KYUNGPOOK Math. J. 63(2023), 11-13
https://doi.org/10.5666/KMJ.2022.63.1.11
pISSN 1225-6951 eISSN 0454-8124

Noether Normalization Implies Full Form of Hilbert Nullstellensatz Theorem

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Abstract. We give a new proof for the full form of Hilbert's Nullstellensatz based on on integral extension and Noether's Normalization Lemma.

1. Introduction

Recall the statements of the the weak and strong forms Hilberts's Nullstellensatz and of Noether's Nomalisation Lemma.

Theorem 1.1. (**Hilbert's Nullstellensatz (weak form)**) *Let K be an algebraically closed field and M be a maximal ideal of $K[X_1,\ldots,X_n]$, then $M =< X_1 - a_1,\ldots,X_n - a_n >$ for some $a_i \in K$.*

Theorem 1.2. (**Hilbert's Nullstellensatz (full form)**) *Let K be an algebraically closed field and $f, g_1,\ldots,g_m \in K[X_1,\ldots,X_n]$, then there exists a natural number k such that $f^k \in< g_1,\ldots,g_m >$ if and only if for each $(a_1,\ldots,a_n) \in K^n$ with $g_i(a_1,\ldots,a_n) = 0$ for $1 \leq i \leq n$, implies that $f(a_1,\ldots,a_n) = 0$.*

Theorem 1.3. (**Noether's Normalization Lemma**) *Let K be an arbitrary field and $T = K[\alpha_1,\ldots,\alpha_n]$ be an integral domain, then there exist $\beta_1,\ldots,\beta_m$ in T such that $\beta_1,\ldots,\beta_m$ are algebraically independent over K and T is a finitely generated as a module over the subring $S := K[\beta_1,\ldots,\beta_m]$. In particular, T is integral over S and m is the transcendence degree of $K(\alpha_1,\ldots,\alpha_n)$ over K.*

Usually, the weak form of Hilbert's Nullstellensatz is proved by some technical and classical facts such as Noether's Normalization Lemma, Zariski's Lemma, Artin-Tate Lemma and the concept of G-domains and Hilbert rings. Good references for this are [5, §31], and the set of references it gives on p. 433; see also [1]. Proofs

Received February 24, 2022; accepted July 25, 2022.
2020 Mathematics Subject Classification: 12E05, 12F20, 13B21.
Key words and phrases: Field, Noether Normalization.
This work was supported by Research Council of Shahid Chamran University of Ahvaz (Ahvaz-Iran) Grant Number: SCU.MM1400.721.

of the full form of Hilbert's Nullstellensatz are usually depend on adding a new variable, or using Rabinowiteh's trick, or by the concept of Hilbert rings, see [7], [4] and [6], respectively. For the general version of Noether's Normalization see [2, Lemma 10.10.1] or [7, Theorem 14.14].

In this short note we prove that Noether's Normalization Lemma also implies the full form of Hilbert's Nullstellensatz Theorem.

Let us recall some standard definitions and facts from commutative ring theory which will be used in this note. For a ring R, the Jacobson radical of R is denoted by $J(R)$. It is clear that if $R \subseteq T$ is an integral extension of rings, then $J(R) = R \cap J(T)$. If $B \subseteq A$ is an algebraic extension of integral domains, and I be a nonzero ideal of A, then one can easily see that $I \cap B \neq 0$. In particular, if $R \subseteq T$ is an integral extension of integral domains with $J(T) \neq 0$, then $J(R) \neq 0$. Finally note that it is manifest that if K is an arbitrary field and $R = K[X_1, \ldots, X_n]$ is the polynomial ring of algebraically independent variables $X_1, \ldots, X_n$ over K, then $J(R) = 0$, for a more general result see [3].

2. The Proof

Proof. **Hilbert's Nullstellensatz (full form)**. Note that it is clear that if there exists a natural number k such that $f^k \in < g_1, \ldots, g_m >$, then for each $(a_1, \ldots, a_n) \in K^n$ with $g_i(a_1, \ldots, a_n) = 0$ for $1 \leq i \leq m$, we have $f(a_1, \ldots, a_n) = 0$. Conversely, assume that for each natural number k, we have $f^k \notin I := < g_1, \ldots, g_m >$. Therefore $I \neq R := K[X_1, \ldots, X_n]$ and so if $S := \{1, f, f^2, \ldots, f^k, \ldots\}$, then we infer that $I \cap S = \emptyset$. Hence there exists a prime ideal P of R such that $I \subseteq P$, $P \cap S = \emptyset$ and P is maximal with respect to these properties. We claim that P is a maximal ideal of R. Otherwise, note that if P is not a maximal ideal of R, then for each maximal ideal Q of R which contains P, we infer that $f \in Q$, by the maximality of P. Hence, $0 \neq f + P \in J(\frac{R}{P}) \neq 0$. Let $A := \frac{R}{P} = K[\alpha_1, \ldots, \alpha_n]$, where $\alpha_i = X_i + P$. Thus by our assumption A is an integral domain which is not a field and therefore A is not algebraic over K. Now by Noether's Normalization Lemma, there exist $\beta_1, \ldots, \beta_r$ in A such that $\beta_1, \ldots, \beta_r$ are algebraically independent over K and B is a finitely generated as a module over the subring $B := K[\beta_1, \ldots, \beta_r]$. Therefore $B \subseteq A$ is an integral extension of integral domains and since A is not algebraic over K we infer that $r \geq 1$. Now since $J(A) \neq 0$, we immediately conclude that $J(B) \neq 0$ which is absurd. Thus P is a maximal ideal of R. Therefore by the weak form of Hilbert Nullstellensatz, we infer that there exist $a_1, \ldots, a_n$ in K such that $P = < X_1 - a_1, \cdots, X_n - a_n >$. Since, $I \subseteq P$, we deduce that for each i, $g_i(a_1, \ldots, a_n) = 0$ and since $f \notin P$, we infer that $f(a_1, \ldots, a_n) \neq 0$ which is absurd. Thus there exists a natural k, such that $f^k \in I$ and hence we are done. □

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