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Some Geometric Constants Related to the Heights and Midlines of Triangles in Banach Spaces

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Abstract. In this paper, we introduce two new geometric constants related to the heights of triangles: $\Delta_H(X)$ and $\Delta_h(X, I)$. We also propose two new geometric constants, $\Delta_m(X)$ and $\Delta_M(X)$, related to the midlines of equilateral triangles, and discuss the relation between the heights and midlines in equilateral triangles. We give estimates for these geometric constants in terms of other geometric parameters, and the geometric constants are used to discuss geometric properties such as uniform non-squareness, uniform normal structure, and the fixed point property.

1. Introduction

Geometric constants has been widely studied, as they make it easier to deal with certain problems in Banach space. They do this because they not only essentially reflect the geometric properties of a space X, but also enables us to study the geometric properties of Banach spaces quantitatively. For example, the modulus of convexity introduced by Clarkson [7] can be used to characterize uniformly convex spaces, the modulus of smoothness proposed by Day [9] can be used to characterize uniformly smooth spaces. We refer the readers to the papers [2–4, 13–15, 22, 28] about the modulus of convexity and the modulus of smoothness.

It is well known that the height and midline of a triangle play an important role in Euclidean geometry, as does the geometric theory of Banach spaces. In 1997, Alsina et al. obtained a collection of new characterizations of inner product spaces by using well-known formulas about the height of a triangle in [1]. In 2009, Ni et al. [27] proposed a new geometric constant $h(X)$ related to the heights of equilateral triangles, which can be used to characterize the uniformly non-square spaces. In Functional Analysis, various fine geometric properties of finite dimensional spaces (i.e. Minkowski spaces) play important

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roles in the so-called local theory of Banach spaces (see [24, 30]). In 2001, Martini et al. [23] obtained the upper bounds of medians for equilateral triangles in any Minkowski planes (i.e. two-dimensional Minkowski spaces).

Inspired by the excellent works mentioned above, in this paper, we will introduce some geometric constants related to the heights and midlines of triangles in non-trivial Banach spaces, and use them to study some properties of Banach spaces. This paper is organized in the following way:

In Section 2, we recall some fundamental concepts and conclusions that we need to use in subsequent discussions.

In Section 3, we consider the following three constants:

$$\Delta_h(X) = \inf\left\{\inf_{\lambda\in\mathbb{R}} \|\lambda x + (1-\lambda)y\| : \|x\| = \|y\| = 1, \|x-y\| = 1\right\},$$

$$\Delta_H(X) = \sup\left\{\inf_{\lambda\in\mathbb{R}} \|\lambda x + (1-\lambda)y\| : \|x\| = \|y\| = 1, \|x-y\| = 1\right\},$$

$$\Delta_h(X, I) = \inf\left\{\inf_{\lambda\in\mathbb{R}} \|\lambda x + (1-\lambda)y\| : \|x\| = \|y\| = 1, x \perp_I y\right\}.$$

The first one is the constant $h(X)$ proposed by Ni et al. [27], and the latter two are proposed by us. These three constants are closely related to the height of the triangle. The two constants $\Delta_h(X)$ and $\Delta_H(X)$ are related to the heights of equilateral triangles, and the constant $\Delta_h(X, I)$ is related to the heights on the hypotenuses of right triangles. The bounds of these constants and the values of these constants for some specific spaces will be given. Further, some estimates of these constants in terms of other constants will also be discussed. In particular, the relationships between these constants and some geometric properties of Banach spaces will be studied, including uniform non-squareness, uniform convexity, strict convexity and uniform normal structure.

In Section 4, we introduce the following third constants:

$$\Delta_m(X) = \inf\left\{\frac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x-y\| = 1\right\},$$

$$\Delta_M(X) = \sup\left\{\frac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x-y\| = 1\right\},$$

$$D_\Delta(X) = \sup\left\{\frac{\|x+y\|}{\inf_{\lambda\in[0,1]} \|\lambda x + (1-\lambda)y\|} : \|x\| = \|y\| = \|x-y\| = 1\right\}.$$

The first two constants are both related to the midlines of equilateral triangles. The bounds of these two constants and the values of these constants for some specific spaces will be given. Further, a necessary and sufficient condition and a sufficient condition for uniformly non-square space will be established. Finally, we introduce a new constant $D_\Delta(X)$ related to the difference between the midilines and heights in equilateral triangles.

2. Preliminaries

Throught the paper, let X be a non-trivial Banach space, that is, $dimX \geq 2$, and using S_X and B_X to represent the unit sphere and closed unit ball of X, respectively.

Recall that the Banach space X is called uniformly non-square [17], if there exists a constant $\delta \in (0, 1)$ such that for any $x, y \in S_X$, either $\frac{\|x+y\|}{2} \leq 1-\delta$ or $\frac{\|x-y\|}{2} \leq 1-\delta$.

The constants

$$J(X)=\sup\{\min\{\|x+y\|,\|x-y\|\}:x,y\in S_X\}$$
$$S(X)=\inf\{\max\{\|x+y\|,\|x-y\|\}:x,y\in S_X\}$$

are defined by Gao [11] in order to measure the degree of uniform non-squareness, and usually called the James constant and Schäffer constant, respectively.

Later, the equivalent definitions of James constant and Schäffer constant

$$J(X)=\sup\{\|x+y\|:x,y\in S_X,\|x+y\|=\|x-y\|\}$$
$$S(X)=\inf\{\|x+y\|:x,y\in S_X,\|x+y\|=\|x-y\|\}$$

are introduced by He and Cui [16].

Now let us collect some properties of the two constants for non-trivial Banach spaces (see [16, 20, 21]):

(1) $1\le S(X)\le\sqrt{2}\le J(X)\le 2$.

(2) $J(X)S(X)=2$.

(3) Let X be a Banach space with $dimX\ge 3$. Then $J(X)=\sqrt{2}$ if and only if X is a Hilbert space.

(4) X is uniformly non-square if and only if either $J(X)<2$ or $S(X)>1$.

Recall that the Banach space X is called uniformly convex [7], if, for any $\epsilon>0$, there exists $\delta>0$, such that for any $x,y\in S_X$ with $\|x-y\|>\epsilon$, then $\|\frac{x+y}{2}\|<1-\delta$.

The Clarkson modulus of convexity of a Banach space X

$$\delta_X(\epsilon)=\inf\left\{1-\frac{\|x+y\|}{2}:x,y\in S_X,\|x-y\|=\epsilon\right\},(0\le\epsilon\le 2)$$

was proposed in [7] and can be used to characterize uniformly convex space.

Some famous conclusions about $\delta_X(\epsilon)$ are listed in [13, 22, 28]:

(1) For all Banach spaces X, then

$$\delta_X(\epsilon)\le 1-\sqrt{1-\frac{\epsilon^2}{4}}.$$

(2) For any $x,y\in X$ such that $\|x\|^2+\|y\|^2=2$, then

$$\|x+y\|^2\le 4-4\delta_X\left(\frac{\|x-y\|}{2}\right).$$

(3) X is uniformly non-square if and only if there exists $\epsilon\in(0,2)$ such that $\delta_X(\epsilon)>0$.

In order to get a better understanding of some geometric properties of Banach spaces, in [14] Gurariy introduced the β_X modulus of as the function

$$\beta_X(\epsilon)=\inf\left\{1-\inf_{a\in[0,1]}\|ax+(1-a)y\|:x,y\in S_X,\|x-y\|=\epsilon\right\},(0\le\epsilon\le 2)$$

and the various properties of this constant were given in [3, 15]:

(1) For any $\epsilon\in[0,2]$, $\delta_X(\epsilon)\le\beta_X(\epsilon)\le 2\delta_X(\epsilon)$.

(2) Let X be a Hilbert space, then for each $\epsilon\in[0,2]$,

$$\beta_X(\epsilon)=1-\left(1-\frac{\epsilon^2}{4}\right)^{\frac{1}{2}}.$$

(3) Banach space X is uniformly convex if and only if $\beta_X(\epsilon) > 0$, for any $\epsilon \in [0, 2]$.

Recall that Banach space X is said to be uniformly smooth [9], whenever given $0 < \epsilon \leq 2$, there exists $\delta > 0$ such that if $x \in S_X$ and $\|y\| \leq \delta$, then

$$\|x + y\| + \|x - y\| < 2 + \epsilon\|y\|.$$

In order to study uniformly smooth space, the modulus of smoothness $\rho_X(\tau) : [0, +\infty) \to [0, +\infty)$ was introduced by Day [9] as follows:

$$\rho_X(\tau) = \sup\left\{\frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : x, y \in S_X\right\}, (\tau \geq 0).$$

The following function $\rho(\epsilon) : [0, 2] \to [0, 1]$, which we call it the modulus of smoothness and can be used to characterize uniformly smooth space, was introduced by Banaś [2] as follows:

$$\rho(\epsilon) = \sup\left\{1 - \frac{\|x + y\|}{2} : x, y \in S_X, \|x - y\| = \epsilon\right\}, (0 \leq \epsilon \leq 2).$$

Later, Baronti and Papini [4] considered the following related modulus:

$$\rho'(\epsilon) = \sup\left\{1 - \|\lambda x + (1 - \lambda)y\| : x, y \in S_X, \|x - y\| = \epsilon, \lambda \in [0, 1]\right\}, (0 \leq \epsilon \leq 2).$$

Some important conclusions about $\rho(\epsilon)$ and $\rho'(\epsilon)$ are listed as follows [4]:

(1) For all Banach space X, then

$$\rho(\epsilon) \geq 1 - \sqrt{1 - \frac{\epsilon^2}{4}}.$$

(2) X is not uniformly non-square if and only if $\rho'(1) = \frac{1}{2}$.

(3) X is not uniformly non-square if and only if $\rho(1) = \frac{1}{2}$.

3. The Heights of Triangles

In this section, we will study two types of constants. The first type is closely related to the heights of equilateral triangles, and the second type is related to the heights on the hypotenuses of right triangles.

3.1. The heights of equilateral triangles

Ni et al. [27] introduced the following constant

$$\Delta_h(X) = \inf\left\{\inf_{\lambda \in \mathbb{R}} \|\lambda x + (1 - \lambda y)\| \;:\; \|x\| = \|y\| = 1,\; \|x - y\| = 1\right\}.$$

Inspired by $\Delta_h(X)$, we consider the following constant

$$\Delta_H(X) = \sup\left\{\inf_{\lambda \in \mathbb{R}} \|\lambda x + (1 - \lambda)y\| \;:\; \|x\| = \|y\| = 1,\; \|x - y\| = 1\right\}.$$

Next, we give the equivalent forms of $\Delta_h(X)$ and $\Delta_H(X)$, which will help us understand their geometric meanings.

Proposition 3.1. *Let X be a Banach space. Then*

(1) $\Delta_h(X) = \inf\left\{\inf_{\lambda \in [0,1]} \|\lambda x + (1 - \lambda)y\| \;:\; \|x\| = \|y\| = 1,\; \|x - y\| = 1\right\}$,

(2) $\Delta_H(X) = \sup\left\{ \inf_{\lambda\in[0,1]} \|\lambda x + (1-\lambda)y\| \ : \ \|x\| = \|y\| = 1, \ \|x-y\| = 1 \right\}.$

Proof. (1) First, for any $x, y \in S_X$ and $\lambda \in [0,1]$, we can get

$$\|\lambda x + (1-\lambda)y\| \le \lambda\|x\| + (1-\lambda)\|y\| = 1.$$

Second, for any $x, y \in S_X$ and $\lambda \in [1, +\infty]$, we can obtain

$$\|\lambda x + (1-\lambda)y\| \ge \lambda\|x\| - (\lambda-1)\|y\| = 1.$$

Further, for any $x, y \in S_X$ and $\lambda \in [-\infty, 0]$, we can also obtain

$$\|\lambda x + (1-\lambda)y\| \ge ||\lambda| - |1-\lambda|| = 1.$$

Thus, we can conclude

$$\begin{aligned}\Delta_h(X) &= \inf\left\{ \inf_{\lambda\in\mathbb{R}} \|\lambda x + (1-\lambda)y\| \ : \ \|x\| = \|y\| = 1, \ \|x-y\| = 1 \right\} \\ &= \inf\left\{ \inf_{\lambda\in[0,1]} \|\lambda x + (1-\lambda)y\| \ : \ \|x\| = \|y\| = 1, \ \|x-y\| = 1 \right\}.\end{aligned}$$

(2) The proof is similar to (1), and we omit it. □

The geometric meanings of $\Delta_h(X)$ and $\Delta_H(X)$: consider the Euclidean plane with $\overrightarrow{OA} = x$, $\overrightarrow{OB} = y$, then we have $\overrightarrow{BA} = x - y$. Assume that $\overrightarrow{OC} \perp \overrightarrow{AB}$, the geometric explanations of $\Delta_h(X)$ and $\Delta_H(X)$ are the infimum and supremun of the heights on the side $\overrightarrow{AB}$ on the equilateral triangles ΔOAB, respectively.

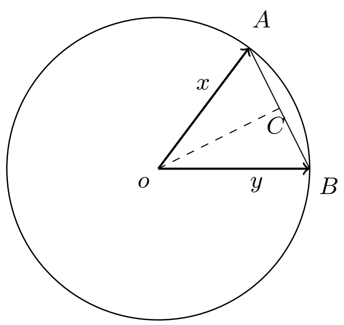

Figure 1. Geometric explanations of $\Delta_h(X)$ and $\Delta_H(X)$.

3.1.1. The bounds of $\Delta_h(X)$ and $\Delta_H(X)$

First, we give the bounds of $\Delta_h(X)$ and $\Delta_H(X)$.

Proposition 3.2. *Let X be a Banach space. Then*

(1) $\frac{1}{2} \le \Delta_h(X) \le \frac{\sqrt{3}}{2}$,

(2) $\sqrt{3} - 1 \le \Delta_H(X) \le 1$.

Proof. (1) In [27], Ni et al. have obtained $\Delta_h(X) \ge \frac{1}{2}$. Moreover, it is clear that

$$\Delta_h(X) \le \inf\left\{ \frac{\|x+y\|}{2} : \|x\| = \|y\| = \|x-y\| = 1 \right\} = 1 - \rho(1).$$

Then, we can obtain

$$\Delta_h(X) \leq 1 - \rho(1) \leq 1 - \left(1 - \sqrt{1 - \frac{1^2}{4}}\right) = \frac{\sqrt{3}}{2},$$

which completes the proof.

(2) From the definition of $\Delta_H(X)$ and $\beta_X(\epsilon)$, we can obtain $\Delta_H(X) = 1 - \beta_X(1)$. Since $\beta_X(\epsilon) \leq 2\delta_X(\epsilon)$ for any $\epsilon \in [0, 2]$, we see that

$$\Delta_H(X) = 1 - \beta_X(1) \geq 1 - 2\delta_X(1) \geq 1 - 2\left(1 - \sqrt{1 - \frac{1^2}{4}}\right) = \sqrt{3} - 1.$$

Note that

$$\|\lambda x + (1-\lambda)y\| \leq \lambda\|x\| + (1-\lambda)\|y\| = \lambda + (1-\lambda) = 1,$$

for each $\lambda \in [0, 1]$ and $\|x\| = \|y\| = \|x - y\| = 1$, which means that $\Delta_H(X) \leq 1$. This completes the proof. □

Next, we give the values of $\Delta_h(X)$ and $\Delta_H(X)$ of some specific spaces. Example 3.3 and Example 3.4 illustrate that the lower bound of $\Delta_h(X)$ can be reached. Example 3.5 shows that the two constants are in general different and the upper bound of $\Delta_H(X)$ can be attained. We can also obtain the exact values of $\Delta_h(X)$ and $\Delta_H(X)$ for Hilbert spaces by Proposition 3.6.

Example 3.3. Let $X = (\mathbb{R}^2, \|\cdot\|_1)$. Then $\Delta_h(X) = \frac{1}{2}$.

Proof. Let $x = (\frac{1}{2}, \frac{1}{2})$ and $y = (-\frac{1}{2}, \frac{1}{2})$. It is easy for us to obtain $\|x\|_1 = \|y\|_1 = \|x-y\|_1 = 1$. Then we can get

$$\frac{1}{2} \leq \Delta_h(X) \leq \frac{\|x+y\|_1}{2} = \frac{1}{2}.$$

Thus, we can obtain $\Delta_h(X) = \frac{1}{2}$. □

Example 3.4. Let $X = (\mathbb{R}^2, \|\cdot\|_\infty)$. Then $\Delta_h(X) = \frac{1}{2}$.

Proof. Let $x = (0, 1)$ and $y = (1, 0)$. It is clear that $\|x\|_\infty = \|y\|_\infty = \|x - y\|_\infty = 1$. Thus, we can obtain

$$\frac{1}{2} \leq \Delta_h(X) \leq \frac{\|x+y\|_\infty}{2} = \frac{1}{2},$$

which means that $\Delta_h(X) = \frac{1}{2}$. □

Example 3.5. Let X be the space $\mathbb{R}^2$ endowed with the norm

$$\|x\| = \|(x_1, x_2)\| = \begin{cases} \|(x_1, x_2)\|_\infty, & (x_1x_2 \geq 0), \\ \|(x_1, x_2)\|_1, & (x_1x_2 \leq 0). \end{cases}$$

Then $\Delta_h(X) = \frac{3}{4}$ and $\Delta_H(X) = 1$.

Proof. From Theorem 4 in [4], we can get $\rho(1) = \rho'(1)$. Then by Example 1 in [4], we can obtain $\rho'(1) = \frac{1}{4}$. Thus $\Delta_h(X) = 1 - \rho'(1) = \frac{3}{4}$.

Moreover, from [3], $\beta_X(\epsilon) = \max\left\{0, 1 - \frac{1}{\epsilon}\right\}$. So, we have $\beta_X(1) = 0$. Thus, we can obtain $\Delta_H(X) = 1 - \beta_X(1) = 1$. □

Proposition 3.6. *Let X be a Hilbert space. Then $\Delta_h(X) = \Delta_H(X) = \frac{\sqrt{3}}{2}$.*

Proof. Let $x, y \in S_X$ such that $\|x-y\| = 1$. Since X is a Hilbert space, then

$$1 = \|x-y\|^2 = \|x\|^2 - 2\langle x,\, y\rangle + \|y\|^2 = 2 - 2\langle x,\, y\rangle,$$

which means that $\langle x,\, y\rangle = \frac{1}{2}$. Then for $\lambda \in [0,1]$, we have

$$\begin{aligned}\|\lambda x + (1-\lambda)y\|^2 &= \lambda^2\|x\|^2 + 2\lambda(1-\lambda)\langle x,\, y\rangle + (1-\lambda)^2\|y\|^2\\ &= \lambda^2 + \lambda(1-\lambda) + (1-\lambda)^2 = \lambda^2 - \lambda + 1\\ &\geq \inf_{\lambda\in[0,1]}\{\lambda^2 - \lambda + 1\} = \frac{3}{4}.\end{aligned}$$

Then, we can conclude that $\Delta_h(X) = \frac{\sqrt{3}}{2}$ from Proposition 3.2 and the above inequality.

On the other hand, since $\Delta_H(X) = 1 - \beta_X(1)$ and X is a Hilbert space, then we can obtain

$$\Delta_H(X) = 1 - \beta_X(1) = 1 - \left[1 - \left(1 - \frac{1^2}{4}\right)^{\frac{1}{2}}\right] = \frac{\sqrt{3}}{2}.$$

This completes the proof. □

3.1.2. The estimates for $\Delta_H(X)$ by other geometric constants

In this section, we will give some estimates for $\Delta_H(X)$ in terms of other geometric constants. Firstly, we give the relation between $\Delta_H(X)$ and the modulus of convexity $\delta_X(\varepsilon)$, the modulus of smoothness $\rho_X(\tau)$.

Proposition 3.7. *Let X be a finite-dimensional Banach space and $\delta = 2\delta_X(1)$. Then*

$$\Delta_H(X) \geq 1 - \delta + \delta^2.$$

Proof. Since X is a finite-dimensional Banach space, we can choose $\|x\| = \|y\| = \|x-y\| = 1$ and $\|x+y\| = 2-\delta$. Notice that if $\|z_1\| = \|z_2\| = \|z_1 - z_2\| = 1$, then $\|z_1 + z_2\| \leq 2-\delta$ and, in particular, $\|2x - y\| \leq 2 - \delta$.

Now for $\lambda \in [\frac{1}{2}, 1]$, we have

$$\begin{aligned}\|\lambda x + (1-\lambda)y\| &= \|\lambda(x+y) + (1-2\lambda)y\|\\ &\geq \lambda\|x+y\| - (2\lambda - 1)\|y\| = \lambda(2-\delta) - 2\lambda + 1 = 1 - \lambda\delta,\end{aligned}$$

and also

$$\begin{aligned}\|\lambda x + (1-\lambda)y\| &= \|(2-\lambda)x - (1-\lambda)(2x-y)\|\\ &\geq (2-\lambda)\|x\| - (1-\lambda)\|2x-y\|\\ &\geq (2-\lambda) - (1-\lambda)(2-\delta) = \lambda(1-\delta) + \delta.\end{aligned}$$

Thus, we can obtain

$$\|\lambda x + (1-\lambda)y\| \geq \max_{\lambda\in[\frac{1}{2},1]}\{1 - \lambda\delta, \lambda(1-\delta) + \delta\},$$

which implies $\|\lambda x + (1-\lambda)y\| \geq 1 - \delta + \delta^2$. The same result is also valid for $\lambda \in [0, \frac{1}{2}]$. Thus, we can get

$$\begin{aligned}\Delta_H(X) &= \sup\left\{\inf_{\lambda\in[0,1]}\{\|\lambda x + (1-\lambda)y\|\} \ :\ \|x\| = \|y\| = 1,\ \|x-y\| = 1\right\}\\ &\geq 1 - \delta + \delta^2.\end{aligned}$$

This completes the proof. □

Proposition 3.8. *Let X be a Banach space. Then $\Delta_H(X)^2 \leq 1 - \delta_X(\frac{1}{2})$.*

Proof. Let $\lambda = \frac{1}{2}$, we have

$$\Delta_H(X) \leq \sup\left\{\frac{\|x+y\|}{2} : \|x\| = \|y\| = \|x-y\| = 1\right\}.$$

Now, let $x, y \in S_X$ such that $\|x-y\| = 1$. Then we can obtain $\|x\|^2 + \|y\|^2 = 2$. Thus, we can get

$$\|x+y\|^2 \leq 4 - 4\delta_X\left(\frac{1}{2}\right),$$

which deduces the desired inequality. □

Proposition 3.9. *Let X be a Banach space. Then $\Delta_H(X) \leq \rho_X(1) + \frac{1}{2}$.*

Proof. Let $\lambda = \frac{1}{2}$, then $\|\lambda x + (1-\lambda)y\| = \frac{\|x+y\|}{2}$. Thus, we can obtain

$$\Delta_H(X) \leq \sup\left\{\frac{\|x+y\|}{2} : \|x\| = \|y\| = \|x-y\| = 1\right\}.$$

Then, we can get

$$\begin{aligned}\Delta_H(X) &\leq \frac{1}{2}\sup\{\|x+y\| + \|x-y\| : \|x\| = \|y\| = \|x-y\| = 1\} - \frac{1}{2}\\ &\leq \frac{1}{2}\sup\{\|x+y\| + \|x-y\| : x, y \in S_X\} - \frac{1}{2}\\ &\leq \rho_X(1) + 1 - \frac{1}{2} = \rho_X(1) + \frac{1}{2},\end{aligned}$$

which completes the proof. □

In order to study the relationship between the constant $\Delta_H(X)$ and the the von Neumann-Jordan constant $C_{NJ}(X)$, we give the definition of the constant $C_{NJ}(X)$ as follows (see [8]):

$$C_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X, (x, y) \neq (0, 0)\right\}.$$

Proposition 3.10. *Let X be a Banach space. Then $\Delta_H(X) \leq \frac{1}{2}\sqrt{4C_{NJ}(X) - 1}$.*

Proof. Let $\lambda = \frac{1}{2}$, we can get

$$\Delta_H(X) \leq \sup\left\{\frac{\|x+y\|}{2} : \|x\| = \|y\| = \|x-y\| = 1\right\}.$$

For any $x, y \in X$, we have

$$\|x+y\|^2 + \|x-y\|^2 \leq 2C_{NJ}(X)(\|x\|^2 + \|y\|^2).$$

Hence, for any $x, y \in S_X$ such that $\|x-y\| = 1$, we can obtain

$$\|x+y\| \leq \sqrt{4C_{NJ}(X) - 1},$$

then we can conclude the desired inequality by a simple calculation. □

3.1.3. The relationships between $\Delta_h(X)$, $\Delta_H(X)$ and some geometric properties of Banach spaces

In this section, we will discuss the relationships between $\Delta_h(X)$, $\Delta_H(X)$ and some geometric properties of Banach spaces, including uniformly non-square, uniform convex and strictly convex.

First, the relationships between $\Delta_h(X)$, $\Delta_H(X)$ and uniformly non-square are shown as follows.

Proposition 3.11. Let X a be Banach space. Then X is uniformly non-square if and only if $\Delta_h(X) > \frac{1}{2}$.

Proof. The Corollary 3 in [4] shows that X is not uniformly non-square if and only if $\rho'(1) = \frac{1}{2}$. Since it is easy for us to obtain $\Delta_h(X) = 1 - \rho'(1)$, then we can get $\Delta_h(X) = \frac{1}{2}$ if and only if X is not uniformly non-square. This completes the proof. □

Remark 3.12. The above conclusion has been proved in [27], but our proof is different and more concise.

Theorem 3.13. *If $\Delta_H(X) < 1$, then X is uniformly non-square.*

Proof. If X is not uniformly non-square, then for any $\epsilon \in (0, 2)$, we have $\delta_X(\epsilon) = 0$. Thus, we can get $\delta_X(1) = 0$. Since for each $\epsilon \in (0, 2)$, $\delta_X(\epsilon) \leq \beta_X(\epsilon) \leq 2\delta_X(\epsilon)$, we can obtain $\beta_X(1) = 0$. Then, we can get $\Delta_H(X) = 1 - \beta_X(1) = 1$. This contradicts $\Delta_H(X) < 1$, hence X is a uniformly non-square Banach space. □

Some sufficient conditions for fixed point property, followed from the fact proved in [12] that uniformly non-square Banach spaces have the fixed point property, are presented in the following corollary.

Corollary 3.14. *Assume X is a Banach space with $\Delta_h(X) > \frac{1}{2}$ or $\Delta_H(X) < 1$. Then X has the fixed point property.*

Proposition 3.15. *If Banach space X is uniformly convex, then $\Delta_H(X) < 1$.*

Proof. If X is uniformly convex, then $\beta_X(1) > 0$. Thus, we can get $\Delta_H(X) = 1 - \beta_X(1) < 1$. □

Recall that the Banach space X is called strictly convex, if for any $x, y \in S_X$ and $x \neq y$, then $\|x + y\| < 2$. Now, we give the relationship between strict convexity and $\Delta_H(X)$.

Proposition 3.16. *Let X be a finite-dimensional Banach space. If $\Delta_H(X) = 1$, then X is not strictly convex.*

Proof. Assume that $\Delta_H(X) = 1$. Since the unit sphere of finite-dimensional Banach space is compact, so there exist $x, y \in S_X$ satisfying $\|x - y\| = 1$, such that for any $\lambda \in [0, 1]$, $\|\lambda x + (1 - \lambda)y\| = 1$. By Hahn-Banach Theorem, there exists $\|f\| = 1$ satisfying $f(\lambda x + (1 - \lambda)y) = 1$. It is easily seen that $f(x) = f(y) = 1$, then we have

$$f\left(\frac{x+y}{2}\right) = \frac{1}{2}[f(x) + f(y)] = 1.$$

Thus we can obtain $\frac{\|x+y\|}{2} = 1$, then X is not strictly convex. □

3.2. The heights on the hypotenuse of right triangles

In order to introduce the new geometric constant and obtain our main results, we firstly give some basic concepts and some related geometric constants.

Birkhoff [5] introduced Birkhoff orthogonality: x is said to be Birkhoff orthogonality to y (denoted by $x \perp_B y$) if $\|x + ty\| \geq \|x\|$, for any $t \in \mathbb{R}$.

Later, James [18] introduced isosceles orthogonality: x is said to be isosceles orthogonality to y (denoted by $x \perp_I y$) if $\|x + y\| = \|x - y\|$.

In [19], Ji and Wu introduced a new geometric constant $D(X)$ to give a quantitative characterization of the difference between Birkhoff orthogonality and isosceles orthogonality as follows:

$$D(X) = \inf\left\{\inf_{\lambda \in \mathbb{R}} \|x + \lambda y\| : x, y \in S_X, x \perp_I y\right\}.$$

Moreover, Mizuguchi [25] further introduced a new geometric constant $IB(X)$ to measure the difference between Birkhoff orthogonality and isosceles orthogonality in the entire normed space X as follows:

$$IB(X) = \inf\left\{\frac{\inf_{\lambda \in \mathbb{R}} \|x + \lambda y\|}{\|x\|} : x, y \in X, x, y \neq 0, x \perp_I y\right\}.$$

Motivated by the two constants $\Delta_h(X)$ and $\Delta_H(X)$ related to the heights of equilateral triangles. In this section, we consider the following geometric constant

$$\Delta_h(X, I) = \inf\left\{\inf_{\lambda \in \mathbb{R}} \|\lambda x + (1 - \lambda) y\| : \|x\| = \|y\| = 1, x \perp_I y\right\}$$

and use it to study some geometric properties.

Similarly, from the proof of Proposition 3.1, we can obtain the equivalent form of $\Delta_h(X, I)$:

$$\Delta_h(X, I) = \inf\left\{\inf_{\lambda \in [0,1]} \|\lambda x + (1 - \lambda) y\| : \|x\| = \|y\| = 1, x \perp_I y\right\}.$$

The geometric meaning of $\Delta_h(X, I)$ is shown in Figure 2: consider the unit sphere in the Euclidean plane with $\overrightarrow{AB} = x$ and $\overrightarrow{AC} = y$. Suppose $\overrightarrow{AB} \perp \overrightarrow{AC}$, $\overrightarrow{AE}$ is the height of the hypotenuse $\overrightarrow{BC}$ on the right triangle ΔABC. Apparently, the $\Delta_h(X, I)$ is the infimum of the heights of the hypotenuses of the right triangles.

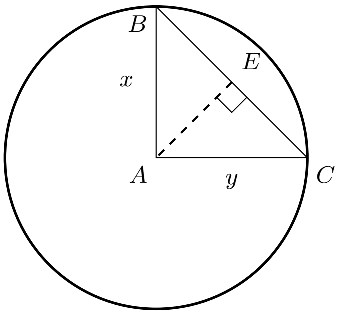

Figure 2. Geometric explanation of $\Delta_h(X, I)$.

Firstly, we give the following result which will help us to give the bounds of $\Delta_h(X,I)$.

Proposition 3.17. *Let X be a Banach space. Then $\frac{\sqrt{2}}{2}IB(X) \le \frac{\sqrt{2}}{2}D(X) \le \Delta_h(X,I) \le \frac{S(X)}{2}$.*

Proof. Since $IB(X) \le D(X)$, we can obtain the first inequality.

On the other hand, from Proposition 3.8 in [25], we have

$$D(X) = 2\inf\left\{\frac{\|\lambda x + (1-\lambda)y\|}{\|x+y\|} : x, y \in S_X, x \perp_I y, 0 \le \lambda \le 1\right\}.$$

Thus, for any $x, y \in S_X$, $x \perp_I y$ and $0 \le \lambda \le 1$, we can get

$$D(X) \le 2\frac{\|\lambda x + (1-\lambda)y\|}{\|x+y\|}.$$

Since

$$J(X) = \sup\{\|x+y\| : x, y \in S_X, x \perp_I y\}$$

and $J(X) \ge \sqrt{2}$, then for any $x, y \in S_X$ with $x \perp_I y$, we have

$$D(X) \le \sqrt{2}\|\lambda x + (1-\lambda)y\|.$$

Then, we can get the second inequality.

Finally, let $\lambda = \frac{1}{2}$, then we can get

$$\Delta_h(X,I) \le \inf\left\{\frac{\|x+y\|}{2} : x, y \in S_X, x \perp_I y\right\} = \frac{S(X)}{2}.$$

Then, we can obtain the third inequality. This completes the proof. □

Corollary 3.18. *If $D(X) > 2(\sqrt{2}-1)$, then $J(X) < 1 + \frac{\sqrt{2}}{2}$.*

Proof. From Proposition 3.17, if $D(X) > 2(\sqrt{2}-1)$, then we can obtain $S(X) > 4 - 2\sqrt{2}$. Applying $J(X)S(X) = 2$, we have

$$J(X) = \frac{2}{S(X)} < \frac{2}{4-2\sqrt{2}} = 1 + \frac{\sqrt{2}}{2},$$

which completes the proof. □

Remark 3.19. The Theorem 3.2 in [29] shows that if $D(X) > 2(\sqrt{2}-1)$, then $J(X) < 2$. Since $1 + \frac{\sqrt{2}}{2} < 2$, we improve the Theorem 3.2 in [29] by Corollary 3.18.

From Proposition 3.17, we can also obtain the following estimate.

Proposition 3.20. *Let X be a Banach space. Then $\frac{\sqrt{2}}{4} \le \Delta_h(X,I) \le \frac{\sqrt{2}}{2}$.*

Proof. From Proposition 3.17 and $IB(X) \ge \frac{1}{2}$ (see Theorem 3.2 in [25]), we can obtain $\Delta_h(X,I) \ge \frac{\sqrt{2}}{2}IB(X) \ge \frac{\sqrt{2}}{4}$. On the other hand, from Proposition 3.17 and $S(X) \le \sqrt{2}$ by Remark 11 in [16], we can get $\Delta_h(X,I) \le \frac{\sqrt{2}}{2}$. This completes the proof.

The following conclusion will show two important things. One is the fact that the upper bound of $\Delta_h(X,I)$ in the Proposition 3.20 is sharp. The other is the Hilbert space can be characterized by $\Delta_h(X,I)$.

Theorem 3.21. *Let X be a Banach space of $dimX \ge 3$. Then $\Delta_h(X,I) = \frac{\sqrt{2}}{2}$ if and only if X is a Hilbert space.*

Proof. If X is a Hilbert space, then $IB(X) = 1$ by Theorem 3.2 in [25]. Thus we can obtain $\Delta_h(X, I) = \frac{\sqrt{2}}{2}$ from Proposition 3.17 and $S(X) \leq \sqrt{2}$.

On the other hand, let $\Delta_h(X, I) = \frac{\sqrt{2}}{2}$. From Proposition 3.17 and $S(X) \leq \sqrt{2}$, we can obtain $S(X) = \sqrt{2}$, then $J(X) = \sqrt{2}$. By Theorem 2.3 in [21], we can get that X is a Hilbert space. □

Now, we will give the relation between uniform non-squareness and $\Delta_h(X, I)$.

Theorem 3.22. *Let X be a Banach space.*

(1) If $\Delta_h(X, I) > \frac{1}{2}$, then X is uniformly non-square.

(2) If X is uniformly non-square, then $\Delta_h(X, I) > \frac{\sqrt{2}}{4}$.

Proof. (1) Let X be not uniformly non-square, then $S(X) = 1$ (see [16]). From Proposition 3.17, we can obtain

$$\Delta_h(X, I) \leq \frac{S(X)}{2} = \frac{1}{2}.$$

This contradicts $\Delta_h(X, I) > \frac{1}{2}$, hence X is a uniformly non-square Banach space.

(2) If X is uniformly non-square, then we can obtain $IB(X) > \frac{1}{2}$ by Corollary 3.6 in [25]. So, we can get $\Delta_h(X, I) > \frac{\sqrt{2}}{4}$ from Proposition 3.17. This completes the proof. □

Corollary 3.23. *Assume that X is a Banach space with $\Delta_h(X, I) > \frac{\sqrt{2}}{4}$. Then X has the fixed point property.*

In the next portion, we will see that the constant $\Delta_h(X, I)$ and the uniform normal structure has a nice relationship. Brodskii and Milman [6] introduced some geometric concepts for the first time in 1948 as:

Definition 3.24. Let K be a non-singleton subset of a Banach space X, if K is closed, bounded as well as convex, then X holds the normal structure, whenever $r(K) < \operatorname{diam}(K)$ for every K, and consequently defined mathematically as is

$$\operatorname{diam}(K) := \sup\{\|x - y\| : x, y \in K\}$$

and

$$r(K) := \inf\{\sup\{\|x - y\| : y \in K\} : X \in K\},$$

where $\operatorname{diam}(K)$ and $r(K)$ are respectively symbolized for diameter as well as for Chebyshev radius. A Banach space X is said to have uniform normal structure if

$$\inf\left\{\frac{\operatorname{diam}(K)}{r(K)}\right\} > 1,$$

with $\operatorname{diam}(K) > 0$.

In order to study the relationship between $\Delta_h(X, I)$ and uniform normal structure. Now, we give the corresponding lemma as follows:

Lemma 3.25. (see [10]) *Let X be a Banach space with $J(X) < \frac{1+\sqrt{5}}{2}$. Then X has uniform normal structure.*

Theorem 3.26. *Let X be a Banach space. If $\Delta_h(X, I) > \frac{\sqrt{5}-1}{2}$, then X has uniform normal structure.*

Proof. Since $J(X)S(X) = 2$, by using Proposition 3.17, we have

$$\Delta_h(X, I) \leq \frac{S(X)}{2} = \frac{1}{J(X)}.$$

Hence, we can obtain

$$J(X) \leq \frac{1}{\Delta_h(X, I)} < \frac{1}{\frac{\sqrt{5}-1}{2}} = \frac{\sqrt{5}+1}{2}.$$

By utilizing Lemma 3.25, we can get that X has uniform normal structure. □

4. The Difference Between the Heights and Midlines in Equilateral Triangles

In this section, we will introduce the two constants $\Delta_m(X)$ and $\Delta_M(X)$ related to the midlines in equilateral triangles and the relationship between the heights and midlines in equilateral triangles. Meanwhile, we also introduce a new constant $D_\Delta(X)$ to study the difference between the heights and midlines in equilateral triangles.

4.1. The midlines in equilateral triangles

In [23], Martin et al. have stated that, in any Minkowski planes,

$$\inf \{\|x + y\| : \|x\| = \|y\| = \|x - y\| = 1\} \leq \sqrt{3}$$

and

$$\sup \{\|x + y\| : \|x\| = \|y\| = \|x - y\| = 1\} \geq \sqrt{3}.$$

This result illustrates that there always exists the equilateral triangle with a median $\leq \frac{\sqrt{3}}{2}$ as well as a median $\geq \frac{\sqrt{3}}{2}$. Motivated by this conclusion, in this section, we will introduce two new constants $\Delta_m(X)$, $\Delta_M(X)$ related to the midlines of equilateral triangles in non-trivial Banach spaces.

Definition 4.1. For a given Banach space X. Let

$$\Delta_m(X) = \inf \left\{ \frac{\|x + y\|}{2} \ : \ \|x\| = \|y\| = 1, \|x - y\| = 1 \right\},$$

$$\Delta_M(X) = \sup \left\{ \frac{\|x + y\|}{2} \ : \ \|x\| = \|y\| = 1, \|x - y\| = 1 \right\}.$$

First, we give their bounds.

Proposition 4.2. *Let X be a Banach space. Then $\frac{1}{2} \leq \Delta_m(X) \leq \frac{\sqrt{3}}{2} \leq \Delta_M(X) \leq 1$.*

Proof. Clearly, $\frac{1}{2} \leq \Delta_m(X)$ and $\Delta_M(X) \leq 1$ can be given the following two inequalities

$$\frac{\|x + y\|}{2} \leq \frac{\|x\| + \|y\|}{2} = 1,$$

$$1 = \|x\| = \frac{\|2x\|}{2} = \frac{\|(x + y) + (x - y)\|}{2} \leq \frac{\|x + y\| + \|x - y\|}{2} = \frac{1}{2} + \frac{\|x + y\|}{2},$$

where $x, y \in S_X$ with $\|x - y\| = 1$.

Then, by the definition of $\delta_X(\epsilon)$ and $\rho(\epsilon)$, we can obtain

$$\Delta_m(X) = 1 - \rho(1) \leq 1 - \left(1 - \sqrt{1 - \frac{1^2}{4}}\right) = \frac{\sqrt{3}}{2}$$

and

$$\Delta_M(X) = 1 - \delta_X(1) \geq 1 - \left(1 - \sqrt{1 - \frac{1^2}{4}}\right) = \frac{\sqrt{3}}{2}$$

which is the desired conclusion. □

Remark 4.3. From the proof of Example 3.3, we can obtain $\Delta_m(\mathbb{R}^2, \|\cdot\|_1) = \Delta_h(\mathbb{R}^2, \|\cdot\|_1) = \frac{1}{2}$. And we can also get $\Delta_M(X) = \Delta_H(X) = 1$, if X is the space $\mathbb{R}^2$ endowed the norm as Example 3.5. Thus, the lower bound and the upper bound of $\Delta_m(X)$ and $\Delta_M(X)$ can be attained, respectively.

With regard to the other bounds of $\Delta_m(X)$ and $\Delta_M(X)$, Proposition 4.4 shows that they can be attained in Hilbert spaces.

Proposition 4.4. *Let X be a Hilbert space, then $\Delta_m(X) = \Delta_M(X) = \frac{\sqrt{3}}{2}$.*

Proof. Let $x, y \in S_X$ such that $\|x - y\| = 1$. By applying the parallelogram law, we can obtain

$$\|x + y\|^2 = 2\|x\|^2 + 2\|y\|^2 - \|x - y\|^2 = 3,$$

it is easy for us to get $\Delta_m(X) = \Delta_M(X) = \frac{\sqrt{3}}{2}$. □

Corollary 4.5. *If X is a Hilbert space, then $\Delta_h(X) = \Delta_m(X)$ and $\Delta_H(X) = \Delta_M(X)$ from Proposition 3.6 and Proposition 4.4, which implies that the heights coincide with the midlines of equilateral triangles in Hilbert spaces. However, the converse is not true by Remark 4.3.*

Now, we will discuss the relation between $\Delta_m(X)$ and the Schäffer constant $S(X)$.

Proposition 4.6. *Let X be a Banach space. Then $\Delta_m(X) \geq \frac{S(X)}{2}$.*

Proof. Let $x, y \in S_X$ such that $\|x - y\| = 1$. From the proof of Proposition 4.2, we can obtain $\|x + y\| \geq 1$. Thus, we can get

$$2\Delta_m(X) = \inf\{\max\{\|x + y\|, \|x - y\|\} : x, y \in S_X, \|x - y\| = 1\}.$$

Then it is easily seen that $2\Delta_m(X) \geq S(X)$, which completes the proof. □

In the following, we can obtain that $\Delta_m(X)$ can be used to characterize uniformly non-square space.

Proposition 4.7. *Let X be a Banach space, then X is uniformly non-square if and only if $\Delta_m(X) > \frac{1}{2}$.*

Proof. Since X is not uniformly non-square if and only if $\rho(1) = \frac{1}{2}$ (see [4]). From the proof of Proposition 4.2, we can get that X is not uniformly non-square if and only if $\Delta_m(X) = \frac{1}{2}$. This completes the proof. □

Next, we will see that uniform non-squareness and $\Delta_M(X)$ has a relationship.

Proposition 4.8. *Let X be Banach space. If $\Delta_M(X) < 1$, then X is uniformly non-square.*

Proof. According to the proof of Proposition 4.2 and $\Delta_M(X) < 1$, we can obtain $\delta_X(1) > 0$, and the proposition follows. □

The converse of Proposition 4.8 is not true. Now, we provide a counterexample as follows:

Example 4.9. Let X be the space $\mathbb{R}^2$ endowed with the norm

$$\|x\| = \|(x_1, x_2)\| = \begin{cases} \|(x_1, x_2)\|_1, & x_1x_2 \geq 0, \\ \|(x_1, x_2)\|_\infty, & x_1x_2 \leq 0. \end{cases}$$

Then $\Delta_M(X) = 1$ and X is uniformly non-square.

Proof. From [26], Mizuguchi have proved that $C_{NJ}(X) = \frac{3+\sqrt{5}}{4} < 2$, then X is uniformly non-square (see [31]). However, fix $x = (1, 0)$, $y = (0, 1)$, then $x - y = (1, -1)$. It is evident that $\|x\| = \|y\| = \|x - y\| = 1$. Now, from Proposition 4.2, we can obtain

$$1 = \left\| \left(\frac{1}{2}, \frac{1}{2} \right) \right\| = \frac{\|x + y\|}{2} \leq \Delta_M(X) \leq 1,$$

which implies $\Delta_M(X) = 1$. □

4.2. The difference between the heights and midlines in equilateral triangles

In this section, in order to study the difference between the heights and midlines of equilateral triangles, we will introduce a new geometric constant.

Definition 4.10. For a given Banach space X, let

$$D_\Delta(X) = \sup \left\{ \frac{\|x + y\|}{\inf_{\lambda \in [0,1]} \|\lambda x + (1 - \lambda)y\|} : \|x\| = \|y\| = \|x - y\| = 1 \right\}.$$

Since $\|\lambda x + (1 - \lambda)y\| \geq \frac{1}{2}$ for any $\lambda \in [0, 1]$ and $\|x\| = \|y\| = \|x - y\| = 1$ by Theorem 1 in [27], then the definition of this constant is meaningful. The geometric meaning of $D_\Delta(X)$ is the supremum of the ratio of twice the midlines to the heights, in equilateral triangles.

It is easy for us to obtain the equivalent form of $D_\Delta(X)$:

$$D_\Delta(X) = \sup \left\{ \frac{\|x + y\|}{\|\lambda x + (1 - \lambda)y\|} : \|x\| = \|y\| = \|x - y\| = 1, 0 \leq \lambda \leq 1 \right\}.$$

First, we give its bounds.

Proposition 4.11. *Let X be a Banach space. Then $2 \leq D_\Delta(X) \leq 4$.*

Proof. Let $\lambda = \frac{1}{2}$, then for any $\|x\| = \|y\| = \|x - y\| = 1$, clearly

$$\frac{\|x + y\|}{\|\lambda x + (1 - \lambda)y\|} = 2,$$

which implies the left inequality.

On the other hand, let $x, y \in S_X$ such that $\|x - y\| = 1$, we have $\|x + y\| \leq \|x\| + \|y\| = 2$. Then, according to Theorem 2 in [27], we can obtain $\|\lambda x + (1 - \lambda)y\| \geq \frac{1}{2}$, for any $\lambda \in [0, 1]$. Thus, we can get

$$\frac{\|x + y\|}{\|\lambda x + (1 - \lambda)y\|} \leq 4.$$

This completes the proof. □

The following result shows that the lower bound of $D_\Delta(X)$ in the above proposition is sharp.

Proposition 4.12. *Let X be a Hilbert space, then $D_\Delta(X) = 2$.*

Proof. Let $\|x\| = \|y\| = \|x - y\| = 1$. Assume that X is a Hilbert space, then

$$1 = \|x - y\|^2 = \|x\|^2 - 2\langle x,\, y\rangle + \|y\|^2 = 2 - 2\langle x,\, y\rangle.$$

Thus, we can obtain $\langle x,\, y\rangle = \frac{1}{2}$. Then, we have

$$\|x + y\|^2 = \|x\|^2 + 2\langle x,\, y\rangle + \|y\|^2 = 3.$$

And, by Proposition 3, we can get $\Delta_h(X) = \frac{\sqrt{3}}{2}$. Hence, we can obtain

$$\begin{aligned} D_\Delta(X) &= \sup\left\{\frac{\|x + y\|}{\|\lambda x + (1-\lambda)y\|} : x, y \in S_X, \|x - y\| = 1, 0 \le \lambda \le 1\right\} \\ &= \frac{\sqrt{3}}{\Delta_h(X)} = 2, \end{aligned}$$

which completes the proof. □

Remark 4.13. From Proposition 4.12, we can also obtain the heights coincide with the midlines for equilateral triangles in Hilbert spaces, which is the same as the statement of Corollary 4.5.

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