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Notes on the Second Tangent Bundle over an Anti -biparaKaehlerian Manifold

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Abstract. In this note, we define a Berger type deformed Sasaki metric as a natural metric on the second tangent bundle of a manifold by means of a biparacomplex structure. First, we obtain the Levi-Civita connection of this metric. Secondly, we get the curvature tensor, sectional curvature, and scalar curvature. Afterwards, we obtain some formulas characterizing the geodesics with respect to the metric on the second tangent bundle. Finally, we present the harmonicity conditions for some maps.

1. Introduction

Many geometric concepts can be defined by a suitable algebraic formalism. This point of view has interest because one can compare different geometric structures having similar algebraic expressions. In this paper, we will consider biparacomplex structures on a smooth $4n-$dimensional manifold. An almost biparacomplex structure on a smooth manifold consists of two almost product structures φ_1 and φ_2 which satisfy [6, 8]

$$\varphi_1 \circ \varphi_2 + \varphi_2 \circ \varphi_1 = 0.$$

The Nijenhuis tensor N_α of φ_α, for $\alpha = 1$ or 2, is defined by

$$N_\alpha(X,Y) = [\varphi_\alpha X, \varphi_\alpha Y] + \varphi_\alpha^2[X,Y] - \varphi_\alpha[X, \varphi_\alpha Y] - \varphi_\alpha[\varphi_\alpha X, Y].$$

It is well known that the structure φ_α is integrable if and only if the corresponding

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Nijenhuis tensor N_α vanishes, $N_\alpha = 0$. Also, the classical definition of integrability of structures can be given in this way: a paracomplex structure φ is integrable if there exists a torsion free connection parallelizing φ [9]. If a torsion free connection parallelizes φ, then $N_\varphi = 0$.

An anti-biparaHermitian metric is a Riemannian metric which is compatible with the (almost) biparacomplex structure φ_α in the sense that the metric g is pure with respect to each φ_α. We called such a structure (almost) anti-biparaHermitian. If φ_α is parallel with respect to the Levi-Civita connection for $\alpha = 1$ and 2, then the manifold is called anti-biparaKaehlerian manifold. The existence of anti-biparaKaehlerian structures on $4n-$dimensional Riemannian manifolds allow one to construct new Riemannian metrics on the second tangent bundle over $4n-$dimensional Riemannian manifolds. This paper aims to construct a new metric on the second tangent bundle over an anti-biparaKaehlerian manifold and study its geometry.

2. The Berger Type Deformed Sasaki Metric on TM

Let (M_n, g) be an $n-$dimensional Riemannian manifold and (TM, π, M) be its tangent bundle. A local chart $(U, x^i)_{i=1\ldots n}$ on M_n induces a local chart $(\pi^{-1}(U), x^i, u^i)_{i=1\ldots n}$ on TM. Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g.

We have two complementary distributions on TM, the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$. Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M_n. The vertical and the horizontal lifts of X are defined by

$$X^V = X^i \frac{\partial}{\partial u^i}, \tag{2.1}$$

$$X^H = X^i \frac{\delta}{\delta x^i} = X^i \{ \frac{\partial}{\partial x^i} - u^j \Gamma^k_{ij} \frac{\partial}{\partial u^k} \} \tag{2.2}$$

(see [11]).

An anti-paraKaehlerian manifold is a triple (M_n, g, φ) such as M_n is a manifold of even dimension $(n = 2k)$ and φ is an integrable almost product structure ($\varphi^2 = I$ and $\nabla\varphi = 0$) verifying

$$g(\varphi(X), Y) = g(X, \varphi(Y)) \tag{2.3}$$

or equivalently

$$g(\varphi(X), \varphi(Y)) = g(X, Y) \tag{2.4}$$

for all vector fields X, Y (for more details on the integrability, see [9]).

Definition 2.1. Let (M_n, g, φ) be an anti-paraKaehlerian manifold. A (φ, δ)-deformed Sasaki metric on TM is defined by

1. $g_{\varphi,\delta}(X^H, Y^H) = \frac{\varepsilon}{2} g(X, Y) \circ \pi$,

2. $g_{\varphi,\delta}(X^H, Y^V) = 0$,
3. $g_{\varphi,\delta}|_{(x,u)}(X^V, Y^V) = g_x(X,Y) + \delta^2 g_x(X, \varphi(u)) g_x(Y, \varphi(u))$,

where X, Y are vector fields on M_n, $(x,u) \in TM$, $\varepsilon \in \{1,2\}$ and δ is a constant.

1. If $\delta = 0$ and $\varepsilon = 2$ then $g_{\varphi,\delta}$ is the Sasaki metric [11],
2. If $\delta = 1$ and $\varepsilon = 2$ then $g_{\varphi,\delta}$ is the Berger type deformed Sasaki metric [1].

Lemma 2.2. *Let (M_n, g, φ) be an anti-paraKaehlerian manifold. For all $x \in M_n$ and $u = u^i \frac{\partial}{\partial x^i} \in T_x M$, we have the following*

1. $X^H(g(u,u))_{(x,u)} = 0$,
2. $X^H(g(Y,u))_{(x,u)} = g(\nabla_X Y, u)_x$,
3. $X^V(g(u,u))_{(x,u)} = 2g(X,u)_x$,
4. $X^V(g(Y,u))_{(x,u)} = g(X,Y)_x$,
5. $X^V(g(Y,\varphi(u))) = g((Y, \varphi(X))$,
6. $X^H(g(Y,\varphi(u))) = g((\nabla_X Y), \varphi(u))$.

Proposition 2.3. *Let (M_n, g, φ) be an anti-paraKaehlerian manifold. Then we have*

$$\begin{aligned} \varphi \circ R(X,Y) &= R(X,Y) \circ \varphi, \\ R(X, \varphi(Y)) &= R(\varphi(X), Y), \\ R(X, \varphi(X)) &= 0 \end{aligned}$$

for all vector fields X, Y on M_n [9].

3. The Berger Type Deformed Sasaki Metric on T^2M

Let (M_n, g) be a Riemannian manifold and ∇ its Levi-Civita connection. The second tangent bundle is the natural bundle of $2-$jets of differentiable curves, defined by

$$T^2M = \{j_0^2\gamma \qquad \gamma : \mathbb{R}_0 \to M, \text{ is a smooth curve at } 0 \in \mathbb{R}\}.$$

Theorem 3.1. ([3]) *If $TM \oplus TM$ denotes the Whitney sum, then*

$$\begin{aligned} S : T^2M &\to TM \oplus TM \\ j_0^2\gamma &\mapsto (\dot\gamma(0), (\nabla_{\dot\gamma(0)}\dot\gamma)(0)) \end{aligned} \tag{3.1}$$

is a diffeomorphism of natural bundles.

In the induced coordinate, we have

$$S : (x^i, u^i, z^i) \mapsto (x^i, u^i, z^i + u^j u^k \Gamma^i_{jk}). \tag{3.2}$$

Definition 3.2. ([3]) Let T^2M be the second tangent bundle endowed with the vectorial structure induced by the diffeomorphism S. For any section $\sigma \in \Gamma(T^2M)$, we define two vector fields on M_n by

$$X_\sigma = P_1 \circ S \circ \sigma \quad \textit{and} \quad Y_\sigma = P_2 \circ S \circ \sigma$$

where P_1 and P_2 denote the first and the second projection from $TM \oplus TM$ onto TM.

Definition 3.3. ([3]) Let (M_n, g) be a Riemannian manifold and $X \in \Gamma(TM)$ be a vector field on M_n. For $\lambda = 0, 1, 2$, the λ-lifts of X to T^2M are defined by

$$\begin{aligned} X^{(0)} &= S_*^{-1}(X^H, X^H), \\ X^{(1)} &= S_*^{-1}(X^V, 0), \\ X^{(2)} &= S_*^{-1}(0, X^V). \end{aligned} \tag{3.3}$$

Theorem 3.4. *Let (M_n, g) be a Riemannian manifold. If R denotes the Riemannian curvature tensor of (M_n, g), then on T^2M we have*

1. $[X^{(0)}, Y^{(0)}] = [X, Y]^{(0)} - (R_x(X, Y)u)^{(1)} - (R_x(X, Y)w)^{(2)}$,
2. $[X^{(0)}, Y^{(i)}] = (\nabla_X Y)^{(i)}$,
3. $[X^{(i)}, Y^{(j)}] = 0$,

where $(x, u, w) = S(p)$ and $i, j = 1, 2$ [3].

Let the quadruple $(M_n, g, \varphi_1, \varphi_2)$ be an anti-biparaKaehlerian manifold such that $n = 4k$ and $\varphi_1(x) \neq \varphi_2(x)$ everywhere on M_n.

Definition 3.5. Let $(M_n, g, \varphi_1, \varphi_2)$ be an anti-biparaKaehlerian manifold. We define a Berger type deformed Sasaki metric g_{BS} on the second tangent bundle by

$$g_{BS} = S_*^{-1}(g_{\varphi_1,\delta} \oplus g_{\varphi_2,\eta}). \tag{3.4}$$

From Definition 3.5 and the formula (3.3), we obtain the following proposition.

Proposition 3.6. *Let $(M_n, g, \varphi_1, \varphi_2)$ be an anti-biparaKaehlerian manifold. If $p \in T^2M$, then for all vector fields X, Y on M_n and $i, j \in \{0, 1, 2\}$ $(i \neq j)$, we obtain*

1. $g_{BS}(X^{(0)}, Y^{(0)})_p = g(X, Y)_x$,
2. $g_{BS}(X^{(i)}, Y^{(j)})_p = 0$,
3. $g_{BS}(X^{(1)}, Y^{(1)})_p = g(X, Y) + \delta^2 g(X, \varphi_1(u))g(Y, \varphi_1(u)))_x$,
4. $g_{BS}(X^{(2)}, Y^{(2)})_p = g(X, Y) + \eta^2 g(X, \varphi_2(w))g(Y, \varphi_2(w))_x$,

where $S(p) = (x, u, w) \in T_xM_n \oplus T_xM_n$.

Theorem 3.7. *Let $(M_n, g, \varphi_1, \varphi_2)$ be an anti-biparaKaehlerian manifold and (T^2M, g) be its second tangent bundle equipped with the Berger type deformed Sasaki metric. If $\widetilde{\nabla}$ denotes the Levi-Civita connection of (T^2M, g_{BS}), then for $p \in T^2M$ and for all vector fields X, Y on M_n we have*

1. $(\widetilde{\nabla}_{X^{(0)}}Y^{(0)})_p = (\nabla_X Y)^{(0)} - \frac{1}{2}(R(X,Y)u)^{(1)} - \frac{1}{2}(R(X,Y)w)^{(2)}$,

2. $(\widetilde{\nabla}_{X^{(0)}}Y^{(1)})_p = (\nabla_X Y)^{(1)} + \frac{1}{2}(R(u,Y)X)^{(0)}$,

3. $(\widetilde{\nabla}_{X^{(0)}}Y^{(2)})_p = (\nabla_X Y)^{(2)} + \frac{1}{2}(R(w,Y)X)^{(0)}$,

4. $(\widetilde{\nabla}_{X^{(1)}}Y^{(0)})_p = \frac{1}{2}(R(u,X)Y))^{(0)}$,

5. $(\widetilde{\nabla}_{X^{(2)}}Y^{(0)})_p = \frac{1}{2}(R(w,X)Y))^{(0)}$,

6. $(\widetilde{\nabla}_{X^{(1)}}Y^{(1)})_p = \frac{\delta^2}{\lambda}g(X, \varphi_1(Y))(\varphi_1(u))^{(1)}$,

7. $(\widetilde{\nabla}_{X^{(2)}}Y^{(2)})_p = \frac{\eta^2}{\beta}g(X, \varphi_2(Y))(\varphi_2(w))^{(2)}$,

8. $(\widetilde{\nabla}_{X^{(1)}}Y^{(2)})_p = (\widetilde{\nabla}_{X^{(2)}}Y^{(1)})_p = 0$,

where $S(p) = (x, u, w)$, $\lambda = 1 + \delta^2|u|^2$, $\beta = 1 + \eta^2|w|^2$, ∇ and R denote the Levi-Civita connection and the Riemannian curvature tensor of (M_n, g), respectively.

Proof. Using the Proposition 3.6, the Lemma 2.2 and the Koszul formula, the Theorem 3.7 follows. □

4.The Riemannian Curvature Tensors

Let $(M_n, g, \varphi_1, \varphi_2)$ be an anti-biparaKaehlerian manifold and (T^2M, g_{BS}) its second tangent bundle equipped with the Berger type deformed Sasaki metric, $F : TM \to TM$ be a smooth bundle endomorphism of TM. The vector fields $F^{(1)}$, $F^{(2)}$ and $F^{(0)}$ on T^2M are defined by

$$F^{(0)}(u) = (Fu)^{(0)} \quad , \quad F^{(1)}(u) = (Fu)^{(1)} \qquad F^{(2)}(u) = (Fu)^{(2)}.$$

Locally, we have

$$F^{(0)}(u) = u^i(F\partial_i)^{(0)} \quad , \quad F^{(1)}(u) = u^i(F\partial_i)^{(1)} \qquad F^{(2)}(u) = u^i(F\partial_i)^{(2)}.$$

Proposition 4.1. *Let $(M_n, g, \varphi_1, \varphi_2)$ be an anti-biparaKaehlerian manifold and (T^2M, g_{BS}) its second tangent bundle equipped with the Berger type deformed Sasaki metric. Then we have the following formulas*

1. $(\widetilde{\nabla}_{X^{(0)}}F^{(0)})_p = \big((\nabla_X F)(y)\big)^{(0)} - \frac{1}{2}\big((R_x(X,Fy)u)\big)^{(1)} - \frac{1}{2}\big((R_x(X,Fy)w)\big)^{(2)}$,

2. $(\widetilde{\nabla}_{X^{(0)}}F^{(1)})_p = \big((\nabla_X F)(u)\big)^{(1)} + \frac{1}{2}\big((R_x(u,Fu)X)\big)^{(0)}$,

3. $(\widetilde{\nabla}_{X^{(0)}}F^{(2)})_p = \big((\nabla_X F)(w)\big)^{(2)} + \frac{1}{2}\big((R_x(w,Fw)X)\big)^{(0)}$,

4. $(\widetilde{\nabla}_{X^{(1)}}F^{(0)})_p = \big((FX)\big)^{(0)} + \frac{1}{2}\big((R_x(u,X)Fu))\big)^{(0)}$,

5. $(\widetilde{\nabla}_{X^{(2)}}F^{(0)})_p = \big((FX)\big)^{(0)} + \frac{1}{2}\big((R_x(w,X)Fw))\big)^{(0)}$,

6. $(\widetilde{\nabla}_{X^{(1)}}F^{(1)})_p = \frac{\delta^2}{\lambda}g(X,\varphi_1(Fu))(\varphi_1(u))^{(1)} + (FX)^{(1)}$,

7. $(\widetilde{\nabla}_{X^{(2)}}F^{(2)})_p = \frac{\eta^2}{\beta}g(X,\varphi_1(Fw))(\varphi_2(w))^{(2)} + (FX)^{(2)}$,

8. $(\widetilde{\nabla}_{X^{(1)}}F^{(2)})_p = (\widetilde{\nabla}_{X^{(2)}}F^{(1)})_p = 0$

for any vector field X *on* M_n, $S(p) = (u,w)$ *and* $y \in \{u,w\}$.

Proof. The results come directly from the Theorem 3.7. □

Theorem 4.2. *Let* $(M_n, g, \varphi_1, \varphi_2)$ *be an anti-biparaKaehlerian manifold and* (T^2M, g_{BS}) *its second tangent bundle equipped with the Berger type deformed Sasaki metric. Then we have the following formulas*

$$\begin{aligned}
\widetilde{R}_p(X^{(0)},Y^{(0)})Z^{(0)} &= \frac{1}{2}\big((\nabla_Z R)(X,Y)u\big)^{(1)} + \frac{1}{2}\big((\nabla_Z R)(X,Y)w\big)^{(2)} \\
&+\Big[R(X,Y)Z + \frac{1}{4}R(u,R(Z,Y)u)X + \frac{1}{4}R(u,R(X,Z)u)Y \\
&+\frac{1}{2}R(u,R(X,Y)u)Z + \frac{1}{4}R(w,R(Z,Y)w)X \\
&+\frac{1}{4}R(w,R(X,Z)w)Y + \frac{1}{2}R(w,R(X,Y)w)Z\Big]^{(0)},
\end{aligned}$$

$$\begin{aligned}
\widetilde{R}_p(X^{(0)},Y^{(0)})Z^{(1)} &= \Big[R(X,Y)Z + \frac{1}{4}R(R(u,Z)Y,X)u - \frac{1}{4}R(R(u,Z)X,Y)u\Big]^{(1)} \\
&+\Big[\frac{1}{4}R(R(u,Z)Y,X)w - \frac{1}{4}R(R(u,Z)X,Y)w\Big]^{(2)} \\
&+\frac{1}{2}\Big[(\nabla_X R)(u,Z)Y - (\nabla_Y R)(u,Z)X\Big]^{(0)} \\
&+\frac{\delta^2}{\lambda}g(\varphi_1(Z),R(X,Y)u)(\varphi_1 u)^{(1)},
\end{aligned}$$

$$
\begin{aligned}
\widetilde{R}_p(X^{(0)},Y^{(0)})Z^{(2)} &= \Big[R(X,Y)Z+\frac{1}{4}R(R(w,Z)Y,X)w-\frac{1}{4}R(R(w,Z)X,Y)w\Big]^{(2)}\\
&+\Big[\frac{1}{4}R(R(w,Z)Y,X)u-\frac{1}{4}R(R(w,Z)X,Y)u\Big]^{(1)}\\
&+\frac{1}{2}\Big[(\nabla_X R)(w,Z)Y-(\nabla_Y R)(w,Z)X\Big]^{(0)}\\
&+\frac{\eta^2}{\beta}g(\varphi_2(Z),R(X,Y)w)(\varphi_2 w)^{(2)},
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{R}(X^{(0)},Y^{(1)})Z^{(0)} &= \Big[\frac{1}{2}R(X,Z)Y+\frac{1}{4}R(R(u,Y)Z,X)u\Big]^{(1)}+\frac{1}{4}\Big[R(R(u,Y)Z,X)w\Big]^{(2)}\\
&+\frac{1}{2}\Big[\nabla_X R)(u,Y)Z\Big]^{(0)}+\frac{\delta^2}{\lambda}g(\varphi_1(Y),R(X,Z)u)(\varphi_1 u)^{(1)},
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{R}_p(X^{(0)},Y^{(2)})Z^{(0)} &= \Big[\frac{1}{2}R(X,Z)Y+\frac{1}{4}R(R(w,Y)Z,X)w\Big]^{(2)}+\frac{1}{4}\Big[R(R(w,Y)Z,X)u\Big]^{(1)}\\
&+\frac{1}{2}\Big[\nabla_X R)(w,Y)Z\Big]^{(0)}+\frac{\eta^2}{\beta}g(\varphi_2(Y),R(X,Z)w)(\varphi_2 w)^{(2)},
\end{aligned}
$$

$$
\widetilde{R}_p(X^{(0)},Y^{(1)})Z^{(1)} = -\frac{1}{2}\Big[R(Y,Z)X+\frac{1}{2}(R(u,Y)R(u,Z)X)\Big]^{(0)},
$$

$$
\widetilde{R}_p(X^{(0)},Y^{(2)})Z^{(2)} = -\frac{1}{2}\Big[R(Y,Z)X+\frac{1}{2}(R(w,Y)R(w,Z)X)\Big]^{(0)},
$$

$$
\widetilde{R}_p(X^{(1)},Y^{(1)})Z^{(0)} = \Big[R(X,Y)Z+\frac{1}{4}R(u,X)R(u,Y)Z-\frac{1}{4}R(u,Y)R(u,X)Z\Big]^{(0)},
$$

$$
\widetilde{R}_p(X^{(2)},Y^{(2)})Z^{(0)} = \Big[R(X,Y)Z+\frac{1}{4}R(w,X)R(w,Y)Z-\frac{1}{4}R(w,Y)R(w,X)Z\Big]^{(0)},
$$

$$
\begin{aligned}
\widetilde{R}_p(X^{(1)},Y^{(1)})Z^{(1)} &= \frac{\delta^4}{\lambda^2}\Big[g(Y,u)g(X,\varphi_1 Z)-g(X,u)g(Y,\varphi_1 Z)\Big](\varphi_1 u)^{(1)}\\
&+\frac{\delta^2}{\lambda}\Big[g(Y,\varphi_1 Z)\varphi_1 X-g(X,\varphi_1 Z)\varphi_1 Y\Big]^{(1)},
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{R}_p(X^{(2)},Y^{(2)})Z^{(2)} &= \frac{\eta^4}{\beta^2}\Big[g(Y,w)g(X,\varphi_2 Z)-g(X,w)g(Y,\varphi_2 Z)\Big](\varphi_2 w)^{(2)}\\
&+\frac{\eta^2}{\beta}\Big[g(Y,\varphi_2 Z)\varphi_2 X-g(X,\varphi_2 Z)\varphi_2 Y\Big]^{(2)}
\end{aligned}
$$

for all vector fields X, Y, Z *on* M_n.

Proof. The results come directly from the Theorem 3.7 and the Proposition 4.1. □

Let $(e_1, ..., e_n)$ (resp $(\overline{e}_1, ..., \overline{e}_n)$) be an orthonormal frame of T_xM where $e_1 =$

$\frac{\varphi_1 u}{\|u\|}$ (resp $\overline{e}_1 = \frac{\varphi_2 w}{\|w\|}$). Then

$$\Big\{E_i = (e_i)^{(0)},\ E_{n+1} = \frac{1}{\sqrt{\lambda}}(e_1)^{(1)},\ E_{n+k} = (e_k)^{(1)},$$
$$E_{2n+1} = \frac{1}{\sqrt{\beta}}(\overline{e}_1)^{(2)},\ E_{2n+k} = (\overline{e}_k)^{(2)}\Big\}_{\substack{i=1\dots n\\ k=2\dots n}}$$

is an orthonormal frame of T_pT^2M, where $p = S^{-1}(x,u,w)$.

Let $\widetilde{K}$ be the sectional curvature of (T^2M, g_{BS}) defined by

$$\widetilde{K} = g_{BS}(\widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{Y},\widetilde{X})$$

for orthonormal vector fields $\widetilde{X},\widetilde{Y}$ on T^2M. From the Theorem 4.2, standard calculations give the following result.

Proposition 4.3. *Let $(M_n, g, \varphi_1, \varphi_2)$ be an anti-biparaKaehlerian manifold and (T^2M, g_{BS}) its second tangent bundle equipped with the Berger type deformed Sasaki metric. We have the following formulas*

1. $\widetilde{K}(E_i,E_j) = K(e_i,e_j) - \frac{3}{4}\|R(e_i,e_j)u\|^2 - \frac{3}{4}\|R(e_i,e_j)w\|^2$,

2. $\widetilde{K}(E_i,E_{n+1}) = \widetilde{K}(E_i,E_{2n+1}) = 0$,

3. $\widetilde{K}(E_i,E_{n+k}) = \frac{1}{4}\|R(u,e_k)e_i\|^2$,

4. $\widetilde{K}(E_i,E_{2n+k}) = \frac{1}{4}\|R(w,\overline{e}_k)e_i\|^2$,

5. $\widetilde{K}(E_{n+1},E_{n+k}) = \frac{\delta^4}{\lambda^2(\lambda-1)}\big[g(\varphi_1(u),u)g(\varphi_1(e_k),e_k) - g(e_k,u)^2\big]$,

6. $\widetilde{K}(E_{2n+1},E_{2n+k}) = \frac{\eta^4}{\beta^2(\beta-1)}\big[g(\varphi_2(w),w)g(\varphi_2(\overline{e}_k),\overline{e}_k) - g(\overline{e}_k,w)^2\big]$,

7. $\widetilde{K}(E_{n+l},E_{n+k}) = \frac{\delta^2}{\lambda}\big[g(\varphi_1(e_l),e_l)g(\varphi_1(e_k),e_k) - g(e_k,\varphi_1(e_l))^2\big]$,

8. $\widetilde{K}(E_{2n+l},E_{2n+k}) = \frac{\eta^2}{\beta}\big[g(\varphi_2(\overline{e}_l),\overline{e}_l)g(\varphi_2(\overline{e}_k),\overline{e}_k) - g(\overline{e}_k,\varphi_2(\overline{e}_l))^2\big]$.

The relationship between the scalar curvature $\widetilde{r}$ of (T^2M, g_{BS}) and the scalar curvature r of (M_n, g) is given in the following theorem.

Theorem 4.4. *Let $(M_n, g, \varphi_1, \varphi_2)$ be an anti-biparaKaehlerian manifold and (T^2M, g_{BS}) its second tangent bundle equipped with the Berger type deformed Sasaki metric. The corresponding scalar curvature $\widetilde{r}$ is given by*

$$
\begin{aligned}
\widetilde{r} \;=\;& r - \sum_{i,j=1}^{n} \frac{3}{4}\Big[\|R(e_i,e_j)u\|^2 + \|R(e_i,e_j)w\|^2\Big] + \frac{\delta^2}{\lambda}A^2 + \frac{2\delta^4}{\lambda^2(\lambda-1)}g(\varphi_1 u,u)A \\
&+ \sum_{i,j=1}^{n} \frac{1}{2}\Big[\|R(u,e_j)e_i\|^2 + \|R(w,\overline{e}_j)e_i\|^2\Big] + \frac{\eta^2}{\beta}B^2 + \frac{2\eta^4}{\beta^2(\beta-1)}g(\varphi_2 w,w)B \\
&- \frac{\delta^6(\lambda+1)(\lambda+2)}{\lambda^2(\lambda-1)^3}g(\varphi_1(u),u)^2 + \frac{\delta^2((2-n)\lambda-2)}{\lambda^2} \\
&- \frac{\eta^6(\beta+1)(\beta+2)}{\beta^2(\beta-1)^3}g(\varphi_2(w),w)^2 + \frac{\eta^2((2-n)\beta-2)}{\beta^2},
\end{aligned}
\tag{4.1}
$$

where $A = \sum_{i=2}^{n} g(\varphi_1(e_i), e_i)$ and $B = \sum_{i=2}^{n} g(\varphi_2(\overline{e}_i), \overline{e}_i)$.

Theorem 4.5. *Let $(M_n, g, \varphi_1, \varphi_2)$ be a flat anti-biparaKaehlerian manifold and (T^2M, g_{BS}) its second tangent bundle equipped with the Berger type deformed Sasaki metric. The corresponding scalar curvature $\widetilde{r}$ is given by*

$$
\begin{aligned}
\widetilde{r} \;=\;& \frac{\delta^2}{\lambda}A^2 + \frac{2\delta^4}{\lambda^2(\lambda-1)}g(\varphi_1 u,u)A + \frac{\eta^2}{\beta}B^2 + \frac{2\eta^4}{\beta^2(\beta-1)}g(\varphi_2 w,w)B \\
&- \frac{\delta^6(\lambda+1)(\lambda+2)}{\lambda^2(\lambda-1)^3}g(\varphi_1(u),u)^2 + \frac{\delta^2((2-n)\lambda-2)}{\lambda^2} \\
&- \frac{\eta^6(\beta+1)(\beta+2)}{\beta^2(\beta-1)^3}g(\varphi_2(w),w)^2 + \frac{\eta^2((2-n)\beta-2)}{\beta^2},
\end{aligned}
$$

where $A = \sum_{i=2}^{n} g(\varphi_1(e_i), e_i)$ and $B = \sum_{i=2}^{n} g(\varphi_2(\overline{e}_i), \overline{e}_i)$.

5. Geodesics on the Second Tangent Bundle

Lemma 5.1. ([10]) *Let (M_n, g) be a Riemannian manifold. If X, Y are vector fields and $(x, u) \in TM$ such that $X_x = u$, then we have*

$$d_x X(Y_x) = Y^H_{(x,u)} + (\nabla_Y X)^V_{(x,u)}.$$

Lemma 5.2. *Let (M_n, g) be a Riemannian manifold. If $Z \in \Gamma(TM)$, $\sigma \in \Gamma(T^2M)$ and $p = \sigma(x)$. Then we have*

$$d_x\sigma(Z_x) = Z^{(0)}_p + (\nabla_Z X_\sigma)^{(1)}_p + (\nabla_Z Y_\sigma)^{(2)}_p. \tag{5.1}$$

Proof. Using the Lemma 5.1, we obtain

$$\begin{aligned} d_x\sigma(Z) &= dS^{-1}(dX_\sigma(Z), dY_\sigma(Z))_{S(p)} \\ &= dS^{-1}(Z^H, Z^H)_{S(p)} + dS^{-1}((\nabla_Z X_\sigma)^V, (\nabla_Z Y_\sigma)^V)_{S(p)} \\ &= Z_p^{(0)} + (\nabla_Z X_\sigma)_p^{(1)} + (\nabla_Z Y_\sigma)_p^{(2)}. \end{aligned}$$

□

Lemma 5.3. *Let $(M_n, g, \varphi_1, \varphi_2)$ be an anti-biparaKaehlerian manifold and (T^2M, g_{BS}) be its second tangent bundle equipped with the Berger type deformed Sasaki metric and $x : I \to M_n$ be a curve on M_n. If $C : t \in I \to C(t) = S^{-1}(x(t), y(t), z(t))$ is a curve in T^2M such that $y(t), z(t)$ are vector fields along $x(t)$ (i.e., $y(t), z(t) \in T_{x(t)}M$), then we have*

$$\dot{C} = \dot{x}^{(0)} + (\nabla_{\dot{x}} y)^{(1)} + (\nabla_{\dot{x}} z)^{(2)}, \tag{5.2}$$

where $\dot{x} = \frac{dx}{dt}$ and $\dot{C} = \frac{dC}{dt}$.

Proof. Locally, if Y, Z are vector fields such that $Y(x(t)) = y(t)$ and $Z(x(t)) = z(t)$, then from the Lemma 5.2 we obtain

$$\dot{C}(t) = dC(t) = d\sigma(\dot{x}(t)) = \dot{x}^{(0)} + (\nabla_{\dot{x}} y)^{(1)} + (\nabla_{\dot{x}} z)^{(2)},$$

where $\sigma = S^{-1}((Y, Z))$.

□

Subsequently, we denote $x' = \dot{x}$, $x'' = \nabla_{\dot{x}}\dot{x}$, $y' = \nabla_{\dot{x}} y$ and $y'' = \nabla_{\dot{x}}\nabla_{\dot{x}} y$.

Theorem 5.4. *Let $(M_n, g, \varphi_1, \varphi_2)$ be an anti-biparaKaehlerian manifold and (T^2M, g_{BS}) be its second tangent bundle equipped with the Berger type deformed Sasaki metric. If $C(t) = S^{-1}(x(t), y(t), z(t))$ is a curve on T^2M such that $y(t), z(t)$ are vector fields along $x(t)$, then*

$$\begin{aligned} \widetilde{\nabla}_{\dot{C}}\dot{C} &= \big(x'' + R(y, y')x' + (R(z, z')x'\big)^{(0)} + \big(y'' + \frac{\delta^2}{\lambda} g(y', \varphi_1(y'))(\varphi_1(y))\big)^{(1)} \\ &\quad + \big(z'' + \frac{\eta^2}{\beta} g(z', \varphi_2(z'))(\varphi_2(z))\big)^{(2)}. \end{aligned} \tag{5.3}$$

Proof. From the formula (5.2) and the Theorem 3.7, we have

$$\begin{aligned} \widetilde{\nabla}_{\dot{C}}\dot{C} &= \widetilde{\nabla}_{[\dot{x}^{(0)} + (\nabla_{\dot{x}} y)^{(1)} + (\nabla_{\dot{x}} z)^{(2)}]}[\dot{x}^{(0)} + (\nabla_{\dot{x}} y)^{(1)} + (\nabla_{\dot{x}} z)^{(2)}] \\ &= (x'')^{(0)} + (y'')^{(1)} + (R(y, y')x')^{(0)} + \frac{\delta^2}{\lambda} g(y', \varphi_1(y'))(\varphi_1(y))^{(1)} \\ &\quad + (z'')^{(2)} + (R(z, z')x')^{(0)} + \frac{\eta^2}{\beta} g(z', \varphi_2(z'))(\varphi_2(z))^{(2)}. \end{aligned}$$

□

From the theorem above we get the following theorem.

Theorem 5.5. *Let $(M_n, g, \varphi_1, \varphi_2)$ be an anti-biparaKaehlerian manifold and (T^2M, g_{BS}) be its second tangent bundle equipped with the Berger type deformed Sasaki metric. If $C(t) = S^{-1}(x(t), y(t), z(t))$ is a curve on T^2M such that $y(t), z(t)$ are vector fields along $x(t)$, then C is a geodesic if and only if*

$$x'' = -R(y, y')x' - R(z, z')x', \tag{5.4}$$

$$y'' = -\frac{\delta^2}{1+\delta^2\|y\|^2} g(y', \varphi_1(y'))(\varphi_1(y)), \tag{5.5}$$

$$z'' = -\frac{\eta^2}{1+\eta^2\|z\|^2} g(z', \varphi_2(z'))(\varphi_2(z)). \tag{5.6}$$

From the Theorem 5.5, we obtain the following results.

Theorem 5.6. *Let $(M_n, g, \varphi_1, \varphi_2)$ be a locally flat anti-biparaKaehlerian manifold and (T^2M, g_{BS}) be its second tangent bundle equipped with the Berger type deformed Sasaki metric. If $C(t) = S^{-1}(x(t), y(t), z(t))$ is a curve on T^2M such that $y(t), z(t)$ are vector fields along $x(t)$, then $C(t)$ is a geodesic on (T^2M, g_{BS}) if and only if $x(t)$ is a geodesic on $(M_n, g, \varphi_1, \varphi_2)$ and*

$$y'' = -\frac{\delta^2}{1+\delta^2\|y\|^2} g(y', \varphi_1(y'))(\varphi_1(y)),$$

$$z'' = -\frac{\eta^2}{1+\eta^2\|z\|^2} g(z', \varphi_2(z'))(\varphi_2(z)).$$

Corollary 5.7. *Let $(M_n, g, \varphi_1, \varphi_2)$ be an anti-biparaKaehlerian manifold and (T^2M, g_{BS}) be its second tangent bundle equipped with the Berger type deformed Sasaki metric. If $C(t) = S^{-1}(x(t), y(t), z(t))$ is the horizontal lift of the curve $x(t)$ (i.e $y' = z' = 0$), then $C(t)$ is a geodesic on (T^2M, g_{BS}) if and only if $x(t)$ is a geodesic on $(M_n, g, \varphi_1, \varphi_2)$.*

Corollary 5.8. *Let $(M_n, g, \varphi_1, \varphi_2)$ be an anti-biparaKaehlerian manifold and (T^2M, g_{BS}) be its second tangent bundle equipped with the Berger type deformed Sasaki metric. The natural lift $C(t) = S^{-1}(x(t), \dot{x}(t), \dot{x}(t))$ of any geodesic $x(t)$ is a geodesic on (T^2M, g_{BS}).*

Theorem 5.9. *Let $(M_n, g, \varphi_1, \varphi_2)$ be a locally symmetric anti-biparaKaehlerian manifold and (T^2M, g_{BS}) be its second tangent bundle equipped with the Berger type deformed Sasaki metric. If $C(t) = S^{-1}(x(t), y(t), z(t))$ is a curve on T^2M such that $y(t), z(t)$ are vector fields along $x(t)$, then we have*

$$x^{(p+1)} = -[R(y, y') + R(z, z')]x^{(p)}, \tag{5.7}$$

$$|x^{(p)}| = const., \tag{5.8}$$

$$g(x^{(p+1)}, x^{(p)}) = 0 \tag{5.9}$$

for all $p \geq 1$.

Proof. Using the formula (5.4), we have

$$x^{(3)} = -[R(y,y'') + (R(z,z'')]x' - [R(y,y') + (R(z,z')]x^{(2)}, \tag{5.10}$$

by substituting (5.5) and (5.6) in (5.10), we obtain

$$\begin{aligned} x^{(3)} &= -[R(y,y') + R(z,z')]x^{(2)} + \big[\frac{\delta^2}{1+\delta^2\|y\|^2} g(y',\varphi_1(y'))R(y,\varphi_1(y)) \\ &\quad + \frac{\eta^2}{1+\eta^2\|z\|^2} g(z',\varphi_2(z'))R(z,\varphi_2(z))\big]x', \end{aligned}$$

thus, from the Proposition 2.3, we obtain

$$x^{(3)} = -[R(y,y') + R(z,z')]x^{(2)}.$$

By induction on p, the formula (5.7) is obtained. On the other hand, we have

$$\nabla_{\dot{x}} g(x^{(p)}, x^{(p)}) = 2g(x^{(p+1)}, x^{(p)}) = -2g([R(y,y') + R(z,z')]x^{(p)}, x^P) = 0,$$

which completes the proof. □

Theorem 5.10. *Let* $(M_n, g, \varphi_1, \varphi_2)$ *be a locally symmetric anti-biparaKaehlerian manifold and* (T^2M, g_{BS}) *be its second tangent bundle equipped with the Berger type deformed Sasaki metric. If* $C(t) = S^{-1}(x(t), \dot{x}(t), \dot{x}(t))$ *is a natural lift of the curve* $\dot{x}(t)$ *on* T^2M*, then all geodesic curvatures of* $\gamma = x(t)$ *are constants.*

Proof. Using the Proposition 3.6, and the formulas (5.2) and (5.9), we obtain

$$\begin{aligned} \|\dot{C}\|^2 &= |\frac{dx}{dt}|^2 + 2|x''|^2 + \delta^2 g(x'', \varphi_1(x'))^2 + \eta^2 g(x'', \varphi_2(x'))^2 \\ &= |x'|^2 + 2|x''|^2 = K^2 = const. \end{aligned}$$

Denote by s an arc length parameter on $x(t)$ and $|x''| = \rho = const.$ Then $x'_t = \frac{dx}{dt} = x'_s \frac{ds}{dt}$ and

$$K^2 = \|\dot{C}\|^2 = |\frac{dx}{dt}|^2 + 2|x''|^2 = |\frac{ds}{dt}|^2 + 2|x''|^2 = |\frac{ds}{dt}|^2 + 2\rho^2.$$

Hence

$$|\frac{d\,s}{dt}| = \sqrt{K^2 - 2\rho^2} = \beta = const., \tag{5.11}$$

where $\beta^2 = K^2 - 2\rho^2$.

Denote by $\nu_1, ..., \nu_{2n-1}$ the Frenet frame along γ and by $k_1, ..., k_{2n-1}$ the geodesic curvatures of γ. From (5.11), we obtain

$$\begin{aligned} x' &= \beta\nu_1 \\ x'' &= \beta^2 k_1 \nu_2 \\ x^{(3)} &= \beta^3 k_1(-k_1\nu_1 + k_2\nu_3) \\ &\vdots \end{aligned}$$

Using the formula (5.8) we deduce $k_1 = const.$, $k_2 = const.$, ..., $k_{2n-1} = const.$, which completes the proof. □

6. Harmonicity

Consider a smooth map $\phi : (M^m, g) \to (N^n, h)$ between two Riemannian manifolds, then the second fundamental form of ϕ is defined by

$$B_\phi(X,Y) = (\nabla d\phi)(X,Y) = \nabla^\phi_X d\phi(Y) - d\phi(\nabla_X Y). \tag{6.1}$$

Here ∇ is the Riemannian connection on M^m and ∇^ϕ is the pull-back connection on the pull-back bundle $\phi^{-1}TN$, and

$$\tau(\phi) = trace_g \nabla d\phi = trace_g B_\phi \tag{6.2}$$

is the tension field of ϕ. A map ϕ is called to be harmonic if and only if $\tau(\phi) = 0$.

If $\psi : (N^n, g) \to (\overline{N}^n, \overline{h})$ is a smooth map between two Riemannian manifolds, then we have

$$\tau(\psi \circ \phi) = d\,\psi(\tau(\phi)) + trace_g \nabla d\,\psi(d\phi, d\phi). \tag{6.3}$$

One can refer to [4], [5], [7], [10] for background on harmonic maps.

6.1. Harmonicity conditions of inclusion

Theorem 6.1. *Let $(M_n, g, \varphi_1, \varphi_2)$ be an anti-biparaKaehlerian manifold and (T^2M, g_{BS}) be its second tangent bundle equipped with the Berger type deformed Sasaki metric. If g_S denotes the Sasaki metric on TM, then the tension field of the inclusion*

$$\begin{aligned} I_2 : (TM, g_S) &\to (T^2M, g_{BS}) \\ (x,u) &\mapsto S^{-1}((x,u,u)) \end{aligned}$$

is given by

$$\begin{aligned} \tau(I_2)_{(x,u)} &= \frac{\delta^2}{1+\delta^2\|u\|^2} trace_g g(*, \varphi_1(*))\big(\varphi_1(u)\big)^{(1)} \\ &\quad + \frac{\eta^2}{1+\eta^2\|u\|^2} trace_g g(*, \varphi_2(*))\big(\varphi_2(u)\big)^{(2)}. \end{aligned} \tag{6.4}$$

Proof. Let X be a vector field on M_n, then we have

$$\begin{aligned} dI_2(X^H) &= dS^{-1}(X^H, X^H) = X^{(0)}, \\ dI_2(X^V) &= dS^{-1}(X^V, X^V) = X^{(1)} + X^{(2)}. \end{aligned}$$

Let $x \in M$, $\{e_i\}_{i=1}^n$ be a local orthonormal frame on M_n and $\overline{\nabla}$ be the Levi-Civita connection of the Sasaki metric g_S. We have

$$\begin{aligned} B_{I_2}(e_i^H, e_i^H) &= \widetilde{\nabla}_{dI_2(e_i^H)} dI_2(e_i^H) - dI_2\big(\overline{\nabla}_{e_i^H} e_i^H\big) = \widetilde{\nabla}_{e_i^0} e_i^0 - \big(\nabla_{e_i} e_i\big)^0 = 0, \\ B_{I_2}(e_i^V, e_i^V) &= \widetilde{\nabla}_{dI_2(e_i^V)} dI_2(e_i^V) - dI_2\big(\overline{\nabla}_{e_i^V} e_i^V\big) \\ &= \widetilde{\nabla}_{e_i^1 + e_i^2}(e_i^1 + e_i^2) = \widetilde{\nabla}_{e_i^1}(e_i^1) + \widetilde{\nabla}_{e_i^2}(e_i^2) \\ &= \frac{\delta^2}{1+\delta^2\|u\|^2} g(e_i, \varphi_1(e_i))\big(\varphi_1(u)\big)^{(1)} \\ &\quad + \frac{\eta^2}{1+\eta^2\|u\|^2} g(e_i, \varphi_2(e_i))\big(\varphi_2(u)\big)^{(2)}. \end{aligned}$$

□

From Theorem 6.1, we have the following corollary.

Corollary 6.2. *Let $(M_n, g, \varphi_1, \varphi_2)$ be an anti-biparaKaehlerian manifold and (T^2M, g_{BS}) be its second tangent bundle equipped with the Berger type deformed Sasaki metric. If g_S denotes the Sasaki metric on TM, the inclusion $I_2 : (TM, g_S) \to (T^2M, g_{BS})$ is a harmonic map if and only if*

$$trace_g g(*, \varphi_1(*)) = trace_g g(*, \varphi_2(*)) = 0.$$

Let (M_n, h, φ) be an anti-biparaKaehlerian manifold and (TM, h_{BS}) be its tangent bundle with the Berger type deformed Sasaki metric $h_{BS} = g_{\varphi,\rho}$ and $\varepsilon = 2$ (see [2]). By a standard calculation we get the following result.

Theorem 6.3. *Let $(M_n, g, \varphi_1, \varphi_2)$ be an anti-biparaKaehlerian manifold and (T^2M, g_{BS}) be its second tangent bundle equipped with the Berger type deformed Sasaki metric. Then the tension field of the inclusion $I_2 : (TM, h_{BS}) \to (T^2M, g_{BS})$ is given by*

$$\begin{aligned} \tau(I_2)_{(x,u)} &= \Big[\overline{h}^{ij}\big(\frac{\delta^2}{1+\delta^2\|u\|_g^2} g(E_i, \varphi_1(E_j))\varphi_1 u - \frac{\rho^2}{1+\rho^2\|u\|_h^2} h(E_i, \varphi(E_j))\varphi u\big)\Big]^{(1)} \\ &+ \Big[\overline{h}^{ij}\big(\frac{\eta^2}{1+\eta^2\|u\|_g^2} g(E_i, \varphi_2(E_j))\varphi_2 u - \frac{\rho^2}{1+\rho^2\|u\|_h^2} h(E_i, \varphi(E_j))\varphi u\big)\Big]^{(2)}, \end{aligned}$$

where $\{E_i\}_{i=1}^n$ *is a local orthonormal frame on* M_n *and* $h_{ij} = h_{BS}(E_i^V, E_j^V) = \delta_{ij} + \rho^2\varphi(u)_i\varphi(u)_j$.

From Theorem 6.3, we obtain the following corollary.

Corollary 6.4. *The inclusion* $I_2 : (TM, h_{BS}) \to (T^2M, g_{BS})$ *is a harmonic map if and only if*

$$\begin{aligned}\overline{h}^{ij}\frac{\delta^2}{1+\delta^2\|u\|_g^2}g(E_i,\varphi_1(E_j))\varphi_1 u &= \overline{h}^{ij}\frac{\rho^2}{1+\rho^2\|u\|_h^2}h(E_i,\varphi(E_j))\varphi u,\\ \overline{h}^{ij}\frac{\eta^2}{1+\eta^2\|u\|_g^2}g(E_i,\varphi_2(E_j))\varphi_2 u &= \overline{h}^{ij}\frac{\rho^2}{1+\rho^2\|u\|_h^2}h(E_i,\varphi(E_j))\varphi u.\end{aligned}$$

6.2. Harmonicity conditions of projections

Let $(E_1, ..., E_n)$ be orthonormal vector fields on M_n. The matrix of Berger type deformed Sasaki metric on T^2M with respect to $(E_1^{(0)}, ..., E_n^{(0)}, E_1^{(1)}, ..., E_n^{(1)}, E_1^{(2)}, ..., E_n^{(2)})$ is as follows

$$g_{BS} = \begin{pmatrix} \delta_{ij} & 0 & 0\\ 0 & aij & 0\\ 0 & 0 & b_{ij}\end{pmatrix}, \tag{6.5}$$

$$g_{BS}^{-1} = \begin{pmatrix} \delta^{ij} & 0 & 0\\ 0 & a^{ij} & 0\\ 0 & 0 & b^{ij}\end{pmatrix}, \tag{6.6}$$

where $a = \left(\delta_{ij} + \delta^2(\varphi_1 u)^i(\varphi_1 u)^j\right)_{i,j\le n}$ and $b = \left(\delta_{ij} + \delta^2(\varphi_{21} w)_i(\varphi_2 w)_j\right)_{i,j\le n}$.

Lemma 6.5. *Let* $(M_n, g, \varphi_1, \varphi_2)$ *be an anti-biparaKaehlerian manifold and* (T^2M, g_{BS}) *be its second tangent bundle equipped with the Berger type deformed Sasaki metric. If* $\pi : (T^2M, g_{BS}) \to (M_n, g)$ *denotes the canonical projection, then we have*

$$\begin{aligned}B_\pi(E_i^0, E_j^0)_p &= B_\pi(E_j^1, E_i^1) = B_\pi(E_j^2, E_i^2) = 0,\\ B_\pi(E_i^0, E_j^1)_p &= -\frac{1}{2}R_x(u, E_j)E_i,\\ B_\pi(E_i^0, E_j^2)_p &= -\frac{1}{2}R_x(w, E_j)E_i,\\ B_\pi(E_i^1, E_j^2)_p &= 0.\end{aligned}$$

Theorem 6.6. *Let $(M_n, g, \varphi_1, \varphi_2)$ be an anti-biparaKaehlerian manifold and (T^2M, g_{BS}) be its second tangent bundle equipped with the Berger type deformed Sasaki metric. The canonical projection $\pi : (T^2M, g_{BS}) \to (M_n, g, \varphi_1, \varphi_2)$ is totally geodesic if and only if ∇ is locally flat. Moreover π is a harmonic map.*

Lemma 6.7. *Let $(M_n, g, \varphi_1, \varphi_2)$ be an anti-biparaKaehlerian manifold and (T^2M, g_{BS}) be its second tangent bundle equipped with the Berger type deformed Sasaki metric. If $\pi : (T^2M, g_{BS}) \to (TM, g_S)$ denotes the canonical projection, then we have*

$$\begin{aligned}
\pi_*(X^0) &= X^H, \quad \pi_*(X^1) = X^V, \quad \pi_*(X^2) = 0. \\
B_\pi(E_i^0, E_j^0)_p &= B_\pi(E_j^2, E_i^2) = B_\pi(E_i^0, E_j^1)_p = 0, \\
B_\pi(E_i^1, E_j^1)_p &= -\frac{\delta^2}{1+\delta^2\|u\|_g^2} g(E_i, \varphi_1 E_j)(\varphi_1 u)^V, \\
B_\pi(E_i^0, E_j^2)_p &= -\frac{1}{2}(R_x(w, E_j)E_i)^H,
\end{aligned}$$

where $(E_1, ..., E_n)$ is a local orthonormal frame on M_n.

Theorem 6.8. *Let $(M_n, g, \varphi_1, \varphi_2)$ be an anti-biparaKaehlerian manifold and (T^2M, g_{BS}) be its second tangent bundle equipped with the Berger type deformed Sasaki metric. The canonical projection $\pi : (T^2M, g_{BS}) \to (TM, g_S)$ is a harmonic map if and only if*

$$a^{ij} g(E_i, \varphi_1 E_j) = 0.$$

Theorem 6.9. *Let $(M_n, g, \varphi_1, \varphi_2)$ be an anti-biparaKaehlerian manifold and (T^2M, g_{BS}) be its second tangent bundle equipped with the Berger type deformed Sasaki metric. The canonical projection $\pi : (T^2M, g_{BS}) \to (TM, h_{BS})$ is a harmonic map if and only if*

$$\frac{\delta^2}{1+\delta^2\|u\|_g^2} a^{ij} g(E_i, \varphi_1 E_j)\varphi_1 u = \frac{\rho^2}{1+\rho^2\|u\|_h^2} a^{ij} h(E_i, \varphi_1 E_j)\varphi\, u.$$

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