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NEW CRITERIA FOR SUBORDINATION AND SUPERORDINATION OF MULTIVALENT FUNCTIONS ASSOCIATED WITH THE SRIVASTAVA-ATTIYA OPERATOR†

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Abstract. The purpose of the present paper is to obtain some subordination and superordination preserving properties with the sandwich-type theorems for multivalent functions in the open unit disk associated with Srivastava-Attiya operator. Moreover, applications for integral operators are also considered.

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1. Introduction

The concept of Löewner chain (subordination chain) plays a vital role in the field of Geometric Function Theory (GFT). It was initially deployed by researchers to find the estimates on the initial coefficients of normalised univalent functions, growth theorems and deriving univalence criterion [22, p. 164–175]. The concept of Löewner chain also serves as a tool in the study of differential subordination.

In what follows, let $\mathcal{H} = \mathcal{H}(\mathbb{D})$ denote the class of functions analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N} = \{1, 2, \cdots\}$, let $\mathcal{H}[a, n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}$. Let f and F be members of $\mathcal{H}$. The function f is said to be subordinate to F, or F is said to be superordinate to f, if there exists a function w analytic in $\mathbb{D}$, with $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathbb{D}$, such that $f(z) = F(w(z))$ $(z \in \mathbb{D})$. In such a case,

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we write $f \prec F$ or $f(z) \prec F(z)$ $(z \in \mathbb{D})$. If the function F is univalent in $\mathbb{D}$, then we have (cf. [16]) $f \prec F \iff f(0) = F(0)$ and $f(\mathbb{D}) \subset F(\mathbb{D})$. Though there were several results on differential implications, a systematic study on this was started by Miller [16]. The first order differential subordination is defined as follows:

Definition 1.1. [16] Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in $\mathbb{D}$. If p is analytic in $\mathbb{D}$ and satisfies the differential subordination

$$\phi(p(z), zp'(z)) \prec h(z) \;\; (z \in \mathbb{D}), \tag{1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant if $p \prec q$ for all p satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1) is said to be the best dominant.

Definition 1.2. [17] Let $\varphi : \mathbb{C}^2 \to \mathbb{C}$ and let h be analytic in $\mathbb{D}$. If p and $\varphi(p(z), zp'(z))$ are univalent in $\mathbb{D}$ and satisfy the differential superordination

$$h(z) \prec \varphi(p(z), zp'(z)) \;\; (z \in \mathbb{D}), \tag{2}$$

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solutions of the differential superordination, or more simply a subordinant if $q \prec p$ for all p satisfying (2). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (2) is said to be the best subordinant.

Definition 1.3. We denote by $\mathcal{Q}$ the class of functions f that are analytic and injective on $\overline{\mathbb{D}} \backslash E(f)$, where

$$E(f) = \left\{ \zeta \in \partial\mathbb{D} : \lim_{z \to \zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{D} \backslash E(f)$.

For the reason, we investigate some properties of the class $\mathcal{A}$ of normalised analytic functions of the form $f(z) = z + a_2z^2 + a_3z^3 + a_4z^4 + \cdots$ defined on the unit disk $\mathbb{D}$ and its subclasses like the class of univalent functions $\mathcal{S}$, starlike functions $\mathcal{S}^*$ and convex functions $\mathcal{K}$ are among the most studies classes in GFT. The class of β-convex functions, defined below, makes a bridge passes between the classes of starlike and convex functions. A functions $f \in \mathcal{H}$ with $f(0) = 0$ and $f'(0) \neq 0$ is said to be β-convex function (not necessary normalised), if it satisfies the following condition:

$$\Re \left[(1 - \beta) \frac{zf'(z)}{f(z)} + \beta \left(1 + \frac{zf''(z)}{f'(z)} \right) \right] > 0 \;\; (\beta \in \mathbb{R};\; z \in \mathbb{D})$$

and we denote this class by $\mathcal{M}^*_\beta$. The class of β-convex functions was introduced by Mocanu [14]. We also note [19] that all β-convex functions univalent and starlike, and

$$\mathcal{M}^*_\beta \subset \mathcal{M}^*_\alpha \subset \mathcal{M}^*_0 \;\; \left(0 \leq \frac{\alpha}{\beta} \leq 1 \right).$$

Moreover, we note that $\mathcal{M}_1^*$ is the class of normalised convex functions in $\mathbb{D}$.

It is interesting to consider the transformation of a class of functions and determine the nature of the resultant function and the class in which it belongs. A generalisation to the class of analytic functions $\mathcal{A}$ can be given as follows: let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \ (p, n \in \mathbb{N};\ z \in \mathbb{D})$$

which are analytic in the open unit disk $\mathbb{D}$. For any complex number κ, we define the multiplier transformations I_λ^κ of functions $f \in \mathcal{A}_p$ by

$$I_\lambda^\kappa f(z) = z^p + \sum_{k=p+n}^{\infty} \left(\frac{k+\lambda}{p+\lambda}\right)^\kappa a_k z^k \ (p, n \in \mathbb{N};\ \lambda \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ \mathbb{Z}_0^- := \{0, -1, -2, \cdots\}). \tag{3}$$

The operator I_λ^κ for $p = 1$ was introduced and studied by Srivastava and Attiya [25], which was called as the Srivastava-Attiya operator [23]. Several interesting operators as special cases of the Srivastava-Attiya operator have been widely studied by (for examples) Cho and Srivastava [7], Jung *et al.*[11], Owa and Srivastava [20], Sălăgean [24], Uralegaddi and Somanatha [33]. Furthermore, it is easily verified from the definition of the operator I_λ^κ that

$$z(I_\lambda^\kappa f(z))' = (\lambda + p) I_\lambda^{\kappa+1} f(z) - \lambda I_\lambda^\kappa f(z). \tag{4}$$

Making use of the principle of subordination, various subordination theorems involving certain integral operators for analytic functions in $\mathbb{D}$ were investigated Bulboacă [3], Miller *et al.* [18] and Owa and Srivastava [21]. Recently, Miller and Mocanu [17] also considered differential superordinations, as the dual problem of differential subordinations (see also [4, 8, 9]. Kumar *et al.* [12] gave an unified approach to study the properties of all these linear operators by considering the aspect that these operators satisfy recurrence relation of some common forms. They studied properties of integral transforms in a similar way. Furthermore, the study of the subordinaton-preserving properties and their dual problems for various operators is a significant role in pure and applied mathematics. For some recent developments one may refer to [1, 5, 6].

The aim of the present paper, motivated by the works mentioned above, is to systematically investigate the subordination- and superordination- preserving results of the multiplier transformation I_λ^κ defined by (3) with certain differential sandwich-type theorems as consequences of the results presented here. We also consider interesting applications to the integral operator. Our results give interesting new properties, and together with other papers that appeared in the last years could emphasize the perspective of the importance of differential subordinations and multiplier transformations. We also note that, in recent years, several authors obtained many interesting results involving various linear and nonlinear operators associated with differential subordinations and their dual problems (for details, see [26, 27, 29, 30, 31, 32, 34]). We use the concept of subordination chain and the following lemmas in our present investigation:

Lemma 1.4. [15] *Let $p \in \mathcal{Q}$ with $p(0) = a$ and let $q(z) = a + a_n z^n + \cdots$ be analytic in $\mathbb{D}$ with $q(z) \not\equiv a$* and $n \in \mathbb{N}$. *If q is not subordinate to p, then there exist points $z_0 = r_0 e^{i\theta} \in \mathbb{U}$* and *$\zeta_0 \in \partial\mathbb{D} \backslash E(f)$, for which $q(\mathbb{D}_{r_0}) \subset p(\mathbb{D})$, $q(z_0) = p(\zeta_0)$ and $z_0 q'(z_0) = m\zeta_0 p'(\zeta_0)$ $(m \geq n)$.*

A function $L(z,t)$ defined on $\mathbb{D} \times [0,\infty)$ is the subordination chain (or Löwner chain) if $L(\cdot,t)$ is analytic and univalent in $\mathbb{D}$ for all $t \in [0,\infty)$, $L(z,\cdot)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{D}$ and

$$L(z,s) \prec L(z,t) \ (z \in \mathbb{D};\ 0 \leq s < t).$$

Lemma 1.5. [17] *Let $q \in \mathcal{H}[a,1]$ and let $\varphi : \mathbb{C}^2 \to \mathbb{C}$. Also set $\varphi(q(z), zq'(z)) \equiv h(z)$ $(z \in \mathbb{D})$. If $L(z,t) = \varphi(q(z), tzq'(z))$ is a subordination chain and $p \in \mathcal{H}[a,1] \cap \mathcal{Q}$, then $h(z) \prec \varphi(p(z), zp'(z))$ $(z \in \mathbb{D})$ implies that $q(z) \prec p(z)$ $(z \in \mathbb{D})$. Furthermore, if $\varphi(q(z), zq'(z)) = h(z)$ has a univalent solution $q \in \mathcal{Q}$, then q is the best subordinant.*

Lemma 1.6. [22] *The function $L(z,t) = a_1(t)z + \cdots$ with*

$$a_1(t) \neq 0 \text{ and } \lim_{t\to\infty} |a_1(t)| = \infty.$$

Suppose that $L(\cdot,t)$ is analytic in $\mathbb{D}$ for all $t \geq 0$, $L(z,\cdot)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{D}$. If $L(z,t)$ satisfies

$$\Re\left\{\frac{\frac{z\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right\} > 0 \ \ (z \in \mathbb{D};\ 0 \leq t < \infty)$$

and

$$|L(z,t)| \leq K_0 |a_1(t)| \ \ (|z| < r_0 < 1;\ 0 \geq t < \infty))$$

for some positive constants K_0 and r_0, then $L(z,t)$ is a subordination chain.

2. Subordination Results

Firstly, we begin by proving the following subordination theorem involving the multiplier transformation I_λ^κ defined by (3).

Theorem 2.1. Let $f, g \in \mathcal{A}_p$. Suppose also that

$$\frac{I_\lambda^\kappa g(z)}{z^{p-1}} \in \mathcal{M}_\beta^* \quad (\beta \geq 0;\ \lambda \geq 0;\ z \in \mathbb{D}) \tag{5}$$

Then the following subordination relation:

$$\left[\frac{I_\lambda^\kappa f(z)}{z^{p-1}}\right]^{1-\beta} \left[\frac{I_\lambda^{\kappa+1} f(z)}{z^{p-1}}\right]^{\beta} \prec \left[\frac{I_\lambda^\kappa g(z)}{z^{p-1}}\right]^{1-\beta} \left[\frac{I_\lambda^{\kappa+1} g(z)}{z^{p-1}}\right]^{\beta} \ \ (z \in \mathbb{D}) \tag{6}$$

implies that

$$\frac{I_\lambda^\kappa f(z)}{z^{p-1}} \prec \frac{I_\lambda^\kappa g(z)}{z^{p-1}} \quad (z \in \mathbb{D}).$$

Moreover, the function $\frac{I_\lambda^\kappa g(z)}{z^{p-1}}$ is the best dominant.

Proof. Let us define the functions F and G by

$$F(z) := \frac{I_\lambda^\kappa f(z)}{z^{p-1}} \quad \text{and} \quad G(z) := \frac{I_\lambda^\kappa g(z)}{z^{p-1}} \quad (f, g \in \mathcal{A}_p;\ z \in \mathbb{D}). \tag{7}$$

By using the equation (4) and (7) and also, by a simple calculation, we have

$$\frac{I_\lambda^{\kappa+1} g(z)}{z^{p-1}} = \frac{(\lambda + p - 1)G(z) + zG'(z)}{\lambda + p}. \tag{8}$$

Hence, combining (7) and (8), we obtain

$$\left[\frac{I_\lambda^\kappa g(z)}{z^{p-1}}\right]^{1-\beta} \left[\frac{I_\lambda^{\kappa+1} g(z)}{z^{p-1}}\right]^{\beta} = G(z) \left[\frac{\lambda + p - 1 + \frac{zG'(z)}{G(z)}}{\lambda + p}\right]^{\beta} \tag{9}$$

Thus, from (9), we need to prove the following subordination implication:

$$\begin{aligned} F(z) \left[\frac{\lambda + p - 1 + \frac{zF'(z)}{F(z)}}{\lambda + p}\right]^{\beta} \prec G(z) \left[\frac{\lambda + p - 1 + \frac{zG'(z)}{G(z)}}{\lambda + p}\right]^{\beta} \quad (z \in \mathbb{D}) \\ \Longrightarrow \quad F(z) \prec G(z)\ (z \in \mathbb{D}). \end{aligned} \tag{10}$$

Since $G \in \mathcal{M}^*(\beta)$, without loss of generality, we can assume that G satisfies the conditions of Theorem 2.1 on the closed disk $\overline{\mathbb{D}}$ and

$$G'(\zeta) \neq 0\ (\zeta \in \partial\mathbb{D}).$$

If not, then we replace F and G by

$$F_r(z) = F(rz) \text{ and, } G_r(z) = G(rz),$$

respectively, where $0 < r < 1$ and then G_r is univalent on $\overline{\mathbb{D}}$. Since

$$F_r(z) \left[\frac{\lambda + p - 1 + \frac{zF_r'(z)}{F_r(z)}}{\lambda + p}\right]^{\beta} \prec G_r(z) \left[\frac{\lambda + p - 1 + \frac{zG_r'(z)}{G_r(z)}}{\lambda + p}\right]^{\beta} \quad (z \in \mathbb{D}),$$

where

$$F_r(z) = F(rz) \quad (0 < r < 1;\ z \in \mathbb{D}),$$

we would then prove that

$$F_r(z) \prec G_r(z) \quad (0 < r < 1;\ z \in \mathbb{D}),$$

and by letting $r \to 1^-$, we obtain

$$F(z) \prec G(z) \quad (z \in \mathbb{D}).$$

If we suppose that the implication (9) is not true, that is,

$$F(z) \not\prec G(z) \quad (z \in \mathbb{D}),$$

then, from Lemma 1.4, there exist points

$$z_0 \in \mathbb{D} \quad \text{and} \quad \zeta_0 \in \partial\mathbb{D}$$

such that

$$F(z_0) = G(\zeta_0) \quad \text{and} \quad z_0 F'(z_0) = m\zeta_0 G'(\zeta_0) \ \ (m \geq 1). \tag{11}$$

To prove the implication (10), we define the function

$$L : \mathbb{D} \times [0, \infty) \longrightarrow \mathbb{C}$$

by

$$\begin{aligned} L(z,t) &= G(z) \left[\frac{\lambda + p - 1 + (1+t)\frac{zG'(z)}{G(z)}}{\lambda + p} \right]^{\beta} \\ &= a_1(t) z + \cdots , \end{aligned}$$

and we will show that $L(z,t)$ is a subordination chain. At first, we note that $L(z,t)$ is analytic in $|z| < r < 1$, for sufficient small $r > 0$ and for all $t \geq 0$. We also have that $L(z,t)$ is continuously differentiable on $[0, \infty)$ for each $|z| < r < 1$. A simple calculation shows that

$$a_1(t) = \frac{\partial L(0,t)}{\partial z} = G'(0) \left[\frac{\lambda + p + t}{\lambda + p} \right]^{\beta}.$$

Hence we obtain

$$a_1(t) \neq 0 \quad (t \geq 0)$$

and also we can see that

$$\lim_{t \to \infty} |a_1(t)| = \infty.$$

While, by a direct computation of $L(z,t)$, we have

$$\Re \left\{ \frac{\frac{z\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}} \right\} = \frac{\lambda + p - 1}{\beta} + \frac{(1+t)}{\beta} \Re \left[(1-\beta)\frac{zG'(z)}{G(z)} + \beta \left(1 + \frac{zG''(z)}{G'(z)} \right) \right]. \tag{12}$$

By using the assumption of Theorem 2.1 condition $\beta > 0$ to (12), we obtain

$$\Re \left\{ \frac{\frac{z\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}} \right\} > 0 \quad (z \in \mathbb{D};\ 0 \leq t < \infty),$$

which completes the proof of the first condition of Lemma 1.6. Moreover, we have

$$\begin{aligned} \left| \frac{L(z,t)}{a_1(t)} \right|^{1/\beta} &= \left| \frac{G(z)}{G'(0)} \right|^{1/\beta} \left| \frac{\lambda + p - 1 + (1+t)\frac{zG'(z)}{G(z)}}{\lambda + p + t} \right| \\ &= \left| \frac{G(z)}{G'(0)} \right|^{1/\beta} \left| \frac{zG'(z)}{G(z)} + \frac{(\lambda + p - 1)\left(1 - \frac{zG'(z)}{G(z)}\right)}{\lambda + p + t} \right| \end{aligned}$$

$$\le \left|\frac{G(z)}{G'(0)}\right|^{1/\beta}\left(\left|\frac{zG'(z)}{G(z)}\right| + \frac{|\lambda+p-1|\left|1+\frac{zG'(z)}{G(z)}\right|}{|\lambda+p+t|}\right).$$

Since G is univalent in $\mathbb{D}$, We have the following sharp growth and distortion results [21]:

$$\frac{r}{(1+r)^2} \le |G(z)| \le \frac{r}{(1-r)^2}\ (|z| = r < 1) \tag{13}$$

and

$$\frac{1-r}{(1+r)^3} \le |G'(z)| \le \frac{1+r}{(1-r)^3}\ (|z| = r < 1). \tag{14}$$

Hence, by applying the equations (13) and (14) to (12), we can find easily an upper bound for the right-hand side of (12). Thus the function $L(z,t)$ satisfies the second condition of Lemma 1.6, which proves that $L(z,t)$ is a subordination chain. In particular, we note from the definition of subordination chain that

$$L(z,0) \prec L(z,t) \quad (z \in \mathbb{D};\ t \ge 0).$$

Now, by using the definition of $L(z,t)$ and the relation (11), we obtain

$$\begin{aligned} L(\zeta_0,t) &= G(\zeta_0)\left[\frac{\lambda+p-1+(1+t)\frac{\zeta_0 G'(\zeta_0)}{G(\zeta_0)}}{\lambda+p}\right]^{\beta} \\ &= F(z_0)\left[\frac{\lambda+p-1+\frac{z_0F'(z_0)}{F(z_0)}}{\lambda+p}\right]^{\beta} \\ &= \left[\frac{I_\lambda^\kappa f(z_0)}{z_0^{p-1}}\right]^{1-\beta}\left[\frac{I_\lambda^{\kappa+1} f(z_0)}{z_0^{p-1}}\right]^{\beta} \in L(\mathbb{D},0) \end{aligned}$$

by virtue of the subordination condition (5). This contradicts the above observation that

$$L(\zeta_0,t) \notin L(\mathbb{D},0).$$

Therefore, the subordination condition (6) must imply the subordination given by (9). Considering $F = G$, we see that the function G is the best dominant. This evidently completes the proof of Theorem 2.1. □

It should be noted that for $p = 1 = n, \lambda = 0$ and $\kappa = 0$, we have $I_0^0 f(z) = f(z), I_1^0 f(z) = zf'(z), I_2^0 f(z) = z(zf''(z) + f'(z))$. In view of Theorem 2.1, we have the following result:

Corollary 2.1. *Let $f \in \mathcal{A}$ and $g \in \mathcal{K}$. Then $f \prec g$, whenever*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\beta} f(z) \prec \left(\frac{zg'(z)}{g(z)}\right)^{\beta} g(z),\ \beta \ge 0.$$

In particular, $zf' \prec zg'$ implies $f \prec g$.

Next, we give another subordination property by using the equation (3) in Theorem 2.2 below:

Theorem 2.2. Let $f, g \in \mathcal{A}_p$ and suppose that

$$\frac{I_\lambda^\kappa g(z)}{z^{p-1}} \in \mathcal{M}_1^* \ (\lambda \geq 0;\ z \in \mathbb{D}). \tag{15}$$

Then the following subordination relation:

$$\beta \frac{I_\lambda^{\kappa+1} f(z)}{z^{p-1}} + (1-\beta)\frac{I_\lambda^\kappa f(z)}{z^{p-1}} \prec \beta \frac{I_\lambda^{\kappa+1} g(z)}{z^{p-1}} + (1-\beta)\frac{I_\lambda^\kappa g(z)}{z^{p-1}} \ (0 \leq \beta \leq 1;\ z \in \mathbb{D})$$

implies that

$$\frac{I_\lambda^\kappa f(z)}{z^{p-1}} \prec \frac{I_\lambda^\kappa g(z)}{z^{p-1}} \ (z \in \mathbb{D}).$$

Moreover, the function $\frac{I_\lambda^\kappa g(z)}{z^{p-1}}$ is the best dominant.

Proof. Let us define the functions F and G as (2.3) and by using the equation (4) to (7), we have (8). Hence, combining (7) and (8), we obtain

$$\beta \frac{I_\lambda^{\kappa+1} g(z)}{z^{p-1}} + (1-\beta)\frac{I_\lambda^\kappa g(z)}{z^{p-1}} = G(z)\left(\frac{\beta\left[\lambda + p - 1 + \frac{zG'(z)}{G(z)}\right]}{\lambda + p} + 1 - \beta\right) \tag{16}$$

Thus, from (16), we need to prove the following subordination implication:

$$F(z)\left(\frac{\beta\left[\lambda + p - 1 + \frac{zF'(z)}{F(z)}\right]}{\lambda + p} + 1 - \beta\right) \prec G(z)\left(\frac{\beta\left[\lambda + p - 1 + \frac{zG'(z)}{G(z)}\right]}{\lambda + p} + 1 - \beta\right)$$
$$\Longrightarrow F(z) \prec G(z) \ (z \in \mathbb{D}). \tag{17}$$

Without loss of generality as in the proof of Theorem 2.1, we can assume that G satisfies the conditions of Theorem 2.1 on the closed disk $\overline{\mathbb{D}}$ and

$$G'(\zeta) \neq 0 \ (\zeta \in \partial\mathbb{D}).$$

To prove the implication (17), we consider the function

$$L : \mathbb{D} \times [0, \infty) \longrightarrow \mathbb{C}$$

by

$$\begin{aligned} L(z,t) &= G(z)\left(\frac{\beta\left[\lambda + p - 1 + (1+t)\frac{zG'(z)}{G(z)}\right]}{\lambda + p} + 1 - \beta\right) \\ &= a_1(t) z + \cdots, \end{aligned}$$

and we want to prove that $L(z,t)$ is a subordination chain. But, the remaining part of the proof in Theorem 2.2 is similar to that of Theorem 2.1 and so we omit the detailed proof. □

We next consider dual problems of Theorem 2.1, in the sense that the subordinations are replaced by superordinations.

Theorem 2.3. Let $f, g \in \mathcal{A}_p$. Suppose that the condition (5) is satisfied and the function

$$\left[\frac{I_\lambda^\kappa f(z)}{z^{p-1}}\right]^{1-\beta}\left[\frac{I_\lambda^{\kappa+1} f(z)}{z^{p-1}}\right]^{\beta}$$

is univalent and $I_\lambda^\kappa f(z)/z^{p-1} \in \mathcal{H}[0,1] \cap \mathcal{Q}$. Then the following subordination relation:

$$\left[\frac{I_\lambda^\kappa g(z)}{z^{p-1}}\right]^{1-\beta}\left[\frac{I_\lambda^{\kappa+1} g(z)}{z^{p-1}}\right]^{\beta} \prec \left[\frac{I_\lambda^\kappa f(z)}{z^{p-1}}\right]^{1-\beta}\left[\frac{I_\lambda^{\kappa+1} f(z)}{z^{p-1}}\right]^{\beta} \quad (z \in \mathbb{D}) \tag{18}$$

implies that

$$\frac{I_\lambda^\kappa g(z)}{z^{p-1}} \prec \frac{I_\lambda^\kappa f(z)}{z^{p-1}} \ (z \in \mathbb{D}).$$

Moreover, the function $\frac{I_\lambda^\kappa g(z)}{z^{p-1}}$ is the best subordinant.

Proof. Let us define the functions F and G by, respectively, (7). By using (7), we have

$$\begin{aligned}\left[\frac{I_\lambda^\kappa g(z)}{z^{p-1}}\right]^{1-\beta}\left[\frac{I_\lambda^{\kappa+1} g(z)}{z^{p-1}}\right]^{\beta} &= G(z)\left[\frac{\lambda+p-1+\frac{zG'(z)}{G(z)}}{\lambda+p}\right]^{\beta} \\ &=: \varphi(G(z), zG'(z)).\end{aligned} \tag{19}$$

Here, we note that the function G is univalent in $\mathbb{D}$ by the condition (5). Next, we prove that the subordination condition (18) implies that

$$F(z) \prec G(z) \ (z \in \mathbb{D}) \tag{20}$$

for the functions F and G defined by (7). Now considering the function $L(z,t)$ defined by

$$L(z,t) := G(z)\left[\frac{\lambda+p-1+t\frac{zG'(z)}{G(z)}}{\lambda+p}\right]^{\beta} \quad (z \in \mathbb{D};\ 0 \le t < \infty).$$

we can prove easily that $L(z,t)$ is a subordination chain as in the proof of Theorem 2.1. Therefore according to Lemma 1.5, we conclude that the superordination condition (18) must imply the superordination given by (20). Furthermore, since the differential equation (19) has the univalent solution G, it is the best subordinant of the given differential superordination. Therefore we complete the proof of Theorem 2.3. □

The proof of Theorem 2.4 below is similar to that of Theorem 2.3, and so the details may be omitted.

Theorem 2.4. Let $f, g \in \mathcal{A}_p$. Suppose that the condition (15) is satisfied and the function

$$\beta\frac{I_\lambda^{\kappa+1}f(z)}{z^{p-1}} + (1-\beta)\frac{I_\lambda^\kappa f(z)}{z^{p-1}}$$

is univalent in $\mathbb{D}$ and $I_\lambda^\kappa f(z)/z^{p-1} \in \mathcal{H}[0,1] \cap \mathcal{Q}$. Then the following subordination relation:

$$\beta\frac{I_\lambda^{\kappa+1}g(z)}{z^{p-1}} + (1-\beta)\frac{I_\lambda^\kappa g(z)}{z^{p-1}} \prec \beta\frac{I_\lambda^{\kappa+1}f(z)}{z^{p-1}} + (1-\beta)\frac{I_\lambda^\kappa f(z)}{z^{p-1}} \ (0 \le \beta \le 1;\ z \in \mathbb{D})$$

implies that

$$\frac{I_\lambda^\kappa g(z)}{z^{p-1}} \prec \frac{I_\lambda^\kappa f(z)}{z^{p-1}} \quad (z \in \mathbb{D}).$$

Moreover, the function $\frac{I_\lambda^\kappa g(z)}{z^{p-1}}$ is the best subordinant.

For $p = 1 = n, \lambda = 0$ and $\kappa = 0$, in view of Theorem 2.1, we have the following result:

Corollary 2.2. *Let $f, g \in \mathcal{A}$ and $\beta z^2 f'' + zf'$ is univalent in $\mathbb{D}$. Then $g \prec f$, whenever*

$$\beta z^2 g'' + zg' \prec \beta z^2 f'' + zf',\ \beta \ge 0.$$

If we combine Theorem 2.1 and Theorem 2.3, and Theorem 2.2 and Theorem 2.4, respectively, then we obtain the following sandwich-type theorems:

Theorem 2.5. Let $f, g_k \in \mathcal{A}_p$ $(k = 1, 2)$. Suppose that

$$\frac{I_\lambda^\kappa g_k(z)}{z^{p-1}} \in \mathcal{M}_\beta^* \quad (\beta \ge 0;\ \lambda \ge 0;\ z \in \mathbb{D};\ k = 1, 2).$$

and the function

$$\left[\frac{I_\lambda^\kappa f(z)}{z^{p-1}}\right]^{1-\beta}\left[\frac{I_\lambda^{\kappa+1}f(z)}{z^{p-1}}\right]^\beta$$

is univalent and $I_\lambda^\kappa f(z)/z^{p-1} \in \mathcal{H}[0,1] \cap \mathcal{Q}$. Then the following subordination relation:

$$\begin{aligned}\left[\frac{I_\lambda^\kappa g_1(z)}{z^{p-1}}\right]^{1-\beta}\left[\frac{I_\lambda^{\kappa+1}g_1(z)}{z^{p-1}}\right]^\beta &\prec \left[\frac{I_\lambda^\kappa f(z)}{z^{p-1}}\right]^{1-\beta}\left[\frac{I_\lambda^{\kappa+1}f(z)}{z^{p-1}}\right]^\beta \\ &\prec \left[\frac{I_\lambda^\kappa g_2(z)}{z^{p-1}}\right]^{1-\beta}\left[\frac{I_\lambda^{\kappa+1}g_2(z)}{z^{p-1}}\right]^\beta \ (z \in \mathbb{D})\end{aligned}$$

implies that

$$\frac{I_\lambda^\kappa g_1(z)}{z^{p-1}} \prec \frac{I_\lambda^\kappa f(z)}{z^{p-1}} \prec \frac{I_\lambda^\kappa g_2(z)}{z^{p-1}} \quad (z \in \mathbb{D}).$$

Moreover, the functions $\frac{I_\lambda^\kappa g_1(z)}{z^{p-1}}$ and $\frac{I_\lambda^\kappa g_2(z)}{z^{p-1}}$ is the best subordinant and the best dominant.

Theorem 2.6. Let $f, g_k \in \mathcal{A}_p$ $(k = 1, 2)$. Suppose that

$$\frac{I_\lambda^\kappa g_k(z)}{z^{p-1}} \in \mathcal{M}_1^* \ (\lambda \geq 0;\ z \in \mathbb{D};\ k = 1, 2).$$

and the function

$$\beta \frac{I_\lambda^{\kappa+1} f(z)}{z^{p-1}} + (1-\beta) \frac{I_\lambda^\kappa f(z)}{z^{p-1}}$$

is univalent in $\mathbb{D}$ and $I_\lambda^\kappa f(z)/z^{p-1} \in \mathcal{H}[0,1] \cap \mathcal{Q}$. Then the following subordination relation:

$$\beta \frac{I_\lambda^{\kappa+1} g_1(z)}{z^{p-1}} + (1-\beta) \frac{I_\lambda^\kappa g_1(z)}{z^{p-1}} \prec \beta \frac{I_\lambda^{\kappa+1} f(z)}{z^{p-1}} + (1-\beta) \frac{I_\lambda^\kappa f(z)}{z^{p-1}}$$
$$\prec \beta \frac{I_\lambda^{\kappa+2} g_2(z)}{z^{p-1}} + (1-\beta) \frac{I_\lambda^\kappa g_2(z)}{z^{p-1}} \ (0 \leq \beta \leq 1;\ z \in \mathbb{D})$$

implies that

$$\frac{I_\lambda^\kappa g_1(z)}{z^{p-1}} \prec \frac{I_\lambda^\kappa f(z)}{z^{p-1}} \prec \frac{I_\lambda^\kappa g_2(z)}{z^{p-1}} \quad (z \in \mathbb{D}).$$

Moreover, the functions $\frac{I_\lambda^\kappa g_1(z)}{z^{p-1}}$ and $\frac{I_\lambda^\kappa g_2(z)}{z^{p-1}}$ is the best subordinant and the best dominant.

3. A related integral transform

Next, we consider the generalized Libera integral operator F_ν $(\nu > -p)$ defined by (cf. [2, 10, 19, 20]).

$$F_\nu(f)(z) := \frac{\nu+p}{z^\nu} \int_0^z t^{\nu-1} f(t) dt \quad (f \in \mathcal{A}_p;\ \Re\{\nu\} > -p). \tag{21}$$

Now, we obtain the following sandwich-type result involving the integral operator defined by (21).

Theorem 3.1. Let $f, g_k \in \mathcal{A}_p (k = 1, 2)$. Suppose also that

$$\frac{I_\lambda^\kappa F_\nu(g_k)(z)}{z^{p-1}} \in \mathcal{M}_\beta^* \ (\nu \geq 0;\ \beta \geq 0;\ z \in \mathbb{D};\ k = 1, 2)$$

and the function

$$\left[\frac{I_\lambda^\kappa F_\nu(f)(z)}{z^{p-1}}\right]^{1-\beta} \left[\frac{I_\lambda^\kappa f(z)}{z^{p-1}}\right]^\beta$$

is univalent in $\mathbb{D}$ and $I_\lambda^\kappa F_\nu(f)(z)/z^{p-1} \in \mathcal{H}[0,1] \cap \mathcal{Q}$. Then the following subordination relation:

$$\left[\frac{I_\lambda^\kappa F_\nu(g_1)(z)}{z^{p-1}}\right]^{1-\beta} \left[\frac{I_\lambda^\kappa g_1(z)}{z^{p-1}}\right]^\beta \prec \left[\frac{I_\lambda^\kappa F_\nu(f)(z)}{z^{p-1}}\right]^{1-\beta} \left[\frac{I_\lambda^\kappa f(z)}{z^{p-1}}\right]^\beta$$
$$\prec \left[\frac{I_\lambda^\kappa F_\nu(g_2)(z)}{z^{p-1}}\right]^{1-\beta} \left[\frac{I_\lambda^\kappa g_2(z)}{z^{p-1}}\right]^\beta \ (z \in \mathbb{D})$$

implies that

$$\frac{I_\lambda^\kappa F_\nu(g_1)(z)}{z^{p-1}} \prec \frac{I_\lambda^\kappa F_\nu(f)(z)}{z^{p-1}} \prec \frac{I_\lambda^\kappa F_\nu(g_2)(z)}{z^{p-1}} \quad (z \in \mathbb{D})$$

Moreover, the functions $\frac{I_\lambda^\kappa F_\nu(g_1)(z)}{z^{p-1}}$ and $\frac{I_\lambda^\kappa F_\nu(g_2)(z)}{z^{p-1}}$ are the best subordinant and the best dominant.

Proof. Let us define the functions F and G by

$$F(z) := \frac{I_\lambda^\kappa F_\nu(f)(z)}{z^{p-1}} \quad \text{and} \quad G(z) := \frac{I_\lambda^\kappa F_\nu(g_k)(z)}{z^{p-1}} \quad (f, g \in \mathcal{A}_p;\ z \in \mathbb{D};\ k = 1, 2). \tag{22}$$

From the definition of the integral operator F_ν defined by (22), we obtain

$$z(I_\lambda^\kappa F_\nu(f)(z))' = (\nu + p) I_\lambda^\kappa f(z) - \nu I_\lambda^\kappa F_\nu(f)(z).$$

Hence, by using (22) and the same method as in the proof of Theorem 2.5, we can prove Theorem 3.1 and so we omit the details involved. Finally, we obtain the following sandwich-type Theorem 3.2 below by using a similar method as in the proof of Theorem 2.6. □

Theorem 3.2. Let $f, g_k \in \mathcal{A}_p$ $(k = 1, 2)$. Suppose that

$$\frac{I_\lambda^\kappa F_\nu(g_k)(z)}{z^{p-1}} \in \mathcal{M}_1^* \ (\nu \geq 0;\ z \in \mathbb{D})$$

and the function

$$\beta \frac{I_\lambda^\kappa f(z)}{z^{p-1}} + (1 - \beta) \frac{I_\lambda^\kappa F_\nu(f)(z)}{z^{p-1}} \ \ (0 \leq \beta \leq 1;\ z \in \mathbb{D})$$

is univalent in $\mathbb{D}$ and $I_\lambda^\kappa F_\nu(f)(z)/z^{p-1} \in \mathcal{H}[0, 1] \cap \mathcal{Q}$. Then the following subordination relation:

$$\begin{aligned} \beta \frac{I_\lambda^\kappa g_1(z)}{z^{p-1}} + (1 - \beta) \frac{I_\lambda^\kappa F_\nu(g_1)(z)}{z^{p-1}} &\prec \beta \frac{I_\lambda^\kappa f(z)}{z^{p-1}} + (1 - \beta) \frac{I_\lambda^\kappa F_\nu(f)(z)}{z^{p-1}} \\ &\prec \beta \frac{I_\lambda^\kappa g_2(z)}{z^{p-1}} + (1 - \beta) \frac{I_\lambda^\kappa F_\nu(g_2)(z)}{z^{p-1}} \end{aligned}$$

implies that

$$\frac{I_\lambda^\kappa F_\nu(g_1)(z)}{z^{p-1}} \prec \frac{I_\lambda^\kappa F_\nu(f)(z)}{z^{p-1}} \prec \frac{I_\lambda^\kappa F_\nu(g_2)(z)}{z^{p-1}} \quad (z \in \mathbb{D}).$$

Moreover, the functions $\frac{I_\lambda^\kappa F_\nu(g_1)(z)}{z^{p-1}}$ and $\frac{I_\lambda^\kappa F_\nu(g_2)(z)}{z^{p-1}}$ are the best subordinant and the best dominant.

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