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Purely Extending Modules and Their Generalizations

Shiv Kumar and Ashok Ji Gupta*
Department of Mathematical Sciences, IIT(B. H. U.) Varanasi-221005, India
E-mail : shivkumar.rs.mat17@itbhu.ac.in *and* agupta.apm@itbhu.ac.in

Abstract. A purely extending module is a generalization of an extending module. In this paper, we study several properties of purely extending modules and introduce the notion of purely essentially Baer modules. A module M is said to be a purely essentially Baer if the right annihilator in M of any left ideal of the endomorphism ring of M is essential in a pure submodule of M. We study some properties of purely essentially Baer modules and characterize von Neumann regular rings in terms of purely essentially Baer modules.

1. Introduction

In this paper, all rings are considered to be associative rings with unity, and all modules are unital right modules unless otherwise stated. Recall that a module M is called an extending module (CS) or said to have the C_1 condition if every submodule of M is essential in a direct summand of M. Extending modules are closely related to injective modules and it has been shown that extending modules generalize such modules as injective modules, quasi-injective modules, and continuous modules (see [11] and [14]). In [5], Clark called a module M purely extending if every submodule of M is essential in a pure submodule of M and proved that closed submodules of a purely extending module are pure submodules. The notion of purely extending modules generalizes the notion of extending modules.

Rizvi and Roman [13] introduced the notion of Baer modules. An R-module M is said to be a Baer module if the right annihilator in M of every left ideal I of $S = End(M)$ is a direct summand of M. In [1], Atani and Khoramdel called a module M purely Baer if the right annihilator in M of every left ideal of S is a pure submodule of M. They also showed that the class of purely Baer modules contains the class of Baer modules. According to Nhan [12], a module is said to be essentially Baer if the right annihilator in M of every left ideal of S is essential in a direct summand of M.

Motivated by the essentially Baer module and purely Baer module, we intro-

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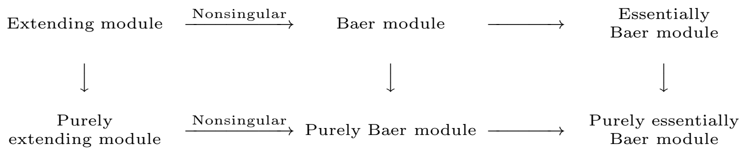
duced the purely essentially Baer module.

The following implications justify the connection among the extending module, purely extending module, Baer module, purely Baer module, essentially Baer module, and purely essentially Baer module.



The converse of any statements in above diagram need not be true (see [1], [2], [5], [11], [12], [13]).

In this paper, several properties of purely extending modules are studied. In general, the direct sum of purely extending modules need not be purely extending, and also submodules of a purely extending module need not be purely extending; counter-examples are given. We study when the direct sum of purely extending modules is purely extending and when the submodules of a purely extending module are purely extending. We prove that a finitely generated torsion-free module over a principal ideal domain is a purely extending module. Also, we discuss when the endomorphism ring of a module is purely extending.

In Section 4, we introduce and study the notion of purely essentially Baer modules. We call a module M purely essentially Baer if the right annihilator in M of every left ideal of $S = End_R(M)$ is essential in a pure submodule of M. It is shown that the purely essentially Baer module is a proper and common generalization of purely Baer module and purely extending module. We prove that a purely essentially Baer module is closed under direct summands. We characterize purely essentially Baer modules in terms of von Neumann regular rings.

2. Preliminaries

The notations $\leq$, $\leq^{\oplus}$, $\leq^{e}$, $\leq^{p}$ and $\leq^{c}$ denote a submodule, a direct summand, an essential submodule, a pure submodule, and a closed submodule, respectively. For an R-module M, $E(M)$, $S = End_R(M)$ and $Cl_M(N) = \{m \in M : (N : m) \leq^{e} R\}$ (or in short, $Cl(N)$) denote the injective hull of a module M, the endomorphism ring of a module M and the closure of a submodule N in a module M, respectively. Let M be an R-module and X be a subset of $S = End_R(M)$, then $r_M(X) = \{m \in M : \varphi(m) = 0, \forall\, \varphi \in X\}$. A regular ring will always mean a von Neumann regular ring.

First we recall some definitions and results which are useful in our further work.

Definition 2.1. A short exact sequence $0 \to N_1 \xrightarrow{\phi} N_2 \to N_3 \to 0$ of right R-modules is said to be pure exact if $0 \to N_1 \otimes F \to N_2 \otimes F \to N_3 \otimes F \to 0$ is an exact sequence (of abelian groups) for any left R-module F [9]. In this case, $\phi(N_1)$ is a pure submodule of M. According to Cohn [6], a submodule N of a right R-module M is said to be a pure submodule of M if and only if $0 \to N \otimes L \to M \otimes L$ is exact for every left R-module L. Also, the condition for a right R-module M to be flat is that whenever $0 \to N_1 \to N_2$ is exact for left R-modules N_1 and N_2, then so is $0 \to M \otimes N_1 \to M \otimes N_2$.

Proposition 2.2.

(i) [9, Proposition 4.29]. A ring R is Noetherian if and only if all finitely generated right R-modules are finitely presented.

(ii) [9, Proposition 4.30] A finitely related R-module M is flat if and only if it is projective .

Lemma 2.3. ([7, Proposition 8.1.]) *The following conditions hold:*

(i) Let N be a submodule of a right R module M. If M/N is flat, then $N \leq^p M$. Moreover, for a flat right R module M, $N \leq^p M$ if and only if M/N is flat.

(ii) If N is a submodule of M such that every finitely generated submodule of N is a pure submodule of M, then $N \leq^p M$.

Lemma 2.4. ([7, Proposition 7.2.]) *Suppose $L \subseteq N \subseteq M$ be right R modules.*

(i) If $L \leq^p N$ and $N \leq^p M$, then $L \leq^p M$.

(ii) If $L \leq^p M$, then $L \leq^p N$.

(iii) If $L \leq^p N$, then $N/L \leq^p M/L$.

(iv) If $L \leq^p M$ and $N/L \leq^p M/L$, then $N \leq^p M$.

In general, pure submodules of a module need not be direct summands. There are modules whose pure submodules are direct summands.

Definition 2.5. A right R-module M is said to be a pure split module if every pure submodule of M is a direct summand of M [7]. Recall that a ring R is said to be a right pure semisimple if every right R-module is a pure injective [17]. Moreover, a ring R is called a pure semisimple ring if every pure exact sequence $0 \to A \to B \to C \to 0$ of right R-modules splits.

Definition 2.6. An R-module M is called Baer if the right annihilator in M of every left ideal of S is a direct summand of M ([2], [13]). Also, M is called a purely Baer module if the right annihilator in M of every left ideal of S is a pure submodule of M [1].

Definition 2.7. A submodule N is called a strongly large submodule of a module M if every $m \in M$ $mI = 0$, then $m(m^{-1}N)I = 0$ where I is an ideal of R. An R-module M is said to be a strongly extending module if every strongly large submodule of M is a direct summand of M. It is seen that every strongly extending module is an extending module [16].

Proposition 2.8. ([1, Theorem 3.]) *A nonsingular purely extending module is a purely Baer module.*

Lemma 2.9. ([15, Lemma 3.1.]) *Let N be a submodule of M. If $N \supseteq Cl_M(0)$, then $Cl_M(N)$ is a closed submodule of M.*

3. Purely Extending Modules

In this section, we study some more properties of purely extending modules. According to Clark [5], a module M is a purely extending module if every submodule of M is essential in a pure submodule of M; equivalently, every closed submodule of M is a pure submodule of M. Also, a ring R is called a right purely extending ring if R_R is a purely extending R-module.

Every extending module is purely extending. The following example shows that a purely extending module need not be an extending module.

Example 3.1.

(i) By [8, Example 13.8], there exists a commutative continuous regular ring F such that $R = M_{2\times2}(F)$ (2 by 2 matrix ring over F) is neither a left nor right continuous ring. Since F is a regular ring, R is a regular ring. So, R_R is a purely extending right R-module while R_R is not a right extending R-module. In fact, by [11, Proposition A.14] a regular ring is right continuous if and only if it is a right extending ring. Therefore, R_R is neither a left nor right extending R-module.

(ii) Let $\mathbb{F}$ be a field and $F_n = \mathbb{F}$ for every $n \in \mathbb{N}$. Consider $R = \Pi_{n=1}^{\infty} F_n$ and $A = \{(x_n)_{n=1}^{\infty} \in R_1 : x_n \text{ is constant eventually}\}$, where A is a subring of R. Clearly, A is a regular ring, but not a Baer ring (see [2, Example 3.1.14(ii)]). So, A is a purely extending ring but not an extending ring. In fact, a nonsingular extending ring is a Baer ring but A is not a Baer ring [2, Lemma 4.1.17]. Hence, A_A is a purely extending A-module which is not an extending A-module.

Now we discuss when a purely extending module to be an extending module.

Theorem 3.2.

(i) A finitely generated flat R-module M over a noetherian ring is purely extending module if and only if it is an extending module.

(ii) An R-module M over a pure semisimple ring R is purely extending if and only if it is an extending module.

(iii) A pure split module M is purely extending module if and only if it is an extending module.

Proof.

(i) Let N be a submodule of a purely extending module M. Then there exists a pure submodule P of M such that $N \leq^e P$, so by Lemma 2.3, M/P is flat. Since M/P is finitely generated and R is a Noetherian ring, M/P is finitely presented. Thus M/P is projective by Proposition 2.2. Therefore, $P \leq^{\oplus} M$. Hence, M is an extending module. The converse is obvious.

(ii) Let R be a pure semisimple ring and N be a submodule of a purely extending module M. Then, there exists a pure submodule L of M such that $N \leq^e L$. Since R is a pure semisimple ring, for any right R-module P, the pure exact sequence $0 \to L \otimes K \to M \otimes K \to P \otimes K \to 0$ splits for every left R-module K. Therefore, the exact sequence $0 \to L \to M \to P \to 0$ also splits. Thus $L \leq^{\oplus} M$. Hence, M is an extending module. The converse is clear.

(iii) It follows from the fact that an R-module M is pure split if every pure submodule of M is a direct summand of M. □

In general, submodules of a purely extending module need not be purely extending.

Example 3.3. Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$, then R_R is a finitely generated and Noetherian R-module which is not extending R-module (see, [4, Example 2.2]). Therefore, R_R is not a purely extending R-module. If $E(R_R)$ is the injective hull of R_R, then $E(R_R)$ is purely extending while R_R is not.

Now, we provide the condition under which submodules of a purely extending module are purely extending.

Proposition 3.4. *Let M be a purely extending module and N be a submodule of M. If for every pure submodule P of M, $N \cap P$ is a pure submodule of N, then N is a purely extending.*

Proof. Let $V \leq N$. Then there exists a pure submodule P of M such that $V \leq^e P$, which implies $V \leq^e P \cap N$. Since $P \cap N$ is a pure submodule of N, so we get $V \leq^e N \cap P \leq^p N$. Hence, N is a purely extending submodule of M. □

Proposition 3.5. *Let M be a module, N be a purely extending submodule of M and P be a pure submodule of M. If $P + N$ is nonsingular, then $P \cap N$ is a pure submodule of M.*

Proof. Let P be a pure submodule of M and $V = P \cap N$. Since $V \leq N$ and N is purely extending, there exists a pure submodule Q of N such that V is essential in Q. Assume that $V \neq Q$, then $P \neq P + Q$. Let $p \in P$ and $q \in Q$ such that $p + q \in P + Q$ and $p + q \notin P$, then $q \neq 0$. Therefore, there exists an essential right ideal S of R such that $0 \neq qS \subseteq V$. Since P is nonsingular, $0 \neq (p+q)S \subseteq P$. Thus P is essential in $P + Q$, which is a contradiction. Therefore, we get $V = Q$. □

Corollary 3.6. *If M is a nonsingular module, N is a purely extending submodule of M and P is a pure submodule of M, then $P \cap N$ is a pure submodule of N.*

Proposition 3.7. *Every direct summand of a purely extending module is a purely extending.*

Proof. Let M be a purely extending module and $N \leq^{\oplus} M$. To prove N is purely extending, it suffices to show that every closed submodule of N is a pure submodule of N. Let $V \leq^c N$. Since every direct summand is closed, $V \leq^c N \leq^c M$. Thus $V \leq^c M$. Also, M is a purely extending module, $V \leq^p M$. Therefore, $V \leq^p N$. □

Now we give an example which shows that the direct sum of purely extending modules need not be purely extending.

Example 3.8. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_p \oplus \mathbb{Z}_{p^3}$ (where p is a prime number). Clearly, M is not an extending R-module. Since M is a finitely generated R-module and R is a Noetherian ring, M is not a purely extending R-module. But $\mathbb{Z}_p$ and $\mathbb{Z}_{p^3}$ are extending R-modules, so $\mathbb{Z}_p$ and $\mathbb{Z}_{p^3}$ are purely extending R-modules.

Now we discuss when the direct sum of purely extending modules is purely extending.

Proposition 3.9. *Let $M = \bigoplus_{i \in I} M_i$ be the direct sum of R-modules M_i $(i \in I)$ for an index set $|I| \geqslant 2$. Then the following statements are equivalent:*

(i) M is purely extending;

(ii) There exists $i, j \in I$, $i \neq j$ such that every closed submodule W of M with $W \cap M_i = 0$ or $W \cap M_j = 0$ is a pure submodule of M;

(iii) There exists $i, j \in I$, $i \neq j$, such that every complement of M_i or M_j in M is a purely extending and a pure submodule of M.

Proof. $(i) \Rightarrow (ii)$. It is clear.
$(ii) \Rightarrow (iii)$. Let N be a complement of M_i in M, so by the hypothesis N is a pure submodule of M . Now, to prove N is purely extending, it suffices to prove that every closed submodule of N is a pure submodule of N. Let $L \leq^c V$, then $L \leq^c M$ and clearly $L \cap M_i = 0$. Therefore, L is a pure submodule of M. Hence, by Lemma 2.4, L is a pure submodule of N.
$(iii) \Rightarrow (i)$. Let $N \leq^c M$, so there exists a closed submodule L of N such that $N \cap M_i \leq^e L$ which implies that $L \cap M_j = 0$. By Zorn's Lemma, there exists a complement H of M_j in M such that $L \leq H$. From which it follows that $L \leq^c M$ and hence $L \leq^c H$. Applying (iii), we see that L is a pure submodule of H and H is a pure submodule of M. So by Lemma 2.4, $L \leq^p M$. Thus $L \leq^p N$. Since $L \subseteq N \subseteq M$, by Lemma 2.4 $N/L \leq^p M/L$. Therefore, $L \leq^p M$ and $N/L \leq^p M/L$. Hence, $N \leq^p M$. □

Theorem 3.10. *Let $M = \bigoplus_{i \in I} M_i$ be the direct sum of right R-modules M_i $(i \in I)$, where I is an index set such that $|I| \geqslant 2$. Then M is an extending module if and only if there exists a subset $\{i_1, , i_2, \ldots, i_n\}$ of I such that every closed submodule N with either $N \cap M_{i_k} \leq^e N$ for some i_k, $1 \leq k \leq n$ or $N \cap M_{i_k} = 0$ for all k, $1 \leq k \leq n$ is a pure submodule.*

Proof. The only if part is trivial.
To prove if part, it is enough to show that there exists $i \neq j \in I$ such that every closed submodule N of M with $N \cap M_i = 0$ or $N \cap M_j = 0$ is a pure submodule. To prove it, let N be a closed submodule with $N \cap M_{i_1} = N \cap M_{i_2} = \ldots = N \cap M_{i_n} = 0$. If $N \cap M_{i_1} = 0$, then by assumption N is a pure submodule of M. Now we consider

$N \cap M_{i_1} \neq 0$ and L be a closed submodule of N such that $N \cap M_{i_1} \leq^e L$. Since $L \leq^c N \leq^c M$, $L \leq^c M$. Therefore, $L \cap M_{i_1} = N \cap M_{i_1} \leq^e L$. So by hypothesis, L is a pure submodule of M. Applying Lemma 2.4, $L \leq^p N$ and $N/L \leq^p M/L$, so again by Lemma 2.4, $N \leq^p M$. Continuing in similar steps, we can prove that whenever N is a closed submodule of M with $N \cap M_{i_n} = 0$, then N is a pure submodule of M. Now there exists $i_1 \neq i_n \in I$ such for every closed submodule N of M with $N \cap M_{i_1} = 0$ or $N \cap M_{i_n} = 0$ is a pure submodule of M. Hence, M is a purely extending module. □

Now we show when finitely generated torsion-free modules and finitely generated flat modules are purely extending. The next result generalizes the Proposition 3.9 of [16].

Proposition 3.11. *Every finitely generated torsion-free module over a principal ideal domain is purely extending.*

Proof. Let M be a finitely generated torsion-free module over a principal ideal domain R and $N \leq M$. Then M/N is either a torsion-free submodule or a torsion submodule of M. Assume first that M/N is a torsion-free module, then $M/N \cong R^n$ for some $n \in \mathbb{N}$, which implies M/N is a projective module. So, M/N is flat and hence N is a pure submodule of M. Now we suppose that M/N is not a torsion-free module, then there exists a submodule $L \leq M$ containing N such that M/L is torsion-free and L/N is torsion. Since M/L is a torsion-free and finitely generated R-module. So, M/L is projective which implies that M/L is a flat module. Therefore, L is a pure submodule of M. Now we show that $N \leq^e L$. For it, let $l \in L \backslash N$ and $r_1 \in R$ with $lr_1 \neq 0$. Suppose $\phi : L \to L/N$ is the natural map. Since L/N is a torsion submodule of M and $\phi(l)$ is non-zero in L/N, there exists $0 \neq r_2 \in R$ such that $\phi(l)r_2 = \phi(lr_2) = 0 \in L/N$ which implies that $lr_2 \in N$. Therefore, $N \leq^e L$ and L is a pure submodule of M. Hence, M is a purely extending module. □

Corollary 3.12. *A finitely generated torsion-free module over a principal ideal domain is an extending module.*

Corollary 3.13. ([16, Proposition 3.9.]) *A finitely generated torsion-free module over a principal ideal domain is a strongly extending module.*

Proposition 3.14. *Finitely generated flat R-module M over a principal ideal domain is purely extending.*

Proof. It follows from Proposition 3.11 and by the fact that a module over principal ideal domain is flat if and only if it is torsion-free. □

Proposition 3.15. *Every finitely generated torsion-free module over a prufer ring is a purely extending module.*

Proof. Let M be a finitely generated torsion-free module over a prufer ring R and N be a closed submodule of M. Then M/N is also torsion-free. In fact, if M/N is not torsion-free, then there exists $m \in M \backslash N$ such that $mr \in N$ for some $0 \neq r \in R$, which contradicts that N is a closed submodule of M. Since M/N is finitely generated torsion-free and R is prufer ring, M/N is flat (see [9, Proposition 4.20]). Hence, N is a pure submodule of M, which proves that M is purely extending. □

Corollary 3.16. *Every finitely generated flat module over a prufer ring is a purely extending module.*

Proposition 3.17. *A nonsingular ring R is purely extending if and only if every torsionless right R-module is flat.*

Proof. Since nonsingular purely extending ring R is purely Baer ring, R is purely extending if and only if every cyclic torsionless right R-module is flat (see [1, Theorem 1]). □

The following proposition tells about the behavior of closures of submodules of a module with purely extending property.

Proposition 3.18. *Let N be a submodule of the purely extending R-module M. Then*

(i) $Cl(Cl(N))$ is always a purely extending.

(ii) $Cl(N)$ is purely extending if $N \supseteq Cl(0)$.

Proof.

(i) Since $Cl(N) \supseteq Cl(0)$, so by Lemma 2.9 $Cl(Cl(N))$ is always closed in M. Thus $Cl(Cl(N))$ is a pure submodule of M and hence a purely extending module.

(ii) Since under the given conditions, $Cl(N)$ is closed, which implies that $Cl(N)$ is a pure submodule of M and hence a purely extending module. □

The following example shows that the endomorphism ring of purely extending module need not be purely extending.

Example 3.19. ([4, Example 2.3.]) Let $R = \begin{pmatrix} \mathbb{C} & \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{R} & \mathbb{C} \\ 0 & 0 & \mathbb{C} \end{pmatrix}$ and $e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Note that R_R is an extending module. Therefore, R_R is purely extending. Take $M = eR$, then $S = End_R(M) \cong \begin{pmatrix} \mathbb{C} & \mathbb{C} & 0 \\ 0 & \mathbb{R} & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Since M is a direct summand of R_R, M is purely extending. But S is not a right purely extending ring. In fact,it is easy to show that closed right ideal $\begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathbb{R} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is not essential in any pure right ideal of S_S.

It is well known that the ring R is called a right V-ring if every simple right R-modules are injective. Recall that a module M is said to be finitely cogenerated if for every set $\{S_i\}_{i\in I}$ of submodules of M, $\cap_{i\in I} S_i = 0$ implies that $\cap_{j\in J} S_j = 0$ for some finite set J of I.

Now we discuss the conditions under which the endomorphism ring of a module is purely extending.

Proposition 3.20.

(i) If M is a finitely generated projective right R-module over a regular ring, then $S = End_R(M)$ is purely extending.

(ii) If M is a finitely cogenerated right R-module over a right V-ring, then $S = End_R(M)$ is purely extending.

Proof.

(i) From [8, Theorem 1.7], the endomorphism ring S of a finitely generated projective R-module M is regular. Therefore, S is purely extending.

(ii) If M is a finitely cogenerated right R-module over a right V-ring R, then by [10, Proposition 2.14] M is endo-regular. Therefore, S is a regular ring so S is purely extending. □

Proposition 3.21. *Let R be a semisimple artinian ring. Then the endomorphism ring of every right R-module M is purely extending.*

Proof. Let R be a semisimple artinian ring. Then every R-module M is an endo-regular module which implies that $S = End_R(M)$ is a regular ring. Hence, S is a purely extending ring. □

4. Purely Essentially Baer Modules

Definition 4.1. An R-module M is called a purely essentially Baer module if for every left ideal I of $S = End_R(M)$, $r_M(I) = \{m \in M : \varphi(m) = 0, \forall\, \varphi \in I\}$ is essential in a pure submodule of M. Further R is called a right purely essentially Baer ring if R_R is a purely essentially Baer R-module.

Proposition 4.2. *Consider the following statements for a right R-module M:*

(i) M is a purely Baer module.

(ii) M is a purely extending module.

(iii) M is a purely essentially Baer module.

Then (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iii), but these implications are not reversible, in general.

Proof. *(i)* $\Rightarrow$ *(iii)* Let M be an R-module, $S = End_R(M)$ and I be a left ideal of S. By *(i)* $r_M(I)$ is a pure submodule of M, so M is a purely essentially Baer module. *(ii)* $\Rightarrow$ *(iii)* It is clear that $r_M(I) \le M$ for every left ideal I of S. Since by assumption, M is a purely extending module, $r_M(I)$ is essential in a pure submodule

of M.
$(iii) \not\Rightarrow (i)$ The $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ (where p is any prime) is a purely essentially Baer module while $\mathbb{Z}_{p^\infty}$ is not a purely Baer $\mathbb{Z}$-module.
$(iii) \not\Rightarrow (ii)$ Let $R = \begin{pmatrix} \mathbb{F} & 0 & \mathbb{F} \\ 0 & \mathbb{F} & \mathbb{F} \\ 0 & 0 & \mathbb{F} \end{pmatrix}$ the $\mathbb{F}$-subalgebra of the ring $M_{3\times 3}(\mathbb{F})$ (3 by 3 matrix ring over $\mathbb{F}$). Clearly, R is a left and right Artinian hereditary ring. Hence, R is a left and right nonsingular ring. So from [1, Theorem 5], R is a purely Baer ring. Thus, R_R is a purely Baer R-module. Hence, R_R is a purely essentially Baer R-module, while R_R is not purely extending. In fact, if R_R is purely extending R-module, then R_R is an extending R-module, but from [3, Example 5.5], R_R is not an extending R-module. □

In the following proposition, we prove when a purely essentially Baer module is a purely Baer module.

Proposition 4.3. *Let M be a nonsingular right R-module with $S = End_R(M)$. If M is a purely essentially Baer module, then M is a purely Baer module.*

Proof. Let M be a purely essentially Baer module and I be a left ideal S. Then $r_M(I) \leq^e P$, where P is a pure submodule of M. Let $U = \{r \in R : pr \in r_M(I)$ for $p \in P\}$. Then $U \leq^e R_R$ and $pU \subseteq r_M(I)$, so for each $\phi \in I$, $\phi(pU) = \phi(p)U = 0$. Since M is nonsingular, $\phi(p) = 0$ for each $\phi \in I$. Therefore, $r_M(I) = P$ is a pure submodule of M. Hence, M is a purely Baer module. □

We have seen that a purely essentially Baer module need not be essentially Baer module. In the following proposition, we show when these two notions are equivalent.

Proposition 4.4.

(i) *Let M be a pure split module with $S = End_R(M)$. Then M is a purely essentially Baer module if and only if M is an essentially Baer module.*

(ii) *Let R be a right noetherian ring and M be a finitely generated flat right R-module. Then M is a purely essentially Baer module if and only if it is an essentially Baer module.*

(iii) *Let R be a right pure semisimple ring. Then a right R-module M is a purely essentially Baer module if and only if M is an essentially Baer module.*

Proof.

(i) Let M be a purely essentially Baer module and I be a left ideal of S. Then $r_M(I) \leq^e P$ for some pure submodule P of M. Since M is pure split, P is a direct summand of M. Hence, M is an essentially Baer module.
The converse is clear.

(ii) Let M be a purely essentially Baer module and I be a left ideal of S. Then $r_M(I) \leq^e P$ for any pure submodule P of M. So by Lemma: 2.3, M/P is flat. Since by hypothesis, M/P is a finitely generated module and R is a right Noetherian ring, by Proposition 2.2 M/P is a projective module. Thus, $P \leq^{\oplus} M$. Hence, M is an essentially Baer module. The converse is clear from the definition.

(iii) The proof follows from the fact that for a pure semisimple ring R, every pure exact sequence of R-modules splits. □

Proposition 4.5. *Direct summand of a purely essentially Baer module is purely essentially Baer.*

Proof. Let $M = M_1 \oplus M_2$ be an R-module. Then $S = End_R(M) = \begin{pmatrix} S_1 & S_{12} \\ S_{21} & S_2 \end{pmatrix}$ where $S_i = End_R(M_i)$ for $i = 1, 2$ and $S_{ij} = Hom_R(M_j, M_i)$ for $i \neq j$, $i, j = 1, 2$. Let I be a left ideal of S_1 and J=$\{\sum_{i=1}^{n} f_i g_i$: $f_i \in S_{21}$ and $g_i \in I$ for all $n \in \mathbb{N}\}$, then $T = \begin{pmatrix} I & 0 \\ J & 0 \end{pmatrix}$ is clearly a left ideal of S. Since M is a purely essentially Baer module, $r_M(T) \leq^e N$ for some pure submodule N of M. Let $N = N_1 \oplus N_2$ such that $N_1 \leq^p M_1$ and $N_2 \leq^p M_2$. For any $(m_1 + m_2) \in M$, where $m_1 \in M_1$ and $m_2 \in M_2$, the element $m_1 + m_2 \in r_M(I)$ if and only if $m_1 \in r_{M_1}(I)$. Therefore, $r_M(T) = r_{M_1}(I) \oplus M_2 \leq^e N_1 \oplus N_2$, which implies that $r_{M_1}(I) \leq^e N_1$. Hence, M_1 is a purely essentially Baer module. □

Theorem 4.6. *Let M be an R-module with $S = End_R(M)$. Then the following statements are equivalent:*

(i) Every purely essentially Baer R-module is purely Baer;

(ii) Every purely extending R-module is purely Baer;

(iii) R is a regular ring.

Proof. $(i) \Rightarrow (ii)$ Let M be a purely extending module and I be a left ideal of S. Then $r_M(I)$ is essential in a pure submodule X of M, which implies that M is a purely essentially Baer module. Therefore, from (i) M is a purely Baer module.
$(ii) \Rightarrow (iii)$ Let M be an R-module and $E(M)$ be the injective hull of M. Then, the homomorphism $\phi : E(M) \to E(E(M)/M)$ defined by $\phi(h) = h + M$ for each $h \in E(M)$, can be extended by the endomorphism $\bar{\phi}$ of $E(M) \oplus E(E(M)/M)$ such that $Ker(\bar{\phi}) = M$. Since $E(M) \oplus E(E(M)/M)$ is a purely extending module, by (ii) it is a purely Baer module. Hence, M is pure in $E(M) \oplus E(E(M)/M)$, which implies that M is pure in $E(M)$. Therefore, M is an absolutely pure R-module. Hence, from [17, 37.6] R is a regular ring.
$(iii) \Rightarrow (i)$ It is clear. □

Proposition 4.7. Let $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$ (where Λ is an index set) be such that $Hom(M_\lambda, M_\mu) = 0$ for every $\lambda \neq \mu \in \Lambda$. Then M is a purely essentially Baer module if and only if each M_λ $(\lambda \in \Lambda)$ is purely essentially Baer.

Proof. If part is clear from Proposition 4.5.

For only if part, let each M_λ is a purely essentially Baer module and $S = End_R(M)$. Since $Hom_R(M_\lambda, M_\mu) = 0$ for every $\lambda \neq \mu \in \Lambda$, S is viewed as a diagonal matrix with S_λ ($\lambda \in \Lambda$) on its diagonal, where $S_\lambda = End_R(M_\lambda)$. Let T be a left ideal of S, then $r_M(T) = \bigoplus_{\lambda\in\Lambda} r_{M_\lambda}(T \cap S_\lambda)$. As each M_λ is a purely essentially Baer module, so $r_{M_\lambda}(T \cap S_\lambda) \leq^e X_\lambda$ for a pure submodule X_λ of M_λ. So, we get $r_M(T) \leq^e \bigoplus_{\lambda\in\Lambda} X_\lambda$. Since each X_λ is pure in M_λ, $\bigoplus_{\lambda\in\Lambda} X_\lambda$ is pure in $\bigoplus_{\lambda\in\Lambda} M_\lambda$. Hence, M is a purely essentially Baer module. □

Proposition 4.8. *Let N be a submodule of a purely essentially Baer module M. If $N \cap X$ is a pure submodule of N for each pure submodule X of M, then N is a purely essentially Baer.*

Proof. Let $T = End_R(N)$ and I be a left ideal of T. As M is a purely essentially Baer module, so $r_M(I) \leq^e X$ where X is a pure submodule of M. Now $r_N(I) = N \cap r_M(I)$, which is essential in X. From the assumption $N \cap r_M(I)$ is a pure submodule of N. Hence, N is a purely essentially Baer. □

Proposition 4.9. *A finitely generated $\mathbb{Z}$-module M is a purely essentially Baer module if M is a semisimple or torsion-free module.*

Proof. If M is a semisimple module, then it is obviously purely essentially Baer module. If M is a finitely generated torsion-free $\mathbb{Z}$-module, then $M \cong \mathbb{Z}^n$, $n \in \mathbb{N}$, which is a purely essentially Baer module. □

The converse of the above proposition need not be true, in general.

Example 4.10. $M = \mathbb{Z} \oplus \mathbb{Z}_p$ be a $\mathbb{Z}$-module, where p is prime. Clearly, M is a purely essentially Baer module but M is neither torsion-free nor semisimple.

Proposition 4.11. *For a finitely generated projective R-module M, the following statements are equivalent:*

(i) M is a purely essentially Baer module;

(ii) The endomorphism ring of M is left purely Baer ring.

Proof. $(i) \iff (ii)$ It follows from the fact that the endomorphism ring of a finitely generated projective module M is von Neumann regular.

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