follows:

$$\frac{1}{2}(\pounds_V g)(X,Y) = (r-\lambda)g(X,Y) + \alpha V^{\#}(X)V^{\#}(Y), \tag{1.3}$$

where $V^{\#}$ is the dual 1-form of V, λ is a constant and α is a smooth function. If V is a gradient of some smooth function f, then the above notion is called quasi Yamabe gradient soliton and then (1.3) can be written as

$$\nabla^2 f = (r-\lambda)g + \alpha df \otimes df. \tag{1.4}$$

The quasi Yamabe soliton is said to be shrinking, steady or expanding according as $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$ respectively. In [8], it is proved that the scalar curvature of a compact quasi Yamabe soliton is constant. In [10], the author proved that a quasi Yamabe gradient soliton on a complete non-compact manifold has warped product structure. In [6], Ghosh proved a similar result for Kenmotsu manifolds. In [13, 15], Wang studied Yamabe solitons on Kenmotsu and $(k,\mu)'$-almost Kenmotsu manifolds. Since a $(k,\mu)'$-almost Kenmotsu manifold is locally isometric to a warped product space and the scalar curvature is also constant (see Theorem 4.2 of [5]), the results proved in [8] and [6] hold trivially in a $(k,\mu)'$-almost Kenmotsu manifold admitting a quasi Yamabe gradient soliton. This motivates us to consider only quasi Yamabe solitons on $(k,\mu)'$-almost Kenmotsu manifolds.

The paper is organized as follows: In Section 2, some preliminary results on almost Kenmotsu manifolds are presented. Section 3 deals with $(k,\mu)'$-almost Kenmotsu manifolds admitting quasi Yamabe soliton. Section 4 contains an example to support the main result.

2. Preliminaries

A $(2n+1)$-dimensional almost contact metric manifold is a differentiable manifold M together with a structure (ϕ,ξ,η,g) satisfying

$$\phi^2 = -I + \eta\otimes\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta\circ\phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

for any vector fields X, Y on M, where ϕ is a $(1,1)$-tensor field, ξ is a unit vector field, η is a 1-form, g is the Riemannian metric and I is the identity endomorphism. The fundamental 2-form Φ on almost contact metric manifolds is defined by $\Phi(X,Y) = g(X,\phi Y)$ for any X, Y on M. Almost contact metric manifold such that η is closed and $d\Phi = 2\eta\wedge\Phi$ are called almost Kenmotsu manifolds (see [5], [9]).

Let M be a $(2n+1)$-dimensional almost Kenmotsu manifold. We denote by $h = \frac{1}{2}\pounds_\xi\phi$ and $l = R(\cdot,\xi)\xi$ on M. The tensor fields l and h are symmetric operators and satisfy the following relations [9]:

$$h\xi = 0,\ l\xi = 0,\ tr(h) = 0,\ tr(h\phi) = 0,\ h\phi + \phi h = 0, \tag{2.3}$$

$$\nabla_X\xi = X - \eta(X)\xi - \phi hX (\Rightarrow \nabla_\xi\xi = 0), \tag{2.4}$$

$$\phi l\phi - l = 2(h^2 - \phi^2), \tag{2.5}$$

$$R(X,Y)\xi = \eta(X)(Y - \phi hY) - \eta(Y)(X - \phi hX) + (\nabla_Y\phi h)X - (\nabla_X\phi h)Y, \tag{2.6}$$

for any vector fields X, Y on M. The $(1,1)$-type symmetric tensor field $h' = h\circ\phi$ is anti-commuting with ϕ and $h'\xi = 0$. Also it is clear that ([5], [11])

$$h = 0 \Leftrightarrow h' = 0,\ \ h'^2 = (k+1)\phi^2 (\Leftrightarrow h^2 = (k+1)\phi^2). \tag{2.7}$$

In [5], Dileo and Pastore introduced the notion of $(k,\mu)'$-nullity distribution, on an almost Kenmotsu manifold (M,ϕ,ξ,η,g), which is defined for any $p\in M$ and $k,\mu\in\mathbb{R}$ as follows:

$$\begin{aligned} N_p(k,\mu)' \ =\ & \{Z\in T_p(M) : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y] \\ & +\mu[g(Y,Z)h'X - g(X,Z)h'Y]\}. \end{aligned} \tag{2.8}$$

In [5], it is proved that in a $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1} with $h'\neq 0$, $k < -1,\ \mu = -2$ and Spec$(h') = \{0,\delta,-\delta\}$, with 0 as simple eigen value and $\delta = \sqrt{-k-1}$.

In [12], Wang and Liu proved that for a $(2n+1)$-dimensional $(k,\mu)'$-almost Kenmotsu manifold M with $h'\neq 0$, the Ricci operator Q of M is given by

$$Q = -2nid + 2n(k+1)\eta\otimes\xi - 2nh'. \tag{2.9}$$

Moreover, the scalar curvature of M^{2n+1} is $2n(k-2n)$. From (2.8), we have

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(X)h'Y], \tag{2.10}$$

where $k,\mu\in\mathbb{R}$. Also we get from (2.10)

$$R(\xi,X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(h'X,Y)\xi - \eta(Y)h'X]. \tag{2.11}$$

Contracting X in (2.10), we have

$$S(Y,\xi) = 2nk\eta(Y). \tag{2.12}$$

Using (2.3), we have

$$(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y) + g(h'X,Y). \tag{2.13}$$

For further details on almost Kenmotsu manifolds, we refer the reader to go through the references ([3]-[5], [9], [14]) and references therein.

3. Quasi Yamabe Soliton

In this section, we characterize $(k,\mu)'$-almost Kenmotsu manifolds admitting quasi Yamabe soliton (g,V,λ,α) such that V is pointwise collinear with the characteristic vector field ξ. In this regard, to prove our main theorem, we need the following definition:

Definition 3.1. A vector field V on an almost contact metric manifold (M,ϕ,ξ,η,g) is said to be an infinitesimal contact transformation if $\pounds_V \eta = f\eta$ for some smooth function f on M. In particular, if $f = 0$, then V is said to be a strict infinitesimal contact transformation.

Theorem 3.2. *If a $(2n+1)$-dimensional $(k,\mu)'$-almost Kenmotsu manifold M with $h' \neq 0$ admits quasi Yamabe soliton (g,V,λ,α) with the soliton vector field V pointwise collinear with ξ, then*

(1) *V is a constant multiple of ξ.*

(2) *V is a strict infinitesimal contact transformation.*

(3) *$(\pounds_V h')X = 0$ for any vector field X on M.*

Proof. If V is pointwise collinear with ξ, then there exist a non-zero smooth function b on M such that $V = b\xi$. Then from (1.3), we have

$$\frac{1}{2}(\pounds_{b\xi} g)(X,Y) = (r-\lambda)g(X,Y) + \alpha b^2 \eta(X)\eta(Y). \tag{3.1}$$

Now,

$$(\pounds_{b\xi} g)(X,Y) = g(\nabla_X b\xi, Y) + g(\nabla_Y b\xi, X).$$

Using (2.4), the foregoing equation reduces to

$$\begin{aligned}(\pounds_{b\xi} g)(X,Y) &= (Xb)\eta(Y) + (Yb)\eta(X) \\ &\quad +2b[g(X,Y) - \eta(X)\eta(Y) - g(\phi hX,Y)].\end{aligned} \tag{3.2}$$

Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of M. Now, substituting (3.2) in (3.1) and taking $X = Y = e_i$ and summing over i, we obtain

$$(\xi b) = (r-\lambda)(2n+1) + \alpha b^2 - 2nb. \tag{3.3}$$

Again substituting (3.2) in (3.1) and then putting $X = Y = \xi$ yields

$$(\xi b) = r - \lambda + \alpha b^2. \tag{3.4}$$

Equating (3.3) and (3.4), we obtain

$$b = r - \lambda. \tag{3.5}$$

Since λ is constant and $r = 2n(k - 2n)$ is constant, b is also a constant. This proves (1).
Now, from (3.4), we have

$$\alpha b^2 = -(r - \lambda), \tag{3.6}$$

which implies $\alpha b = -1$ and hence, α is also a constant. Now, since $V = b\xi$, we write (1.3) as

$$\frac{1}{2}(\pounds_V g)(X,Y) = (r - \lambda)g(X,Y) + \alpha b^2 \eta(X)\eta(Y). \tag{3.7}$$

Differentiating the above equation covariantly along any vector field Z, we get

$$\frac{1}{2}(\nabla_Z \pounds_V g)(X,Y) = \alpha b^2[\eta(Y)(\nabla_Z \eta)X + \eta(X)(\nabla_Z \eta)Y]. \tag{3.8}$$

It is well known that (see [16])

$$(\pounds_V \nabla_X g - \nabla_X \pounds_V g - \nabla_{[V,X]} g)(Y,Z) = -g((\pounds_V \nabla)(X,Y),Z) - g((\pounds_V \nabla)(X,Z),Y).$$

Since $\nabla g = 0$, then the above relation becomes

$$(\nabla_X \pounds_V g)(Y,Z) = g((\pounds_V \nabla)(X,Y),Z) + g((\pounds_V \nabla)(X,Z),Y). \tag{3.9}$$

Since $\pounds_V \nabla$ is symmetric, then it follows from (3.9) that

$$\begin{aligned} g((\pounds_V \nabla)(X,Y),Z) &= \frac{1}{2}(\nabla_X \pounds_V g)(Y,Z) + \frac{1}{2}(\nabla_Y \pounds_V g)(X,Z) \\ &\quad -\frac{1}{2}(\nabla_Z \pounds_V g)(X,Y). \end{aligned} \tag{3.10}$$

Using (2.13) and (3.8) in (3.10) we have

$$g((\pounds_V \nabla)(X,Y),Z) = 2\alpha b^2[g(X,Y) - \eta(X)\eta(Y) + g(h'X,Y)]\eta(Z),$$

which implies

$$(\pounds_V \nabla)(X,Y) = 2\alpha b^2[g(X,Y) - \eta(X)\eta(Y) + g(h'X,Y)]\xi. \tag{3.11}$$

Substituting $Y = \xi$ in (3.11), we get $(\pounds_V \nabla)(X,\xi) = 0$. From which we obtain $\nabla_Y(\pounds_V \nabla)(X,\xi) = 0$. This gives

$$(\nabla_Y \pounds_V \nabla)(X,\xi) = -(\pounds_V \nabla)(\nabla_Y X,\xi) - (\pounds_V \nabla)(X,\nabla_Y \xi).$$

Using $(\pounds_V\nabla)(X,\xi)=0$, (3.11) and (2.4) in the foregoing equation, we infer that

$$(\nabla_Y\pounds_V\nabla)(X,\xi) = -2\alpha b^2[g(X,Y)-\eta(X)\eta(Y) +2g(h'X,Y)+g(h'^2X,Y)]\xi. \tag{3.12}$$

Using the foregoing equation in the following formula (see [16])

$$(\pounds_V R)(X,Y)Z=(\nabla_X\pounds_V\nabla)(Y,Z)-(\nabla_Y\pounds_V\nabla)(X,Z),$$

we obtain

$$(\pounds_V R)(X,\xi)\xi=(\nabla_X\pounds_V\nabla)(\xi,\xi)-(\nabla_\xi\pounds_V\nabla)(X,\xi)=0. \tag{3.13}$$

Now, substituting $Y=\xi$ in (3.7) and using (3.6), we get $(\pounds_V g)(X,\xi)=0$, which implies

$$(\pounds_V\eta)X=g(X,\pounds_V\xi). \tag{3.14}$$

Since $V=b\xi$ and b is a constant, then we can easily obtain $\pounds_V\xi=0$ and hence, from (3.14), we have $(\pounds_V\eta)X=0$ for any vector field X on M. Therefore, V is a strict infinitesimal contact transformation. This proves (2).
From (2.10), we have

$$R(X,\xi)\xi=k(X-\eta(X)\xi)-2h'X. \tag{3.15}$$

Now, using (3.14)-(3.15) and (2.10)-(2.11), we obtain

$$(\pounds_V R)(X,\xi)\xi=-2(\pounds_V h')X. \tag{3.16}$$

Equating (3.13) and (3.16), we obtain

$$(\pounds_V h')X=0, \tag{3.17}$$

for any vector field X on M. This proves (3) and the proof is complete. □

Remark 3.3. From (3.5), we can see that $\lambda=r-b$. Since $r=2n(k-2n)<0$ as $k<-1$, then $\lambda<0$ when $b>0$, $\lambda=0$ when $b=2n(k-2n)$ and $\lambda>0$ when $b<2n(k-2n)$. Hence, the quasi Yamabe soliton is shrinking, steady or expanding according as $b>0$, $b=2n(k-2n)$ or $b<2n(k-2n)$ respectively.

Remark 3.4. It is known that for smooth tensor field T, $\pounds_X T=0$ if and only if ϕ_t is a symmetric transformation for T, where $\{\phi_t : t\in\mathbb{R}\}$ is the 1-parameter group of diffeomorphisms corresponding to the vector filed X on a manifold [1]. Since h' is a smooth tensor field of type $(1,1)$ on M, then $(\pounds_V h')X=0$ if and only if ψ_t is a symmetric transformation for V, where $\{\psi_t : t\in\mathbb{R}\}$ is the 1-parameter group of diffeomorphisms corresponding to the vector filed V.

4. Example of a Shrinking Quasi Yamabe Soliton

In [5], Dileo and Pastore presented an example of a $(2n+1)$-dimensional $(k,\mu)'$-almost Kenmotsu manifold. In [4], the authors studied this for 5-dimensional case and obtained $k=-2$.
We now mention the necessary results from that example to verify our result.

$$[\xi, e_2] = -2e_2,\ [\xi, e_3] = -2e_3,\ [\xi, e_4] = 0,\ [\xi, e_5] = 0,$$

$[e_i, e_j] = 0$, where $i, j = 2, 3, 4, 5$. The Riemannian metric is defined by

$$g(\xi,\xi) = g(e_2,e_2) = g(e_3,e_3) = g(e_4,e_4) = g(e_5,e_5) = 1$$

and $g(\xi, e_i) = g(e_i, e_j) = 0$ for $i \neq j$; $i, j = 2, 3, 4, 5$.
The 1-form η is defined by $\eta(Z) = g(Z, \xi)$, for any $Z \in T(M)$.
Also, $h'\xi = 0,\ h'e_2 = e_2,\ h'e_3 = e_3,\ h'e_4 = -e_4,\ h'e_5 = -e_5$.
The scalar curvarure is $r = -80$.
Now, it is easy to see that

$$(\pounds_\xi g)(\xi,\xi) = (\pounds_\xi g)(e_4,e_4) = (\pounds_\xi g)(e_5,e_5) = 0,$$

$$(\pounds_\xi g)(e_2,e_2) = (\pounds_\xi g)(e_3,e_3) = 4.$$

Now, it can be easily verified from (1.3) that $(g, V, \lambda, \alpha) = (g, \xi, -41, 121)$ is a shrinking quasi Yamabe soliton, where $\lambda = -49 < 0$ and $b = 1 > 0$.
Also, $(\pounds_\xi \eta)X = 0$ and $(\pounds_\xi h')X = 0$ for all $X \in \{e_2, e_3, e_4, e_5\}$.
This verifies our Theorem 3.2.

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