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On Two Versions of Cohen's Theorem for Modules

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Abstract. Parkash and Kour obtained a new version of Cohen's theorem for Noetherian modules, which states that a finitely generated R-module M is Noetherian if and only if for every prime ideal $\mathfrak{p}$ of R with $\mathrm{Ann}(M) \subseteq \mathfrak{p}$, there exists a finitely generated submodule $N^{\mathfrak{p}}$ of M such that $\mathfrak{p}M \subseteq N^{\mathfrak{p}} \subseteq M(\mathfrak{p})$, where $M(\mathfrak{p}) = \{x \in M \mid sx \in \mathfrak{p}M$ for some $s \in R \setminus \mathfrak{p}\}$. In this paper, we generalize the Parkash and Kour version of Cohen's theorem for Noetherian modules to S-Noetherian modules and w-Noetherian modules.

1. Introduction

Throughout this article, all rings are commutative rings with identity and all modules are unitary. Let R be a ring and M an R-module. For a subset U of M, we denote by $\langle U \rangle$ the submodule of M generated by U. Early in 1950, Cohen showed that a ring R is Noetherian if and only if every prime ideal of R is finitely generated [4, Theorem 2]. Let $\mathfrak{p}$ be a prime ideal of R. Following [10], we set $M(\mathfrak{p}) := \{x \in M \mid sx \in \mathfrak{p}M$ for some $s \in R \setminus \mathfrak{p}\}$. Then $M(\mathfrak{p})$ is obviously a submodule of M. In 1994, Smith extended Cohen's theorem from rings to modules, showing that a finitely generated R-module M is Noetherian if and only if the submodules $\mathfrak{p}M$ of M are finitely generated for every prime ideal $\mathfrak{p}$ of R, if and only if $M(\mathfrak{p})$ is finitely generated for each prime ideal $\mathfrak{p}$ of R with $\mathfrak{p} \supseteq \mathrm{Ann}(M)$ [12].

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Recently, Parkash and Kour generalized Smith's result on Noetherian modules as follows:

Theorem. ([11, Theorem 2.1.]) *Let R be a ring and M a finitely generated R-module. Then M is Noetherian if and only if for every prime ideal $\mathfrak{p}$ of R with* $\mathrm{Ann}(M) \subseteq \mathfrak{p}$*, there exists a finitely generated submodule $N^{\mathfrak{p}}$ of M such that $\mathfrak{p}M \subseteq N^{\mathfrak{p}} \subseteq M(\mathfrak{p})$.*

In the past few decades, some generalizations of Noetherian rings or Noetherian modules have been extensively studied, especially via some multiplicative subsets S of R and the w-operation (see [1, 6, 7, 8, 9, 15, 16] for example). And the related Cohen's theorem has also been considered by many authors (see [1, 3, 5, 14] for example). In 2002, Anderson and Dumitrescu gave an analogue of Cohen's theorem for S-Noetherian modules, which states that an S-finite module M is S-Noetherian if and only if the submodules of the form $\mathfrak{p}M$ are S-finite for each prime ideal $\mathfrak{p}$ of R (disjoint from S) [1, Proposition 4]. In 1997, Wang and McCasland obtained an analogue of Cohen's theorem for strong Mori (SM) modules M over integer domains for which M satisfies the ascending chain condition on w-submodules of M. In fact, they showed that a w-module M is an SM module if and only if each w-submodule of M is w-finite type, if and only if M and every prime w-submodule of M are w-finite type [14, Theorem 4.4]. In this paper, we give both an S-analogue and a w-analogue of Parkash and Kour's result on Noetherian modules, which can be seen as generalizations of Cohen's theorem for modules.

2. Cohen's Theorem for S-Noetherian Modules

Let R be a ring and S a multiplicative subset of R, that is $1 \in S$ and $s_1 s_2 \in S$ for any $s_1 \in S$, $s_2 \in S$. Let M be an R-module. Recall from [1] that M is called *S-finite* if $sM \subseteq F$ for some $s \in S$ and some finitely generated submodule F of M. Also, M is called *S-Noetherian* if each submodule of M is an S-finite R-module. Then R is called an S-Noetherian ring if R is S-Noetherian as an R-module. Anderson and Dumitrescu obtained a Cohen-type theorem for S-Noetherian modules: An S-finite R-module M is S-Noetherian if and only if the submodules of the form $\mathfrak{p}M$ are S-finite for each prime ideal $\mathfrak{p}$ of R (disjoint from S) [1, Proposition 4]. Now we give a "stronger" version of Cohen's theorem for S-Noetherian modules which can be seen as an S-analogue of Parkash and Kour's result [11, Theorem 2.1].

Theorem 2.1. *Let R be a ring and S a multiplicative subset of R. Let M be an S-finite R-module. Then M is S-Noetherian if and only if for every prime ideal $\mathfrak{p}$ of R with* $\mathrm{Ann}(M) \subseteq \mathfrak{p}$*, there exists an S-finite submodule $N^{\mathfrak{p}}$ of M such that $\mathfrak{p}M \subseteq N^{\mathfrak{p}} \subseteq M(\mathfrak{p})$.*

Proof. Suppose that M is an S-Noetherian R-module and let $\mathfrak{p}$ be a prime ideal with $\mathrm{Ann}(M) \subseteq \mathfrak{p}$. If we take $N^{\mathfrak{p}} := \mathfrak{p}M$, then $N^{\mathfrak{p}}$ is certainly an S-finite submodule of M satisfying $\mathfrak{p}M \subseteq N^{\mathfrak{p}} \subseteq M(\mathfrak{p})$.

Conversely, suppose on the contrary that M is not S-Noetherian. Let $\mathcal{N}$ be the set of all submodules of M which are not S-finite. Then $\mathcal{N}$ is non-empty. Make a partial order on $\mathcal{N}$ by defining $N_1 \leq N_2$ if and only if $N_1 \subseteq N_2$ in $\mathcal{N}$. Let $\{N_i \mid i \in \Lambda\}$ be a chain in $\mathcal{N}$. Set $N := \bigcup_{i\in\Lambda} N_i$. Then N is not S-finite. Indeed, suppose $sN \subseteq \langle x_1, \ldots, x_n\rangle \subseteq N$ for some $s \in S$. Then there exists $i_0 \in \Lambda$ such that $\{x_1, \ldots, x_n\} \subseteq N_{i_0}$. Thus $sN_{i_0} \subseteq sN \subseteq \langle x_1, \ldots, x_n\rangle \subseteq N_{i_0}$ implies that N_{i_0} is S-finite, which is a contradiction. Then by Zorn's Lemma, $\mathcal{N}$ has a maximal element, which is also denoted by N. Set $\mathfrak{p} := (N : M) = \{r \in R \mid rM \subseteq N\}$.

We claim that $\mathfrak{p}$ is a prime ideal of R. Assume on the contrary that there exist $a, b \in R \setminus \mathfrak{p}$ such that $ab \in \mathfrak{p}$. Since $a, b \in R \setminus \mathfrak{p}$, we have $aM \not\subseteq N$ and $bM \not\subseteq N$. Therefore $N + aM$ is S-finite. Let $\{y_1, \ldots, y_m\}$ be a subset of $N + aM$ such that $s_1(N + aM) \subseteq \langle y_1, \ldots, y_m\rangle$ for some $s_1 \in S$. Write $y_i = w_i + az_i$ for some $w_i \in N$ and $z_i \in M$ $(1 \leq i \leq m)$. Set $L := \{x \in M \mid ax \in N\}$. Then $N + bM \subseteq L$, and hence L is also S-finite. Let $\{x_1, \ldots, x_k\}$ be a subset of L such that $s_2L \subseteq \langle x_1, \ldots, x_k\rangle$ for some $s_2 \in S$. Let $n \in N$ and write

$$s_1 n = \sum_{i=1}^{m} r_i y_i = \sum_{i=1}^{m} r_i w_i + a \sum_{i=1}^{m} r_i z_i.$$

Then $\sum_{i=1}^{m} r_i z_i \in L$. Thus $s_2 \sum_{i=1}^{m} r_i z_i = \sum_{i=1}^{k} r_i' x_i$ for some $r_i' \in R$ $(i = 1, \ldots, k)$. So $s_1 s_2 n = \sum_{i=1}^{m} s_2 r_i w_i + \sum_{i=1}^{k} r_i' a x_i$. Thus $s_1 s_2 N \subseteq \langle w_1, \ldots, w_m, ax_1, \ldots, ax_k\rangle$ implies that N is S-finite, which is a contradiction.

We also claim that $M(\mathfrak{p}) \subseteq N$. Suppose on the contrary that there exists $y \in M(\mathfrak{p})$ such that $y \notin N$. Then there exists $t \in R \setminus \mathfrak{p}$ such that $ty \in \mathfrak{p}M = (N : M)M \subseteq N$. As $t \notin \mathfrak{p} = (N : M)$, it follows that $tM \not\subseteq N$. Therefore $N + tM$ is S-finite. Let $\{u_1, \ldots, u_m\}$ be a subset of $N + tM$ such that $s_3(N + tM) \subseteq \langle u_1, \ldots, u_m\rangle$ for some $s_3 \in S$. Write $u_i = w_i + tz_i$ $(i = 1, \ldots, m)$ with $w_i \in N$ and $z_i \in M$. Set $T := \{x \in M \mid tx \in N\}$. Then $N \subset N + Ry \subseteq T$, and hence T is S-finite. Then there exists a subset $\{v_1, \ldots, v_l\}$ of T such that $s_4T \subseteq \langle v_1, \ldots, v_l\rangle$ for some $s_4 \in S$. Let n be an element in N. Then

$$s_3 n = \sum_{i=1}^{m} r_i u_i = \sum_{i=1}^{m} r_i w_i + t \sum_{i=1}^{m} r_i z_i.$$

Thus $\sum_{i=1}^{m} r_i z_i \in T$. So $s_4 \sum_{i=1}^{m} r_i z_i = \sum_{i=1}^{l} r_i' v_i$ for some $r_i' \in R$ $(i = 1, \ldots, l)$. Hence $s_3 s_4 n = \sum_{i=1}^{m} s_4 r_i w_i + \sum_{i=1}^{l} r_i' t v_i$. Thus $s_3 s_4 N \subseteq \langle w_1, \ldots, w_m, tv_1, \ldots, tv_l\rangle$ implies that N is S-finite, which is a contradiction.

Let $F = \langle m_1, \ldots, m_k\rangle$ be a submodule of M such that $sM \subseteq F$ for some $s \in S$. Claim that $\mathfrak{p} \cap S = \emptyset$. Indeed, if $s' \in \mathfrak{p}$ for some $s' \in S$, then $s'M \subseteq N \subseteq M$. So $ss'N \subseteq ss'M \subseteq s'F \subseteq s'M \subseteq N$ implies that N is S-finite, which is a contradiction.

Note that $\mathfrak{p} = (N : M) \subseteq (N : F) \subseteq (N : sM) = (\mathfrak{p} : s) = \mathfrak{p}$ as $\mathfrak{p}$ is a prime ideal of R. So $\mathfrak{p} = (N : F) = (N : \langle m_1, \dots, m_k\rangle) = \bigcap_{i=1}^{k} (N : Rm_i)$. By [2, Proposition 1.11], $\mathfrak{p} = (N : Rm_j)$ for some $1 \leq j \leq k$. Since $m_j \notin N$, it follows that $N + Rm_j$ is S-finite. Let $\{y_1, \dots, y_m\}$ be a subset of $N + Rm_j$ such that $s_5(N + Rm_j) \subseteq \langle y_1, \dots, y_m\rangle$ for some $s_5 \in S$. Write $y_i = w_i + a_i m_j$ for some $w_i \in N$ and $a_i \in R$ $(i = 1, \dots, m)$. Let $n \in N$. Then $s_5 n = \sum_{i=1}^{m} r_i(w_i + a_i m_j) = \sum_{i=1}^{m} r_i w_i + (\sum_{i=1}^{m} r_i a_i)m_j$. Thus $(\sum_{i=1}^{m} r_i a_i)m_j \in N$. So $\sum_{i=1}^{m} r_i a_i \in \mathfrak{p}$. Thus $s_5 N \subseteq \langle w_1, \dots, w_m\rangle + \mathfrak{p}m_j$. As $\mathrm{Ann}(M) \subseteq (N : M) = \mathfrak{p}$, there exists an S-finite submodule $N^{\mathfrak{p}}$ of M such that $\mathfrak{p}M \subseteq N^{\mathfrak{p}} \subseteq M(\mathfrak{p})$. Thus

$$\begin{aligned} s_5 N &\subseteq \langle w_1, \dots, w_m\rangle + \mathfrak{p}m_j \\ &\subseteq \langle w_1, \dots, w_m\rangle + \mathfrak{p}M \\ &\subseteq \langle w_1, \dots, w_m\rangle + N^{\mathfrak{p}} \\ &\subseteq \langle w_1, \dots, w_m\rangle + M(\mathfrak{p}) \\ &\subseteq N \end{aligned}$$

Since $N^{\mathfrak{p}} + \langle w_1, \dots, w_m\rangle$ is S-finite, it follows that N is also S-finite, which is a contradiction. Hence M is S-Noetherian. □

Taking $S = \{1\}$, we can recover the following result of Parkash and Kour.

Corollary 2.2. ([11, Theorem 2.1]) *Let R be a ring and M a finitely generated R-module. Then M is Noetherian if and only if for every prime ideal $\mathfrak{p}$ of R with* $\mathrm{Ann}(M) \subseteq \mathfrak{p}$, *there exists a finitely generated submodule $N^{\mathfrak{p}}$ of M such that $\mathfrak{p}M \subseteq N^{\mathfrak{p}} \subseteq M(\mathfrak{p})$.*

3. Cohen's Theorem for w-Noetherian Modules

We recall some basic knowledge on the w-operation over a commutative ring. One can refer to [13] for more details. Let R be a commutative ring and J a finitely generated ideal of R. Then J is called a GV-*ideal* if the natural homomorphism $R \to \mathrm{Hom}_R(J, R)$ is an isomorphism. The set of GV-ideals is denoted by $\mathrm{GV}(R)$. Let M be an R-module. Define

$$\mathrm{tor}_{\mathrm{GV}}(M) := \{x \in M \mid Jx = 0 \text{ for some } J \in \mathrm{GV}(R)\}.$$

An R-module M is said to be GV-*torsion* (resp., GV-*torsion-free*) if $\mathrm{tor}_{\mathrm{GV}}(M) = M$ (resp., $\mathrm{tor}_{\mathrm{GV}}(M) = 0$). A GV-torsion-free module M is called a w-module if $\mathrm{Ext}_R^1(R/J, M) = 0$ for any $J \in \mathrm{GV}(R)$. A DW *ring* R is a ring for which every R-module is a w-module. A *maximal w-ideal* is an ideal of R which is maximal among the w-submodules of R. The set of all maximal w-ideals is denoted by w-$\mathrm{Max}(R)$. Each maximal w-ideals is a prime ideal (see [13, Theorem 6.2.14]).

An R-homomorphism $f : M \to N$ is said to be a *w-monomorphism* (resp., *w-epimorphism*, *w-isomorphism*) if for any $\mathfrak{p} \in w\text{-Max}(R)$, $f_\mathfrak{p} : M_\mathfrak{p} \to N_\mathfrak{p}$ is a monomorphism (resp., an epimorphism, an isomorphism). Note that f is a w-monomorphism (resp., w-epimorphism) if and only if $\text{Ker}(f)$ (resp., $\text{Coker}(f)$) is GV-torsion. An R-module M is said to be *w-finite type* if there exist a finitely generated free module F and a w-epimorphism $g : F \to M$. Obviously, an R-module M is w-finite type if and only if there is a finitely generated submodule N of M such that M/N is GV-torsion.

Lemma 3.1. *Let N be a w-submodule of a GV-torsion-free w-finite type module M. Then $(N :_R M)_\mathfrak{p} = (N_\mathfrak{p} :_{R_\mathfrak{p}} M_\mathfrak{p})$ for any prime w-ideal $\mathfrak{p}$ of R.*

Proof. Let $\mathfrak{p}$ be a prime w-ideal of R. Obviously, $(N :_R M)_\mathfrak{p} \subseteq (N_\mathfrak{p} :_{R_\mathfrak{p}} M_\mathfrak{p})$. On the other hand, since M is a w-finite type R-module, there exists a finitely generated submodule $F = \langle m_1, \ldots, m_n\rangle$ of M satisfying that for any $m \in M$ there exists $J \in \text{GV}(R)$ such that $Jm \subseteq F$. Let $\frac{r}{s}$ be an element in $(N_\mathfrak{p} :_{R_\mathfrak{p}} M_\mathfrak{p})$. Then for each $i = 1, \ldots, n$, there exists $s_i \in R \setminus \mathfrak{p}$ such that $s_i r m_i \in N$. Thus $s_1 \cdots s_n r F \subseteq N$. So $s_1 \cdots s_n r J m \subseteq N$ for all $m \in M \subseteq E(M)$, where $E(M)$ is the injective envelope of M. By [13, Theorem 6.16], $s_1 \cdots s_n r M \subseteq N$ since N is a w-module. Hence $s_1 \cdots s_n r \in (N :_R M)$. Consequently, $\frac{r}{s} = \frac{s_1 \cdots s_n r}{s_1 \cdots s_n s} \in (N :_R M)_\mathfrak{p}$. □

Let M be an R-module. Recall from [13, Definition 8.1] that M is called a *w-Noetherian module* if every submodule of M is w-finite type. And R is called a *w-Noetherian ring* if R is w-Noetherian as an R-module.

Theorem 3.2. *Let R be a ring and M a GV-torsion-free w-finite type R-module. Then M is a w-Noetherian module if and only if for every prime w-ideal $\mathfrak{p}$ of R with $\text{Ann}(M) \subseteq \mathfrak{p}$, there exists a w-finite type submodule $N^\mathfrak{p}$ of M such that $\mathfrak{p}M \subseteq N^\mathfrak{p} \subseteq M(\mathfrak{p})$.*

Proof. Suppose that M is a w-Noetherian R-module and let $\mathfrak{p}$ be a prime w-ideal with $\text{Ann}(M) \subseteq \mathfrak{p}$. If we take $N^\mathfrak{p} := \mathfrak{p}M$, then $N^\mathfrak{p}$ is certainly a w-finite type submodule of M satisfying $\mathfrak{p}M \subseteq N^\mathfrak{p} \subseteq M(\mathfrak{p})$.

Conversely, suppose on the contrary that M is not w-Noetherian. Let $\mathcal{N}$ be the set of all w-submodules of M which are not w-finite type. Then $\mathcal{N}$ is non-empty. Make a partial order on $\mathcal{N}$ by defining $N_1 \leq N_2$ if and only if $N_1 \subseteq N_2$ in $\mathcal{N}$. Let $\{N_i \mid i \in \Lambda\}$ be a chain in $\mathcal{N}$. Set $N := \bigcup_{i\in\Lambda} N_i$. Then N is not w-finite type. Indeed, suppose there is an exact $0 \to F \to N \to T \to 0$ with T GV-torsion and $F = \langle x_1, \ldots, x_n\rangle$ finitely generated. Then there exists $i_0 \in \Lambda$ such that $F \subseteq N_{i_0}$. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & N_{i_0} & \longrightarrow & T' & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F & \longrightarrow & N & \longrightarrow & T & \longrightarrow & 0 \end{array}$$

Since T' is a submodule of T, we have that T' being GV-torsion implies that N_{i_0} is w-finite type, which is a contradiction. Since N is a w-submodule of M, it follows that $N \in \mathcal{N}$. So by Zorn's Lemma, $\mathcal{N}$ has a maximal element, which is also denoted by N. Set $\mathfrak{p} := (N : M) = \{r \in R \mid rM \subseteq N\}$. Then $\mathfrak{p}$ is a w-ideal by [13, Section 6.10, Exercise 6.8].

We claim that $\mathfrak{p}$ is a prime ideal of R. Assume on the contrary that there exist $a, b \in R \setminus \mathfrak{p}$ such that $ab \in \mathfrak{p}$. Since $a, b \in R \setminus \mathfrak{p}$, we have $aM \nsubseteq N$ and $bM \nsubseteq N$. Therefore $N + aM$ is w-finite type. Let $\{y_1, \dots, y_m\}$ be a subset of $N + aM$ such that $0 \to F_1 \to N + aM \to T_1 \to 0$ be an exact sequence with T_1 GV-torsion and $F_1 = \langle y_1, \dots, y_m\rangle$ finitely generated. Write $y_i = w_i + az_i$ for some $w_i \in N$ and $z_i \in M$ $(1 \le i \le m)$. Set $L := \{x \in M \mid ax \in N\}$. Then $N + bM \subseteq L$, and hence L is w-finite type. Let $0 \to F_2 \to L \to T_2 \to 0$ be an exact sequence with T_2 GV-torsion and $F_2 = \langle x_1, \dots, x_k\rangle$ finitely generated. Let n be an element in N. Then there is a GV-ideal $J_1 = \langle j_1^1, \dots, j_1^p\rangle$ such that $J_1 n \subseteq F_1$. So there is $\{r_i^t \mid t = 1, \dots, p; i = 1, \dots, m\} \subseteq R$ such that

$$j_1^t n = \sum_{i=1}^{m} r_i^t y_i = \sum_{i=1}^{m} r_i^t w_i + a \sum_{i=1}^{m} r_i^t z_i \ (t = 1, \dots, p).$$

Then $\sum\limits_{i=1}^{m} r_i^t z_i \in L$ $(t = 1, \dots, p)$. Thus there exists a GV-ideal $J_2 = \langle j_2^1, \dots, j_2^l\rangle$ such that $j_2^s \sum\limits_{i=1}^{m} r_i^t z_i = \sum\limits_{i=1}^{k} r_i'^{t,s} x_i$ for some $\{r_i'^{t,s} \mid i = 1, \dots, k; t = 1, \dots, p; s = 1, \dots, l\} \subseteq R$. So $j_1^t j_2^s n = \sum\limits_{i=1}^{m} j_2^s r_i^t w_i + \sum\limits_{i=1}^{k} r_i'^{t,s} a x_i$ $(t = 1, \dots, k; s = 1, \dots, l)$. Thus

$$J_1 J_2 n \subseteq \langle w_1, \dots, w_m, ax_1, \dots, ax_k\rangle$$

implies that N is w-finite type, which is a contradiction.

We claim that $M(\mathfrak{p}) \subseteq N$. Assume on the contrary that there exists an element $y \in M(\mathfrak{p})$ such that $y \notin N$. Then there exists $t' \in R \setminus \mathfrak{p}$ such that $t'y \in \mathfrak{p}M = (N : M)M \subseteq N$. As $t' \notin \mathfrak{p} = (N : M)$, it follows that $t'M \nsubseteq N$. Therefore $N + t'M$ is w-finite type. Let $0 \to F_3 \to N + t'M \to T_3 \to 0$ be an exact sequence with T_3 GV-torsion and $F_3 = \langle u_1, \dots, u_m\rangle$ a finitely generated submodule of $N + t'M$. Write $u_i = w_i + t'z_i$ $(i = 1, \dots, m)$ with $w_i \in N$ and $z_i \in M$. Set $L := \{x \in M \mid tx \in N\}$. Then $N \subset N + Ry \subseteq L$, and hence L is w-finite type. Let $0 \to F_4 \to L \to T_4 \to 0$ be an exact sequence with T_4 GV-torsion and $F_4 = \langle u_1, \dots, u_n\rangle$ a finitely generated submodule of L. Let n be an element in N. Then there is a GV-ideal $J_3 = \langle j_3^1, \dots, j_3^k\rangle$ such that $J_3 n \subseteq F_3$. So there is $\{r_i^t \mid t = 1, \dots, p; i = 1, \dots, m\} \subseteq R$ such that

$$j_3^t n = \sum_{i=1}^{m} r_i^t u_i = \sum_{i=1}^{m} r_i^t w_i + t' \sum_{i=1}^{m} r_i^t z_i \ (t = 1, \dots, p).$$

So $\sum\limits_{i=1}^{m} r_i^t z_i \in L$ $(t = 1, \dots, p)$. Thus there exists a GV-ideal $J_4 = \langle j_4^1, \dots, j_4^l\rangle$ such that $j_4^s \sum\limits_{i=1}^{m} r_i^t z_i = \sum\limits_{i=1}^{n} r_i'^{t,s} u_i$ for some $\{r_i'^{t,s} \mid i = 1, \dots, m; t = 1, \dots, p; s =$

$1,\dots,l\} \subseteq R$. So $j_3^t j_4^s n = \sum_{i=1}^{m} j_4^s r_i^t w_i + \sum_{i=1}^{k} r_i'^{t,s} t' u_i$ $(t = 1,\dots,k; s = 1,\dots,l)$. Thus $J_3 J_4 n \subseteq \langle w_1,\dots,w_m, t'u_1,\dots,t'u_k\rangle$ implies that N is w-finite type, which is a contradiction.

Let $\mathfrak{m}$ be a maximal w-ideal of R and $F = \langle m_1,\dots,m_k\rangle$ a submodule of M such that M/F is GV-torsion. So $M_\mathfrak{m} = F_\mathfrak{m}$. Then $(N :_R M)_\mathfrak{m} = (N_\mathfrak{m} :_{R_\mathfrak{m}} M_\mathfrak{m}) = (N_\mathfrak{m} :_{R_\mathfrak{m}} F_\mathfrak{m}) = (N :_R F)_\mathfrak{m}$ by Lemma 3.1. By [13, Section 6.10, Exercise 6.8], $(N :_R M)$ and $(N :_R F)$ are all w-ideals. So we have $\mathfrak{p} = (N :_R M) = (N :_R F) = \bigcap_{i=1}^{k}(N : Rm_i)$. By [2, Proposition 1.11], $\mathfrak{p} = (N :_R Rm_j)$ for some $1 \leq j \leq k$. Since $m_j \notin N$, it follows that $N+Rm_j$ is w-finite type. Let $0 \to F_5 \to N+Rm_j \to T_5 \to 0$ be an exact sequence with T_5 GV-torsion and $F_5 = \langle y_1,\dots,y_m\rangle$ a finitely generated submodule of $N + Rm_j$. Write $y_i = w_i + a_i m_j$ for some $w_i \in N$ and $a_i \in R$ $(i = 1,\dots,m)$. Let n be an element in N. Then there is a GV-ideal $J_5 = \langle j_5^1,\dots,j_5^l\rangle$ such that $J_5 n \subseteq F_5$. So there is $\{r_i^t \mid t = 1,\dots,p; i = 1,\dots,m\} \subseteq R$ such that $j_5^t n = \sum_{i=1}^{m} r_i^t y_i = \sum_{i=1}^{m} r_i^t w_i + (\sum_{i=1}^{m} r_i^t a_i) m_j$ $(t = 1,\dots,l)$. So $\sum_{i=1}^{m} r_i^t a_i \in \mathfrak{p}$. Thus $J_5 N \subseteq \langle w_1,\dots,w_m\rangle + \mathfrak{p} m_j$. As $\mathrm{Ann}(M) \subseteq (N : M) = \mathfrak{p}$, there exists a w-finite type submodule $N^\mathfrak{p}$ of M such that $\mathfrak{p}M \subseteq N^\mathfrak{p} \subseteq M(\mathfrak{p})$. Thus

$$\begin{aligned} J_5 N &\subseteq \langle w_1,\dots,w_m\rangle + \mathfrak{p} m_j \\ &\subseteq \langle w_1,\dots,w_m\rangle + \mathfrak{p} M \\ &\subseteq \langle w_1,\dots,w_m\rangle + N^\mathfrak{p} \\ &\subseteq \langle w_1,\dots,w_m\rangle + M(\mathfrak{p}) \\ &\subseteq N \end{aligned}$$

Since $N^\mathfrak{p} + \langle w_1,\dots,w_m\rangle$ is w-finite type, it follows that N is also w-finite type, which is a contradiction. Hence M is w-Noetherian. □

Taking $M := R$, we have the following characterization of w-Noetherian rings.

Corollary 3.3. ([15, Theorem 4.7(1)]) *Let R be a ring. Then R is a w-Noetherian ring if and only if each prime w-ideal of R is w-finite type.*

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