J. Appl. Math. & Informatics Vol. **41**(2023), No. 2, pp. 363 - 372
https://doi.org/10.14317/jami.2023.363

BOUNDS ON THE GROWTH RATE FOR THE KUO PROBLEM

S. LAVANYA*, V. GANESH, G. VENKATA RAMANA REDDY

Abstract. We consider Kuo problem of hydrodynamic stability which deals with incompressible, inviscid, parallel shear flows in the β-plane. For this problem, we derived instability region without any approximations and which intersects with Howard semi-circle region under certain condition. Also, we derived upper bound for growth rate and amplification factor of an unstable mode and proved Howard's conjecture.

AMS subject classification : 76E05.
Key words and phrases : Parallel shear flows, inviscid, incompressible fluids, β-plane.

1. Introduction

Kuo problem of hydrodynamic stability deals with incompressible, inviscid, parallel shear flows in the β-plane. When the Coriolis force become zero then it reduces to standard Rayleigh problem of hydrodynamic stability. For this problem, [8] derived Rayleigh inflexion point theorem . [7] derived semi-circle theorem in the upper half of $c_r - c_i$ plane whose diameter is same as range of the basic velocity profile. [10] extended and derived an instability region which includes Cariolis force. [15] deduced semi-circle for zonal shear flows. [12] derived a bound for estimate for the growth rate of an unstable mode. [18] derived growth rate of an unstable mode. [2] proved Howard's conjecture for standard Rayleigh problem, [18] proved Howard's conjecture for the baroclinic zonal flows. [15] derived an upper bound for the growth rate of a unstable mode for free surface. [1] derived two parabolic instability regions under some approximations for the standard Rayleigh problem. [9] extended their work to Kuo problem and derived two parabolic instability regions under some approximations. [13]obtained a sufficient condition for stability. [17] derived a parabolic

Received June 6, 2022. Revised August 15, 2022. Accepted September 30, 2022.
*Corresponding author.

instability region which depend on minimum of the basic velocity should be positive for Kuo problem. [11] observed that the time spend by the wave and phase speed are related. Hence it is necessary to know the location of eigen values. Howard semi-circle depends on the basic velocity profile only. It does not give boundedness of the growth rate of an unstable mode. However, boundedness of growth rate is proved here. Amplification factor tends to zero as wave number tends to zero is known as Howard's conjecture(cf. [7]). But it is likelihood that growth rate tends to zero as wave number tends to infinity. [3], [14] proved Howard's conjecture for Taylor Goldstein problem, [4] proved for extended Taylor Goldstein problem. [6], [12] derived estimates for growth rate of an unstable mode. In this paper, we derived a parabolic instability region which does not depend on any approximations like [9], [17]. New parabolic instability region depends on shear term, wave number apart from minimum and maximum of velocity profile. New instability region is unbounded and it will be useful if they intersect with Howard semi-circle. We have derived condition for the parabolic instability region intersect with Howard semi-circle region. Also, we derived an estimate for the growth rate and upper bound for the amplification factor of an unstable mode and proved Howard's conjecture namely, growth rate approaches to zero as wave number approaches to infinity.

2. Kuo Problem:

Kuo problem [cf. [8]] of hydrodynamic stability is given by

$$\phi'' - \left[\frac{U'' - \beta}{U - c} + k^2\right]\phi = 0, \tag{1}$$

with boundary conditions

$$\phi(z_1) = 0 = \phi(z_2). \tag{2}$$

Here U is the basic velocity profile, ϕ is the complex eigen function, $k > 0$ is the wave number , c is the complex phase velocity and β is the derivative of Cariolis force in the latitudinal direction.

Using the transformation $\phi = (U - c)^{\frac{1}{2}}\varphi$, then (1), (2) becomes

$$[(U - c)\varphi']' - k^2(U - c)\varphi - \left(\frac{U''}{2} - \beta\right)\varphi - \frac{\frac{(U')^2}{4}}{(U - c)}\varphi = 0, \tag{3}$$

with boundary conditions

$$\varphi(z_1) = 0 = \varphi(z_2). \tag{4}$$

Theorem 2.1. *For an unstable mode $c_i > 0$, we have ${c_i}^2 \leq \lambda\,[c_r + U_{\max}]$, where*

$$\lambda = \frac{|U'|^2 \max}{2\,|3U_{\min} + U_{\max}|\left[\frac{\pi^2}{(z_2 - z_1)^2} + k^2\right]} \text{ and } U_m = \frac{U_{\min} + U_{\max}}{2}.$$

Proof. Multiplying (3) with φ^*, integrating using by parts and applying (4), we get

$$\int_{z_1}^{z_2}(U-c)\left[|\varphi'|^2+k^2|\varphi|^2\right]dz+\int_{z_1}^{z_2}\left(\frac{U''}{2}-\beta\right)|\varphi|^2dz \\ +\frac{1}{4}\int_{z_1}^{z_2}\frac{(U')^2}{(U-c)}|\varphi|^2dz=0.$$

Equating real parts, we get

$$\int_{z_1}^{z_2}(U-c_r)\left[|\varphi'|^2+k^2|\varphi|^2\right]dz+\int_{z_1}^{z_2}\left(\frac{U''}{2}-\beta\right)|\varphi|^2dz \\ +\frac{1}{4}\int_{z_1}^{z_2}\frac{(U')^2}{|U-c|^2}(U-c_r)|\varphi|^2dz=0. \tag{5}$$

Equating imaginary parts, we get

$$-c_i\int_{z_1}^{z_2}\left[|\varphi'|^2+k^2|\varphi|^2\right]dz+\frac{c_i}{4}\int_{z_1}^{z_2}\frac{(U')^2}{|U-c|^2}|\varphi|^2dz=0. \tag{6}$$

Multiplying (6) with $\frac{(c_r+U_m)}{c_i}$ and subtracting from (5), we get

$$\int_{z_1}^{z_2}(U+U_m)\left[|\varphi'|^2+k^2|\varphi|^2\right]dz+\int_{z_1}^{z_2}\left(\frac{U''}{2}-\beta\right)|\varphi|^2dz \\ +\frac{1}{4}\int_{z_1}^{z_2}\frac{(U')^2}{|U-c|^2}(U-2c_r-U_m)|\varphi|^2dz=0. \tag{7}$$

Multiplying (6) with $\left(\frac{U_{\max}-U_{\min}}{2c_i}\right)$ and adding with (5), we get

$$\int_{z_1}^{z_2}\left(U-c_r-\frac{U_{\max}}{2}+\frac{U_{\min}}{2}\right)\left[\left|\varphi'\right|^2+k^2\left|\varphi\right|^2\right]dz \\ +\int_{z_1}^{z_2}\left(\frac{U''}{2}-\beta\right)|\varphi|^2dz \\ +\frac{1}{4}\int_{z_1}^{z_2}\frac{(U')^2}{|U-c|^2}\left(U-c_r+\frac{U_{\max}}{2}-\frac{U_{\min}}{2}\right)|\varphi|^2dz=0.$$

Since $U_{\min}<c_r<U_{\max}$, and $\left(U-c_r-\frac{U_{\max}}{2}+\frac{U_{\min}}{2}\right)<0$ and hence dropping the first term, we get

$$\int_{z_1}^{z_2}\left(\frac{U''}{2}-\beta\right)|\varphi|^2dz\geq\frac{1}{4}\int_{z_1}^{z_2}\frac{(U')^2}{|U-c|^2}\left(c_r-U+\frac{U_{\min}}{2}-\frac{U_{\max}}{2}\right)|\varphi|^2dz. \tag{8}$$

Substituting (8) in (7), we get

$$\int_{z_1}^{z_2} (U + U_m) \left[|\varphi'|^2 + k^2 |\varphi|^2 \right] dz$$
$$+ \frac{1}{4} \int_{z_1}^{z_2} \frac{(U')^2}{|U - c|^2} \left(-c_r - U_m + \frac{U_{\min}}{2} - \frac{U_{\max}}{2} \right) |\varphi|^2 dz \leq 0.$$

Since $\frac{1}{|U-c|^2} \leq \frac{1}{c_i^2}$ and $U_m = \frac{U_{\min}+U_{\max}}{2}$ and using Rayleigh-Ritz inequality,

$$\int_{z_1}^{z_2} \left|\phi'\right|^2 dz \geq \frac{\pi^2}{(z_2 - z_1)^2} \int_{z_1}^{z_2} |\phi|^2 \, dz,$$

we get

$$c_i^2 \leq \lambda \left[c_r + U_{\max} \right], \tag{9}$$

where

$$\lambda = \frac{|U'|^2 \max}{2 \left| 3U_{\min} + U_{\max} \right| \left[\frac{\pi^2}{(z_2 - z_1)^2} + k^2 \right]}.$$

□

Unlike [9], [17], newly derived parabolic instability region does not depend on any approximations. Now we shall check the condition for intersection with Howard semi circle.

Remark 2.1. The result is true for constant basic velocity profile and exchange flows also. Even if the minimum velocity is zero, the result is true. The result is true for all basic velocity profiles.

Theorem 2.2. *If $\lambda < \lambda_c$, where $\lambda_c = (U_{\min} + 3U_{\max}) - 2\sqrt{2U_{\max}^2 + 2U_{\min}U_{\max}}$ then the parabola $c_i^2 \leq \lambda \left[c_r + U_{\max} \right]$ intersect Howard semi circle.*

Proof. Howard semi circle [cf. [7]] is given by

$$\left[c_r - \frac{U_{\min} + U_{\max}}{2} \right]^2 + c_i^2 \leq \left[\frac{U_{\max} - U_{\min}}{2} \right]^2. \tag{10}$$

Substituting (9) in (10) we get

$$c_r^2 + \left[\lambda - U_{\min} - U_{\max} \right] c_r + \left[U_{\min}U_{\max} + \lambda U_{\max} \right] \leq 0.$$

For real roots, the discriminant part of the above equation should be greater than or equal to zero and hence we have

$$\lambda^2 - (2U_{\min} + 6U_{\max}) \lambda + \left[U_{\max} - U_{\min} \right]^2 \geq 0.$$

Solving for λ, and λ with positive sign implies $c_r < U_{\min}$ and hence, we have

$$\lambda_c = (U_{\min} + 3U_{\max}) - 2\sqrt{2U_{\max}^2 + 2U_{\min}U_{\max}}.$$

Hence if $\lambda < \lambda_c$, where $\lambda_c = (U_{\min} + 3U_{\max}) - 2\sqrt{2U_{\max}^2 + 2U_{\min}U_{\max}}$ then the parabola given in (9) will intersect semi-circle given in (10). □

Now we shall illustrate the example.

Example 1. Let us consider the exchange flow $U = \left[z - \frac{1}{2}\right], z \in [0,1]$, as considered in [16], [5]. In this case $U_{\min} = -0.5, U_{\max} = 0.5, \lambda = 0.0506, \lambda_c = 1.$ Since $\lambda < \lambda_c$, the parabola ${c_i}^2 \leq 0.0506\,[c_r + 0.5]$, intersects with Howard semi-circle.

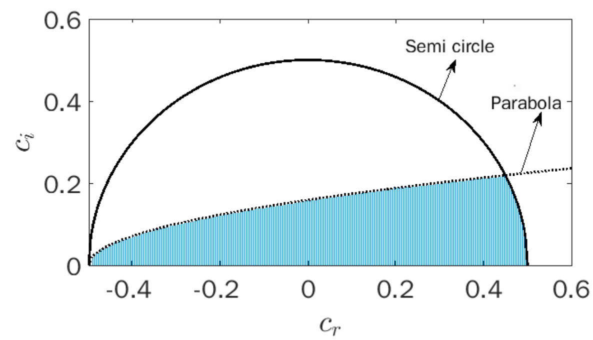

FIGURE 1. c_r vs c_i (Intersection of parabola with Howard semi circle)

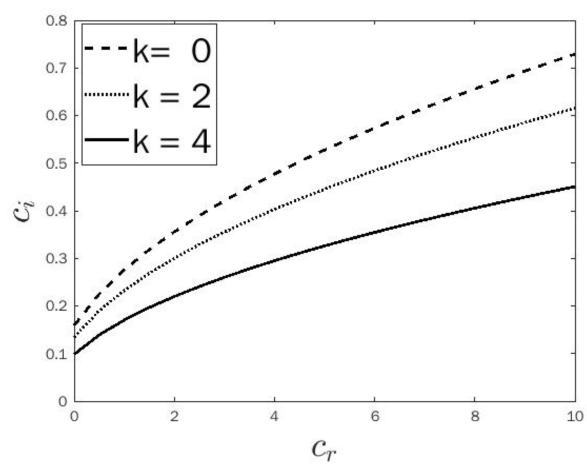

FIGURE 2. c_r vs c_i (parabolic instability region for different values of k)

From fig.2, it is clear that as wave number k increases, the instability region is further reduced. Example 1 is not valid for [17] because minimum velocity is not positive.

Example 2. Let us consider the flow $U = \sin z$, $z = [1, 2]$. In this case

$$U_{\min} = 0.8415,\ U_{\max} = 0.9975,\ \lambda = \ 0.00298804, \lambda_c = 0.003175.$$

Since $\lambda < \lambda_c$, the parabola ${c_i}^2 \leq 0.002988\,[c_r + 0.9975]$, intersects with Howard's semi-circle.

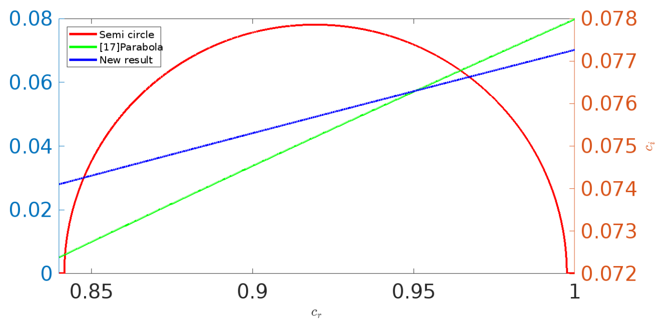

FIGURE 3. $c_r\,\mathrm{vs}\,c_i$ (Intersection of parabolas with Howard semi circle)

Theorem 2.3. *Upper bounds for the growth rate is given by*

$$kc_i \leq \frac{(U'' - \beta)_{\max}}{2k\left[\frac{\pi^2}{(z_2 - z_1)^2 k^2} + 1\right]}.$$

Proof. Multiplying (1) with ϕ^*, integrating and applying (2),we get

$$\int_{z_1}^{z_2} \left[|\phi'|^2 + k^2|\phi|^2\right] dz + \int_{z_1}^{z_2} \frac{U'' - \beta}{(U - c)}|\phi|^2 dz = 0. \tag{11}$$

Equating real parts, we get

$$\int_{z_1}^{z_2} \left[|\phi'|^2 + k^2|\phi|^2\right] dz + \int_{z_1}^{z_2} \frac{U'' - \beta}{|U - c|^2}\left(U - c_r\right)|\phi|^2 dz = 0;$$

$$\int_{z_1}^{z_2} \left[|\phi'|^2 + k^2|\phi|^2\right] dz = \left|- \int_{z_1}^{z_2} \frac{U'' - \beta}{|U - c|^2}\left(U - c_r\right)|\phi|^2 dz\right|.$$

i,e.,

$$\int_{z_1}^{z_2} \left[|\phi'|^2 + k^2|\phi|^2\right] dz \leq \int_{z_1}^{z_2} \frac{U'' - \beta}{|U - c|^2}\left(U - c_r\right)|\phi|^2 dz.$$

Using the inequality $\frac{(U - c_r)}{|U - c|^2} \leq \frac{1}{2c_i}$, we get

$$\int_{z_1}^{z_2} |\phi'|^2\, dz + k^2 \int_{z_1}^{z_2} |\phi|^2 dz \leq \int_{z_1}^{z_2} \frac{(U'' - \beta)}{2c_i}|\phi|^2 dz. \tag{12}$$

By using Rayleigh Ritz inequality,

$$\int_{z_1}^{z_2} \left|\phi^{'}\right|^2 dz \geq \frac{\pi^2}{(z_2 - z_1)^2} \int_{z_1}^{z_2} |\phi|^2 \, dz,$$

we have

$$\left[\frac{\pi^2}{(z_2 - z_1)^2} + k^2\right] \leq \frac{|U'' - \beta|_{\max}}{2c_i};$$

i.e.,

$$kc_i \leq \frac{|U'' - \beta|_{\max}}{2k\left[\frac{\pi^2}{(z_2-z_1)^2 k^2} + 1\right]}.$$

□

Theorem 2.4. *Upper bound for the growth rate is given by*

$$kc_i \leq \frac{|U'' - \beta|_{\max}}{2k}.$$

Proof. In (12) the first term is non negative, hence dropping, we get

$$k^2 \int_{z_1}^{z_2} |\phi|^2 dz \leq \int_{z_1}^{z_2} \frac{(U'' - \beta)}{2c_i} |\phi|^2 dz;$$

i.e.,

$$kc_i \leq \frac{|U'' - \beta|_{\max}}{2k}.$$

□

Theorem 2.5. *Growth rate kc_i tends to zero as $k \to \infty$.*

Proof. From (12), the first term is non-negative, hence dropping we get

$$k^2 \leq \frac{|U'' - \beta|_{\max}}{2c_i};$$

i.e.,

$$kc_i \leq \frac{|U'' - \beta|_{\max}}{2k}.$$

$$\lim_{k \to \infty} kc_i = 0.$$

□

Theorem 2.6. *The bounds of amplification factor c_i is*

$$0 < c_i \leq \left[\frac{2k^2 (U'' - \beta)(U - U_m) + (U'' - \beta)^2}{\frac{\pi^4}{(z_2-z_1)^4} - k^4}\right]_{\max}^{\frac{1}{2}},$$

where

$$0 \leq k \leq \frac{\pi}{(z_2 - z_1)}.$$

Proof.
Multiplying (1) with $(\phi^*)''$, integrating and applying (2), we get

$$\int_{z_1}^{z_2} |\phi''|^2 \, dz - \int_{z_1}^{z_2} \left(\frac{U'' - \beta}{U - c} + k^2 \right) \phi \, (\phi^*)'' \, dz = 0. \tag{13}$$

Taking complex conjugate of (1), we get

$$(\phi^*)'' = \left(\frac{U'' - \beta}{U - c^*} + k^2 \right) \phi^*. \tag{14}$$

Substituting (14) in (13), we get

$$\begin{aligned} &\int_{z_1}^{z_2} |\phi''|^2 \, dz - k^4 \int_{z_1}^{z_2} |\phi|^2 dz - k^2 \int_{z_1}^{z_2} \frac{U'' - \beta}{U - c} |\phi|^2 dz \\ &-k^2 \int_{z_1}^{z_2} \frac{U'' - \beta}{U - c^*} |\phi|^2 dz - \int_{z_1}^{z_2} \frac{(U'' - \beta)^2}{|U - c|^2} |\phi|^2 dz = 0. \end{aligned}$$

Equating real part, we get

$$\begin{aligned} \int_{z_1}^{z_2} |\phi''|^2 \, dz - k^4 \int_{z_1}^{z_2} |\phi|^2 dz - 2k^2 \int_{z_1}^{z_2} \frac{(U'' - \beta)(U - c_r)}{|U - c|^2} |\phi|^2 dz \\ - \int_{z_1}^{z_2} \frac{(U'' - \beta)^2}{|U - c|^2} |\phi|^2 dz = 0. \end{aligned} \tag{15}$$

Equating imaginary parts of (11), we get

$$c_i \int_{z_1}^{z_2} \frac{U'' - \beta}{|U - c|^2} |\phi|^2 dz = 0. \tag{16}$$

Multiplying (16) with $2k^2 \frac{(c_r - U_m)}{c_i}$ and subtracting from (15), we get

$$\begin{aligned} \int_{z_1}^{z_2} |\phi''|^2 \, dz - k^4 \int_{z_1}^{z_2} |\phi|^2 dz - 2k^2 \int_{z_1}^{z_2} \frac{(U'' - \beta)(U - U_m)}{|U - c|^2} |\phi|^2 dz \\ - \int_{z_1}^{z_2} \frac{(U'' - \beta)^2}{|U - c|^2} |\phi|^2 dz = 0. \end{aligned}$$

Using Rayleigh-Ritz inequality and $\frac{1}{|U-c|^2} \leq \frac{1}{c_i^2}$, we get

$$\frac{\left[\frac{\pi^4}{(z_2 - z_1)^4} - k^4 \right] c_i^2 - \left[2k^2 (U'' - \beta)(U - U_m) + (U'' - \beta)^2 \right]}{c_i^2} \int_{z_1}^{z_2} |\phi|^2 dz \leq 0.$$

$$\left[\frac{\pi^4}{(z_2 - z_1)^4} - k^4 \right] c_i^2 \leq \left[2k^2 (U'' - \beta)(U - U_m) + (U'' - \beta)^2 \right].$$

$${c_i}^2 \leq \frac{2k^2 (U'' - \beta)(U - U_m) + (U'' - \beta)^2}{\left[\frac{\pi^4}{(z_2 - z_1)^4} - k^4 \right]}.$$

$$c_i \le \left[\frac{2k^2 \left(U'' - \beta\right)(U - U_m) + \left(U'' - \beta\right)^2}{\frac{\pi^4}{(z_2 - z_1)^4} - k^4} \right]_{\max}^{\frac{1}{2}},$$

where

$$0 \le k \le \frac{\pi}{(z_2 - z_1)}.$$

3. Conclusion

In this paper, we consider Kuo Problem of hydrodynamic stability which deals with incompressible, inviscid, parallel shear flows in β-plane. For this problem, we derived parabolic instability region which intersects with Howard semi-circle under some condition. Unlike [9], [17] new parabolic instability region does not depend on any approximations. When $\beta = 0$, new result reduces to parabolic instability region for standard Rayleigh problem of hydrodynamic stability. Furthermore, we derived upper bound for the growth rate and amplification factor of an unstable mode and Howard's conjecture namely growth rate approaches to zero as wave number approaches to infinity.

Conflicts of interest : The author's declare no conflict of interest.

Data availability : Not applicable

Acknowledgments : The author's are thankful to reviewer for valuable suggestion to improve the manuscript.

References

1. M.B. Banerjee, J.R. Gupta, and M. Subbiah, *On reducing Howard's semi-circle for homogeneous shear flows*, J. Math. Anal. Appl. **130** (1988), 398-402.
2. M.B. Banerjee, R.G. Shandil, Vinay kanwar, *A proof of Howard's Conjecture in homogeneous parallel shear flow*, Proc. Indian Acad. Sci. (Math Sci.) **104** (1994), 593-595.
3. M.B. Banerjee, R.G. Shandil, Jyoti Prakash, Balraj Singh Bandral and Prem Lal, *On Howard's Conjecture in heterogeneous shear flow instability of Modified s-waves*, Indian J. Pure App. Math. **28** (1997), 825-830.
4. V. Ganesh, M. Subbiah, *On Upper bounds for the growth rate in the extended Taylor-Goldstein Problem of hydrodynamic stability*, Proc. Indian Acad. Sci.(Math. Sci.) **119** (2009), 119-135.
5. J.R. Gupta, R.G. Shandil, S.D. Rana, *On the limitations of the complex wave velocity in the instability problem of heterogeneous shear flows*, J. Math. Anal. Appl. **144** (1989), 367-376.
6. F.J. Hickernell, *An upper bound on the growth rate of a linear instability in a homogeneous shear flow*, Stud. in Appl. Math. **72** (1985), 87-93.
7. L.N. Howard, *Note on a paper of John*, W. Miles, J. Fluid Mech. **10** (1961), 509.
8. H.L. Kuo, *Dynamic instability of two-dimensional non-divergent flow in a barotropic atmosphere*, Journal of Metereology **6** (1949), 105-122.
9. M. Padmini, M. Subbiah, *Note on Kuo's problem*, J. Math. Anal. Appl. **173** (1993), 659-665.

10. J. Pedlosky, *Geophysical Fluid Dynamics*, Springer-Verlog, New York, Berlin, 1979.
11. L.J. Pratt, H.E. Deese, S.P. Murray and W. Johnes, *Continuous dynamical modes in straits having arbitrary cross section with application to the Bab al Mandab*, J. Physical Oceanography **30** (2000), 2515-2534.
12. D.H. Sattinger, *On the Rayleigh Problem in hydrodynamic stability*, SIAM J. Appl. Math. **15** (1967), 419.
13. R.G. Shandil, *Linear Instability of Inviscid Incompressible zonal flows on a β-Plane*, Ganita **51** (2000), 165-168.
14. R.G. Shandil, *Jagjith Singh, On Howard's Conjecture in heterogeneous Shear flow Problem*, Proc. Indian Acad. Sci. (Math.Sci.) **113** (2003), 451-456.
15. B.K. Shivamoggi, David. K. Rollini, *Linear stability theory of zonal shear flows with a free surface*, Geophys. Astrophys. Fluid Dynamics **95** (2001), 31-53.
16. M. Subbiah, V. Ganesh, *Bounds on the phase speed and growth rate of the extended Taylor Goldstein Problem*, Fluid Dynamic Research **40** (2008), 364-377.
17. R. Thenmozhy, N. Vijayan, *On the Linear Stability of Shear Flows in the β-Plane*, International Journal of Mathematics Trends and Technology **66** (2020), 10-16.
18. Vinay Kanwar, A.K. Sinha, *A new upper bound on the growth rate of linear instability of baroclinic zonal flows in the two-layer model on a β-Plane*, Phys. Fluids **11** (1999), 2925-2927.

S. Lavanya received M.Sc. from Osmania University, from Hyderabad, and Part time, Research Program, Department of Mathematics, Koneru Lakshmaiah Education Foundation, Vaddeswaram,522502, Andhra pradesh, India. Her research interest include Hydrodynamic stability. Currently working as faculty in Engineering Department, University of Technology and Applied Sciences, Muscat, Sultanate of Oman.

Research Scholar, Department of Mathematics, Koneru Lakshmaiah Education Foundation, Vaddeswaram,522502, Andhra pradesh, India.
e-mail: `lavanya.sharmas@gmail.com`

V. Ganesh received Doctorate degree from Madurai Kamaraj University, India. He did Post Doctorate work with Faculty of Mechanical Engineering and Automation, Zhejiang Sci-Tech University, Hangzhou, P.R. China. He worked in a project funded by Natural Science foundation, ZSTU, P.R. China. He has more than 24 years of teaching experience. He has authored and co-authored 25 journal papers to his credit. His area of interest include Hydrodynamic stability, Magneto hydrodynamic stability. Currently working as faculty in Engineering Department, University of Technology And Applied Sciences, Muscat, Sultanate of Oman.

Engineering Department, University of Technology and Applied Sciences, Muscat, 133, Sultanate of Oman.
e-mail: `profvganesh@gmail.com`

G. Venkata Ramana Reddy received Doctorate degree Philosophy from Sri. Venkateshwara University, India. He is currently working as Professor in the Department of Mathematics, Koneru Lakshmaiah Education Foundation . His research interests area is Fluid Mechanics. He published 150 papers in national and international reputed journals.

Department of Mathematics, Koneru Lakshmaiah Education Foundation, Vaddeswaram, 522502, Andhra pradesh, India.
e-mail: `gvrr1976@kluniversity.in`