J. Appl. Math. & Informatics Vol. **41***(2023), No. 2, pp. 345 - 362*
https://doi.org/10.14317/jami.2023.345

PROUHET ARRAY MORPHISM AND PARIKH q-MATRIX

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Abstract. Prouhet string morphism has been a well investigated morphism in different studies on combinatorics on words. In this paper we consider Prouhet array morphism for the images of binary picture arrays in terms of Parikh $\mathtt{q}-$matrices. We state the formulae to calculate $\mathtt{q}-$counting scattered subwords of the images of any arrays under this array morphism and also investigate the properties such as $\mathtt{q}-$weak ratio property and commutative property under this array morphism in terms of Parikh $\mathtt{q}-$ matrices of arrays.

AMS Mathematics Subject Classification : 68R15, 68Q42, 68R99, 68Q45, 68Q15, 68Q25.
Key words and phrases : Subwords, Parikh matrix, Parikh $\mathtt{q}-$matrix, picture array, Prouhet array morphism.

1. Introduction

The Parikh matrix mapping [12] was originally introduced as an extension of the Parikh vector [14]. Parikh matrices provide more structural information about words than Parikh vectors and is a useful tool in studying subword occurrences. Since the introduction of Parikh matrices a number of studies on various properties related to Parikh matrices have been extensively investigated in [1, 15, 18]. A notable improvement seems to be the use of morphisms that distinguish amiable binary words based on their Parikh matrices. Specifically, properties of subwords and Parikh matrices of image words under some mappings known as morphisms on words have been widely analyzed in [2, 3, 16]. Since the characterization of words with the same Parikh matrix is an open problem, Atanasiu [2] exploited the Parikh matrices of their images under Istrail morphism to separate $M-$equivalent words. However in [3], Atanasiu showed that these morphisms distinguish the sequences in many classes of $M-$equivalent words

Received May 12, 2022. Revised October 31, 2022. Accepted November 3, 2022.
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but not totally. Teh shown in [19] that no morphism can completely separate $M-$equivalent words over a given alphabet.

As an extension of the Parikh matrix, O.Egecioglu et al. introduced the Parikh $\mathtt{q}-$matrix in [10], which transfers words to matrices with polynomial elements in $\mathtt{q}$. The Parikh $\mathtt{q}-$matrix provides more details about words than Parikh matrix. Two words having the same Parikh matrix can have different Parikh $\mathtt{q}-$matrices. Since the Parikh $\mathtt{q}-$matrix mapping is not injective, several investigations on injectivity in relation to the Parikh $\mathtt{q}-$matrix have been analyzed [5, 6, 7]. A conspicuous development seems to be the use of certain morphisms which separate $M-$equivalent words by their Parikh matrices but not completely. This leads to consider Prouhet morphism on words to identify $\mathtt{q}-$equivalent binary words based on their Parikh $\mathtt{q}-$matrix.

Mathematics and computer science have developed an interest in picture languages. In addition, it is integral to the theory of image analysis, a well-established and active field with numerous applications. A two-dimensional language or picture language has been defined since 1967 by extending the results and techniques of string (one-dimensional) languages to the two-dimensional case. Pictures are generally used to understand things better than other modes. So, there is a lot of technology to compute pictures using computers. As a result of this computation of pictures, picture-generating devices were introduced. A picture array is an arrangement of symbols from a finite alphabet in rows and columns. For example, a digitized binary picture array describing the chessboard pattern with each pixel having white square or black square is shown in Figure 1.

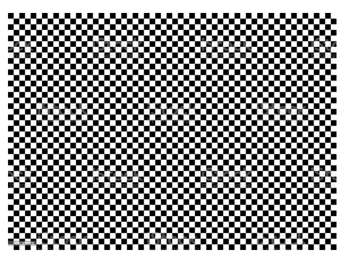

FIGURE 1. Chessboard Pattern

A picture language is a set of such picture arrays. To recognize or generate these picture arrays, various formal models are employed. These strategies were derived from the problems associated with image processing and pattern recognition. Extensive investigations on picture languages studying the problem of generation of such languages and other properties have been done by different researchers. In addition, several combinatorial properties of picture arrays have been extensively investigated by several researchers [8, 9]. In [17], two types of Parikh matrices are defined, namely a row Parikh matrix and a column Parikh matrix for a picture array. These matrices extend the notion of a Parikh matrix to arrays. The notion of M-ambiguity of a picture array is introduced in [17] by considering two picture arrays to be M-equivalent if their row Parikh matrices and their column Parikh matrices are the same. More specifically, conditions that ensure $M-$ambiguity are established for binary and ternary arrays. The problem of reconstruction of two dimensional binary images has been studied [11] based on Parikh matrices. In [13], Nithya Vani et al. introduced Prouhet array morphism and studied the properties of images of binary and ternary arrays under this array morphism interms of Parikh matrices. Based on the notion of Parikh $\mathtt{q}-$matrix and Parikh matrices of picture array, in [4] two types of Parikh $\mathtt{q}-$matrices are defined, namely row Parikh $\mathtt{q}-$matrix and column Parikh $\mathtt{q}-$matrix, leading to the notions of $\mathtt{q}-$row and $\mathtt{q}-$column equivalences of two arrays and also several properties relating to $\mathtt{q}-$ambiguity including conditions for $\mathtt{q}-$ambiguity of row or column products for binary arrays are derived. Different kinds of morphisms have been considered in the study of combinatorics on words. A conspicuous development seems to be the use of certain morphisms which separate $M-$equivalent words by their Parikh matrices but not completely. This leads us to consider about Prouhet morphism on words and picture arrays to identify $\mathtt{q}-$equivalent binary words and binary picture arrays based on their Parikh $\mathtt{q}-$matrix.

The remainder of this paper is structured as follows. Section 2 provides basic definitions which are used in subsequent sections. In section 3, we state the formulae to calculate $\mathtt{q}-$counting of scattered subwords of the image of any picture array under Prouhet array morphism. We also establish several properties of image of picture arrays under this array morphism in the sense of Parikh $\mathtt{q}-$matrices and scattered subwords.

2. Preliminaries

In this section we recollect fundamental definitions and notations of words, scattered subwords, Parikh matrix and Parikh $\mathtt{q}-$matrix for two dimensional words.

2.1. Subwords. Consider an alphabet $\Sigma = \{a_1, a_2, \cdots, a_k\}$ and the set of all words over Σ is Σ^*. For any word $x \in \Sigma^*$, the length of x is denoted by $|x|$. An ordered alphabet is an alphabet $\Sigma = \{a_1, a_2, \cdots, a_k\}$ with the total order relation $a_1 < a_2 < \cdots < a_k$ and it is denoted by Σ_k. The empty word is denoted by λ. A word $y \in \Sigma^*$ is called a *scattered subword* of x if there exist words $y_1, y_2, \cdots, y_n$ and $x_0, x_1, x_2, \cdots, x_n$ over Σ such that $y = y_1y_2 \cdots y_n$ and $x = x_0y_1x_1y_2 \cdots y_nx_n$. The number of occurrences of the word y as a scattered subword of the word x is denoted by $|x|_y$. For instance $|abbbaaab|_{aab} = 6$. Let a_{ij} be the word $a_ia_{i+1} \cdots a_j$ for $1 \le i < j \le k$ and if $i = j$ then $a_{ij} = a_i$.

2.2. Parikh matrix. Let $\mathcal{M}_k$ denote the set of all $k \times k$ upper triangular matrices with entries $\mathbb{N}$ and unit diagonal where $\mathbb{N}$ is the set of all non-negative integers.

Definition 2.1. [12] Let $\Sigma_k = \{a_1, a_2, \cdots, a_k\}$ be an ordered alphabet where $k \geq 1$. The *Parikh matrix mapping* denoted by ψ_k is the morphism $\psi_k : \Sigma_k^* \to \mathcal{M}_{k+1}$ defined as $\psi_k(a_l) = (m_{ij})_{1 \leq i, j \leq k+1}$ where

- $m_{ii} = 1$ for $1 \leq i \leq k+1$
- $m_{l,(l+1)} = 1$

and all other entries are zero.

Two words $x, y \in \Sigma_k^*$ are said to be *M−equivalent* denoted by $x \sim_{\mathtt{M}} y$ if and only if $\psi_k(x) = \psi_k(y)$. A word $z \in \Sigma_k^*$ is said to be *M−ambiguous* if there exists a word $w \neq z$ such that $z \sim_{\mathtt{M}} w$. Otherwise z is called *M−unambiguous*.

2.3. Parikh q−matrix. The notion of Parikh matrices is extended to a mapping called Parikh q−matrix mapping which takes its values in matrices with polynomial entries. The entries of the Parikh q−matrices are obtained by q−counting the number of occurrences of certain words as scattered subwords of a given word. The *q−counting of a scattered subword* a_{ij} of a word x represented by $S_{x,aij}$ is defined as follows:

Definition 2.2. [10] Let $\Sigma_k = \{a_1, a_2, \cdots, a_k\}$ be an ordered alphabet where $k \geq 1$, $x \in \Sigma_k^*$ and a_{ij} be a scattered subword of x for $1 \leq i \leq j < k$. Then

$$\mathtt{S}_{x,a_{ij}}(\mathtt{q}) = \sum_{x = u_i a_i u_{i+1} \cdots u_j a_j u_{j+1}} \mathtt{q}^{|u_i|_{a_i} + |u_{i+1}|_{a_{i+1}} + \cdots + |u_j|_{a_j} + |u_{j+1}|_{a_{j+1}}}.$$

For any word $x \in \Sigma_k^*$, $\mathtt{S}_{x,a_{ij}}(1) = |x|_{a_{ij}}$ for $1 \leq i \leq j \leq k$. Let $\mathcal{M}_k(\mathtt{q})$ denote the set of all $k \times k$ upper triangular matrices with entries $\mathbb{N}(\mathtt{q})$ and unit diagonal where $\mathbb{N}(\mathtt{q})$ is the set of all polynomials in the variable $\mathtt{q}$ with coefficients from $\mathbb{N}$.

Definition 2.3. [10] Let $\Sigma_k = \{a_1, a_2, \cdots, a_k\}$ be an ordered alphabet and $x \in \Sigma_k^*$ then the *Parikh q−matrix mapping* denoted by $\psi_{\mathtt{q}}$ is the morphism $\psi_{\mathtt{q}} : \Sigma_k^* \to \mathcal{M}_k(\mathtt{q})$ defined as $\psi_{\mathtt{q}}(a_l) = (m_{ij})_{1 \leq i, j \leq k+1}$ where

- $m_{ll} = \mathtt{q}$
- $m_{ii} = 1$ for $1 \leq i \leq k$, $i \neq l$
- $m_{l(l+1)} = 1$ if $l < k$

and all other entries are zero.

Note that the Parikh vector of x is given by the formal derivative of $(\mathtt{q}^{|x|_{a_1}}, \mathtt{q}^{|x|_{a_2}}, \cdots, \mathtt{q}^{|x|_{a_k}})$ with respect to $\mathtt{q}$ at $\mathtt{q} = 1$. The entries of the q−matrices are obtained by q−counting the number of occurrences of certain words as scattered subwords of a given word. By comparing Parikh matrix with Parikh q−matrix, add a new symbol d to Σ_3 to get $\Sigma_4 = \{a, b, c, d\}$ and compute the Parikh q−matrix of the word x treating it as a word over Σ_4. Two words $x, y \in \Sigma_k^*$ are said to be *q−equivalent* denoted by $x \sim_{\mathtt{q}} y$ if and only if $\psi_{\mathtt{q}}(x) = \psi_{\mathtt{q}}(y)$. A word $z \in \Sigma_k^*$ is said to be *q−ambiguous* if there exists a word $w \neq z$ such that $z \sim_{\mathtt{q}} w$. Otherwise z is called *q−unambiguous*. Note that if two words x, y are q−equivalent then they have same Parikh vector.

Throughout the paper if there is a word x from Σ_k, we assume x to be a word from Σ_{k+1} and compute the Parikh $\mathtt{q}-$matrix of x in $\mathcal{M}_{k+1}(\mathtt{q})$.

2.4. Two dimensional words. Let $\Sigma_k = \{a_1 < a_2 < \cdots < a_k\}$ be an ordered alphabet and h, v be two positive integers. A two dimensional word (or picture array) X is a rectangular array of symbols over Σ_k in h rows and v columns which is in the form of

$$\begin{matrix} a_{11} & \cdots & a_{1v} \\ \vdots & \ddots & \vdots \\ a_{h1} & \cdots & a_{hv} \end{matrix}$$

where $a_{ij} \in \Sigma$ and $1 \leq i \leq h$, $1 \leq j \leq v$. The set of all picture arrays over Σ_k is denoted by $\wp$. Let $\circ$ and $\diamond$ be the symbol of column concatenation and row concatenation of picture arrays respectively in $\wp$. For $X, Y \in \wp$, $X \circ Y$ is defined if and only if X and Y have same number of rows and $X \diamond Y$ is defined if and only if X and Y have same number of columns.

2.4.1. Parikh $\mathtt{q}-$matrices of a Picture array.

Definition 2.4. [4] For $h, v \geq 1$, let $X \in \wp$ be a $h \times v$ array over Σ_k. Let x_i be the horizontal words in the h rows and y_j be the vertical words in the v columns. Let $\psi_{\mathtt{q}}(x_i)$ and $\psi_{\mathtt{q}}(y_j^t)$ be the Parikh $\mathtt{q}-$matrix of x_i and y_j^t respectively. Then the row and column Parikh $\mathtt{q}-$matrix $R_q(X)$ and $C_q(X)$ respectively are defined as

$$R_q(X) = \bigoplus_{i=1}^{h} \overline{\psi}_{\mathtt{q}}(x_i) \text{ and } C_q(X) = \bigoplus_{j=1}^{v} \overline{\psi}_{\mathtt{q}}(y_j^t).$$

Definition 2.5. For $h, v \geq 1$, let $X \in \wp$ be a $h \times v$ array over Σ_k. Let x_i be the horizontal words in the h rows and y_j be the vertical words in the v columns. Let $u_{i,j}$ be a scattered subword of X where $1 \leq i \leq j \leq k$. Then the row and column $\mathtt{q}-$counting scattered subword $u_{i,j}$ of an array X denoted by $R(\mathtt{S}_{X,u_{i,j}}(\mathtt{q}))$ and $C(\mathtt{S}_{X,u_{i,j}}(\mathtt{q}))$ respectively and defined as

$$R(\mathtt{S}_{X,u_{i,j}}(\mathtt{q})) = \sum_{i=1}^{h} S_{x_i,u_{i,j}}(\mathtt{q}) \text{ and } C(\mathtt{S}_{X,u_{i,j}}(\mathtt{q})) = \sum_{j=1}^{v} S_{y_j^t,u_{i,j}}(\mathtt{q}).$$

In this work, we will be dealing mostly with Σ_3 and without loss of generality, let $\Sigma_3 = \{a, b, c\}$ and $a < b < c$ be the corresponding total order.

3. Parikh $\mathtt{q}$-matrix under Prouhet Array Morphism

In this section, we consider Prouhet array morphism and investigate the properties under this array morphism in terms of Parikh $\mathtt{q}-$matrices of binary arrays.

Definition 3.1. [16] A Prouhet morphism is a mapping $\mathtt{P} : \Sigma_3^* \to \Sigma_3^*$ defined by

$$\mathtt{P}(a) = abc,\ \mathtt{P}(b) = bca,\ \mathtt{P}(c) = cab.$$

The following theorem states the formulae to calculate $\mathtt{q}$-counting scattered subwords of the image of any words under Prouhet morphism.

Lemma 3.2. *Suppose $\Sigma_2 = \{a < b\}$ and $x \in \Sigma_2^*$. Then the $\mathtt{q}$-counting of a image word in x under Prouhet morphism are as follows:*

(i) $\mathtt{S}_{\mathtt{P}(x),a}(\mathtt{q}) = |x|_a \mathtt{q}^{|x|} + |x|_b \mathtt{q}^{|x|-1}$

(ii) $\mathtt{S}_{\mathtt{P}(x),b}(\mathtt{q}) = |x| \mathtt{q}^{|x|}$

(iii) $\mathtt{S}_{\mathtt{P}(x),c}(\mathtt{q}) = \sum_{i=1}^{|x|} \mathtt{q}^{i-1}$

(iv) $\mathtt{S}_{\mathtt{P}(x),ab}(\mathtt{q}) = \left[\frac{1}{2}|x|_a \left(|x|_a + 1\right) + |x|_{ab}\right] \mathtt{q}^{|x|} + \left[\frac{1}{2}|x|_b \left(|x|_b - 1\right) + |x|_{ba}\right] \mathtt{q}^{|x|-1}$

(v) $\mathtt{S}_{\mathtt{P}(x),bc}(\mathtt{q}) = 1 + 2\mathtt{q} + 3\mathtt{q}^2 + ... + (|x|)\mathtt{q}^{|x|-1}$

(vi) $$\mathtt{S}_{\mathtt{P}(x),abc}(\mathtt{q}) = 1 + 3\mathtt{q} + 6\mathtt{q}^2 + ... + \left(\frac{1}{2}|x|(|x|-1)\right) \mathtt{q}^{|x|-2} + \left(\frac{1}{2}|x|_a(|x|_a + 1) + |x|_{ab}\right) \mathtt{q}^{|x|-1}.$$

The following defines Prouhet array morphism, an extension of Prouhet morphism on words.

Definition 3.3. [13] A Prouhet array morphism is a mapping $\mathtt{P} : \Sigma_3^{++} \to \Sigma_3^{++}$ defined by

$$\mathtt{P}(a) = \begin{matrix} a & b & c \\ b & c & a \\ c & a & b \end{matrix}, \ \mathtt{P}(b) = \begin{matrix} b & c & a \\ c & a & b \\ a & b & c \end{matrix}, \ \mathtt{P}(c) = \begin{matrix} c & a & b \\ a & b & c \\ b & c & a \end{matrix}.$$

Example 3.4. Consider an array

$$X = \begin{matrix} a & b & a \\ a & a & b \end{matrix}$$

over Σ_2. Then the image of X under Prouhet array morphism is

$$\mathtt{P}(X) = \begin{matrix} a & b & c & b & c & a & a & b & c \\ b & c & a & c & a & b & b & c & a \\ c & a & b & a & b & c & c & a & b \\ a & b & c & a & b & c & b & c & a \\ b & c & a & b & c & a & c & a & b \\ c & a & b & c & a & b & a & b & c \end{matrix}.$$

Now obtain certain properties of arrays in the context of Prouhet array morphism so we consider the following two string morphisms $\mathtt{P}_1, \mathtt{P}_2$ as follows:

Definition 3.5. [13] A string morphism is a mapping $\mathtt{P}_1 : \Sigma_3^* \to \Sigma_3^*$ defined by

$$\mathtt{P}_1(a) = bca, \ \mathtt{P}_1(b) = cab, \ \mathtt{P}_1(c) = abc.$$

Definition 3.6. [13] A string morphism is a mapping $\mathtt{P}_2 : \Sigma_3^* \to \Sigma_3^*$ defined by

$$\mathtt{P}_2(a) = cab,\ \mathtt{P}_2(b) = abc,\ \mathtt{P}_2(c) = bca.$$

Using the definition of $\mathtt{P}_1$, we get the following results:

Lemma 3.7. *Suppose $\Sigma_2 = \{a < b\}$ and $x \in \Sigma_2^*$. Then the $\mathtt{q}$-counting of image word in x under Prouhet string morphism P_1 are as follows:*

(i) $S_{P_1(x),a}(q) = |x|_a\, q^{|x|-1} + |x|_b\, q^{|x|}$

(ii) $S_{P_1(x),b}(q) = |x|_a\, q^{|x|} + |x|_b\, q^{|x|-1}$

(iii) $S_{P_1(x),c}(q) = \sum_{i=1}^{|x|} q^{i-1}$

(iv) $$S_{P_1(x),ab}(q) = |x|_{ba}\, q^{|x|} + \left[|x|_b + \frac{1}{2}|x|_a\,(|x|_a - 1) + \frac{1}{2}|x|_b(|x|_b - 1)\right] q^{|x|-1} + |x|_{ab}\, q^{|x|-2}$$

(v) $S_{P_1(x),bc}(q) = 1 + 2q + 3q^2 + ... + (|x| - 1)\, q^{|x|-2} + |x|_a\, q^{|x|-1}$

(vi) $$S_{P_1(x),abc}(q) = 1 + 3q + 6q^2 + ... + \left(\frac{1}{2}(|x| - 1)(|x| - 2)\right) q^{|x|-3} + \left(\frac{1}{2}(|x|)(|x| - 1) - |x|_{ab}\right) q^{|x|-2} + |x|_{ba}\, q^{|x|-1}.$$

Proof. Let $x = x_1x_2\cdots x_n$ be the word over Σ_2. Then the image of x under Prouhet string morphism of is $\mathtt{P}_1(x) = \mathtt{P}_1(x_1)\mathtt{P}_1(x_2)\cdots\mathtt{P}_1(x_{i-1})\mathtt{P}_1(x_i)\mathtt{P}_1(x_{i+1})\cdots\mathtt{P}_1(x_n)$.

(1) Consider an occurrence of a in $\mathtt{P}_1(x)$. Then a must be in some $\mathtt{P}_1(x_i)$ for some $1 \le i \le n$.

If $x_i = a$, since $\mathtt{P}_1(a) = bca$, the number of $a's$ (in $\mathtt{P}_1(x)$) on the left of the a in $\mathtt{P}_1(x_i)$ is $i-1$ (since each $\mathtt{P}_1(x_k)$ has one a) and the number of b on the right of this a is $n - i$. Therefore, the monomial due this occurrence of a in $\mathtt{P}_1(x)$ is $\mathtt{q}^{|x|-1}$. As there are $|x|_a$ number of $a's$ in x, the $\mathtt{q}-$counting polynomial due to the occurrences of a in $\mathtt{P}_1(x)$ is $|x|_a\mathtt{q}^{|x|-1}$.

If $x_i = b$, the number of $a's$ (in $\mathtt{P}_1(x)$) on the left of the a in $\mathtt{P}_1(x_i)$ is $i - 1$ (since each $\mathtt{P}_1(x_k)$ has one a) and the number of b on the right of this a is $1+n-i$ (since each $\mathtt{P}_1(x_k)$ has one a and $\mathtt{P}_1(a) = abc$ contributes 1 to the sum). Therefore, the monomial due this occurrence of b in $\mathtt{P}_1(x)$ is $\mathtt{q}^{|x|}$. As there are $|x|_b$ number of $a's$ in x, the $\mathtt{q}-$counting polynomial due to the occurrences of b in $\mathtt{P}_1(x)$ is $|x|_b\mathtt{q}^{|x|}$. Therefore the $\mathtt{q}$-counting a of $\mathtt{P}_1(x)$ as

$$\begin{aligned}\mathtt{S}_{\mathtt{P}_1(x),a}(\mathtt{q}) &= \text{the } \mathtt{q}\text{-counting polynomial due to the occurrences of } a \text{ in } \mathtt{P}_1(x)\\ &\quad + \text{the } \mathtt{q}\text{-counting polynomial due to the occurrences of } b \text{ in } \mathtt{P}_1(x)\\ &= |x|_a\mathtt{q}^{|x|-1} + |x|_b\mathtt{q}^{|x|}.\end{aligned}$$

(2) The proof can be omitted as the argument is similar to (1).

(3) The argument is similar as in (i) and in fact regardless of whether $x_i = a$ $(or\ b)$, the number of $c's$ (in P(x)) on the left of the a (or b) in $P(x_i)$ is $i-1$ (since each $P(x_k)$ has one c). Therefore, the monomial due to this occurrence of c in $P(x)$ is $\mathtt{q}^{i-1}$. As there are $|x|$ number of $c's$ in $\mathtt{P}(x)$, the $\mathtt{q}-$counting polynomial due to the occurrences of c in $P(x)$ is $\sum_{i=1}^{|x|} \mathtt{q}^{i-1}$.

(4) We argue by induction on length of x to show that $\mathtt{S}_{\mathtt{P}_1(x),ab}(\mathtt{q}) = |x|_{ba}\mathtt{q}^{|x|} + \left[|x|_b + \frac{1}{2}|x|_a\,(|x|_a - 1) + \frac{1}{2}|x|_b(|x|_b - 1)\right]\mathtt{q}^{|x|-1} + |x|_{ab}\mathtt{q}^{|x|-2}$. Clearly the word of length 1 satisfies $\mathtt{S}_{\mathtt{P}_1(x),ab}(\mathtt{q})$ and thus the base step holds. Consider the induction step. At this step there are two cases depending on the last letter.

Case : 3 Consider $x' = ya$. By applying Prouhet string morphism $\mathtt{P}_1$ we get $\mathtt{P}_1(x') = \mathtt{P}_1(y)bca$. Note that for each occurrences of ab in $\mathtt{P}_1(y)$, the contribution of it to $\mathtt{S}_{\mathtt{P}_1(x'),ab}(\mathtt{q})$ either increases by 1 due to occurrence of a after y or decreases by 1 due to occurrence of $b's$ after $a's$ in x'. There are new occurrence of ab in $\mathtt{P}_1(x')$ due to new b (after $\mathtt{P}_1(y)$). However notice that for each a in $\mathtt{P}_1(y)$, together with the new b, the contribution of it to $\mathtt{S}_{\mathtt{P}_1(x'),ab}(\mathtt{q})$ is the same as its contribution in $\mathtt{S}_{\mathtt{P}_1(y),a}(\mathtt{q})$ except the power increase by one due to c after the new b. Here using result by (1) and by the induction hypothesis

$$\begin{aligned}
& \mathtt{S}_{\mathtt{P}_1(x'),ab}(\mathtt{q}) \\
= \;& |y|_{ba}\mathtt{q}^{|y|+1} + [|y|_b + |y|_{aa} + |y|_{bb}]\,\mathtt{q}^{|y|} + |y|_{ab}\mathtt{q}^{|y|-1} + |y|_a\mathtt{q}^{|y|} \\
& + |y|_b\mathtt{q}^{|y|+1} \\
= \;& [|x'|_{ba} - |x'|_b]\,\mathtt{q}^{|x'|} + [|x'|_b + |x'|_{aa} - (|x'|_a - 1) + |x'|_{bb}]\,\mathtt{q}^{|x'|-1} \\
& + |x'|_{ab}\mathtt{q}^{|x'|-2} + (|x'|_a - 1)\mathtt{q}^{|x'|-1} + |x'|_b\mathtt{q}^{|x'|} \\
= \;& |x'|_{ba}\mathtt{q}^{|x'|} + [|x'|_b| + x'|_{aa} + |x'|_{bb}]\,\mathtt{q}^{|x'|-1} + |x'|_{ab}\mathtt{q}^{|x'|-2} \\
= \;& |x'|_{ba}\mathtt{q}^{|x'|} + \left[|x'|_b| + \frac{1}{2}|x'|_a\,(|x'|_a - 1) + \frac{1}{2}|x'|_b(|x'|_b - 1)\right]\mathtt{q}^{|x'|-1} \\
& + |x'|_{ab}\mathtt{q}^{|x'|-2}.
\end{aligned}$$

Case : 4 Consider $x' = yb$. This is similar to *Case* 3.

(5) Here also we argue by induction on length of x to show that $\mathtt{S}_{\mathtt{P}_1(x),bc}(\mathtt{q}) = 1 + 2\mathtt{q} + 3\mathtt{q}^2 + ... + (|x| - 1)\mathtt{q}^{|x|-2} + |x|_a\mathtt{q}^{|x|-1}$. The base step holds since the word of length 1 satisfies $\mathtt{S}_{\mathtt{P}_1(x),bc}(\mathtt{q})$. At this induction step there are two cases depending on the last letter.

Case : 5 Consider $x' = ya$. By applying Prouhet string morphism $\mathtt{P}_1$ we get $\mathtt{P}_1(x') = \mathtt{P}_1(y)bca$. For each occurrences of bc in $\mathtt{P}_1(y)$, the contribution of it to $\mathtt{S}_{\mathtt{P}_1(x'),bc}(\mathtt{q})$ is similar to $\mathtt{S}_{\mathtt{P}_1(y),c}(\mathtt{q})$ that is $\sum_{i=1}^{|y|}(|y| + 1 - i)\mathtt{q}^{|y|-i}$. There are new occurrence of bc in $\mathtt{P}_1(x')$ due to new c (after $\mathtt{P}_1(y)$). However notice that for each b in $\mathtt{P}_1(y)$ (which is corresponds to a in y), together with the new

c, the contribution of it to $\mathtt{S}_{\mathtt{P}_1(x'),bc}(\mathtt{q})$ is $(|y|_a+1)\mathtt{q}^{|y|}$. Now by the induction hypothesis

$$\begin{aligned}
\mathtt{S}_{\mathtt{P}_1(x'),bc}(\mathtt{q}) &= \sum_{i=1}^{|y|}(|y|+1-i)\mathtt{q}^{|y|-i}+(|y|_a+1)\mathtt{q}^{|y|} \\
&= \sum_{i=1}^{|x'|-1}(|x'|-1+1-i)\mathtt{q}^{|x'|-1-i}+(|x'|_a-1+1)\mathtt{q}^{|x'|-1} \\
&= \sum_{i=1}^{|x'|-1}(|x'|-i)\mathtt{q}^{|x'|-1-i}+(|x'|_a)\mathtt{q}^{|x'|-1} \\
&= 1+2\mathtt{q}+\cdots+(|x'|-1)\mathtt{q}^{|x'|-2}+|x'|_a\mathtt{q}^{|x'|-1}.
\end{aligned}$$

Case : 6 Consider $x'=yb$. This is similar to *Case* 5.

(6) By using the result for part (4) and (5) and by induction hypothesis, we can prove (vi) and hence the proof part is omitted. □

Using the definition of $\mathtt{P}_2$, we get the following results:

Lemma 3.8. *Suppose $\Sigma_2=\{a<b\}$ and $x\in\Sigma_2^*$. Then the $\mathtt{q}$-counting of image word in x under Prouhet morphism are as follows:*

(i) $\mathtt{S}_{\mathtt{P}_2(x),a}(\mathtt{q})=|x|\mathtt{q}^{|x|}$

(ii) $\mathtt{S}_{\mathtt{P}_2(x),b}(\mathtt{q})=|x|_a\mathtt{q}^{|x|-1}+|x|_b\mathtt{q}^{|x|}$

(iii) $\mathtt{S}_{\mathtt{P}_2(x),c}(\mathtt{q})=\sum_{i=1}^{|x|}\mathtt{q}^{i-1}$

(iv) $\mathtt{S}_{\mathtt{P}_2(x),ab}(\mathtt{q})=\left[\frac{1}{2}|x|_a(|x|_a+1)+|x|_{ba}\right]\mathtt{q}^{|x|-1}+\left[\frac{1}{2}|x|_b(|x|_b+1)+|x|_{ab}\right]\mathtt{q}^{|x|}$

(v) $\mathtt{S}_{\mathtt{P}_2(x),bc}(\mathtt{q})=1+2\mathtt{q}+3\mathtt{q}^2+....+(|x|-1)\mathtt{q}^{|x|-2}+|x|_b\mathtt{q}^{|x|-1}$

(vi) $$\mathtt{S}_{\mathtt{P}_2(x),abc}(\mathtt{q})=1+3\mathtt{q}+6\mathtt{q}^2+....+\left(\frac{1}{2}|x|(|x|-1)\right)\mathtt{q}^{|x|-2}+\left(\frac{1}{2}|x|_b(|x|_b+1)+|x|_{ab}\right)\mathtt{q}^{|x|-1}.$$

Proof. Let $x=x_1x_2\cdots x_n$ be the word over Σ_2. Then the image of x under Prouhet string morphism of is $\mathtt{P}_2(x)=\mathtt{P}_2(x_1)\mathtt{P}_2(x_2)\cdots\mathtt{P}_2(x_{i-1})\mathtt{P}_2(x_i)\mathtt{P}_2(x_{i+1})\cdots\mathtt{P}_2(x_n)$.

(1)Consider an occurrence of a in $\mathtt{P}_2(x)$. Then this a must be in some $\mathtt{P}_2(x_i)$ for some $1\le i\le n$.

If $x_i=a$, since $\mathtt{P}_2(a)=cab$, the number of $a's$ $(in\mathtt{P}_2(x))$ on the left of the b in $\mathtt{P}_2(x_i)$ is $i-1$ (since each $\mathtt{P}_2(x_k)$ has one b) and the number of b on the right of this b is $1+n-i$ (since each $\mathtt{P}_2(x_k)$ has one b and $\mathtt{P}_2(a)=cab$ contributes

1 to the sum). Therefore, the monomial due this occurrence of a in $\mathtt{P}_2(x)$ is $\mathtt{q}^{i-1+1+n-i} = \mathtt{q}^n = \mathtt{q}^{|x|}$. As there are $|x|_a$ number of $a's$ in x, the $\mathtt{q}-$counting polynomial due to the occurrences of a in $P(x)$ is $|x|_a\mathtt{q}^{|x|}$.

If $x_i = b$, since $\mathtt{P}_2(b) = abc$, the number of $a's$ on the left of the b in $\mathtt{P}_2(x_i)$ is $i-1$ (since each $\mathtt{P}_2(x_k)$ has one b) and the number of b on the right of this a is $1+n-i$. Therefore, the monomial due this occurrence of b in $\mathtt{P}_2(x)$ is $\mathtt{q}^{i-1+1+n-i} = \mathtt{q}^n = \mathtt{q}^{|x|}$. As there are $|x|_b$ number of $b's$ in x, the $\mathtt{q}-$counting polynomial due to the occurrences of b in $\mathtt{P}_2(x)$ is $|x|_b\mathtt{q}^{|x|}$. Therefore the $\mathtt{q}$-counting a of $\mathtt{P}_2(x)$ as $\mathtt{S}_{\mathtt{P}_2(x),a}(\mathtt{q})$

$$\begin{aligned}\mathtt{S}_{\mathtt{P}_2(x),a}(\mathtt{q}) &= \text{the } \mathtt{q}\text{-counting polynomial due to the occurrences of } a \text{ in } \mathtt{P}_2(x)\\ &\quad +\text{the } \mathtt{q}\text{-counting polynomial due to the occurrences of } b \text{ in } \mathtt{P}_2(x)\\ &= |x|_a\mathtt{q}^{|x|} + |x|_b\mathtt{q}^{|x|}\\ &= |x|\mathtt{q}^{|x|}.\end{aligned}$$

(2) The proof can be omitted as the argument is similar to (1).

(3) The argument is similar as in (i) and in fact regardless of whether $x_i = a \text{ or } b$, the i^{th} occurrence of c in $\mathtt{P}_2(x)$ contributes clearly to $\mathtt{q}^{i-1}$. As there are $|x|$ number of $c's$ in $\mathtt{P}_2(x)$, the $\mathtt{q}$-counting c of $\mathtt{P}_2(x)$ is $\sum_{i=1}^{|x|}\mathtt{q}^{i-1}$.

(4) We argue by induction on length of x to show that

$$\mathtt{S}_{\mathtt{P}_2(x),ab}(\mathtt{q}) = \left[\frac{1}{2}|x|_a\left(|x|_a+1\right) + |x|_{ba}\right]\mathtt{q}^{|x|-1} + \left[\frac{1}{2}|x|_b\left(|x|_b+1\right) + |x|_{ab}\right]\mathtt{q}^{|x|}.$$

Clearly the word of length 1 satisfies $\mathtt{S}_{\mathtt{P}_2(x),ab}(\mathtt{q})$ and thus the base step holds. Consider the induction step. At this step there are two cases depending on the last letter.

Case : 7 Consider $x' = ya$. By applying Prouhet string morphism $\mathtt{P}_2$ we get $\mathtt{P}_2(x') = \mathtt{P}_2(y)cab$. Consider $\mathtt{S}_{\mathtt{P}_2(x),ab}(\mathtt{q})$. It should be noted that for each occurrences of ab in $\mathtt{P}_2(y)$, the contribution of it to $\mathtt{S}_{\mathtt{P}_2(x'),ab}(\mathtt{q})$ is increase by 1 interms of the power of $\mathtt{q}$ due to occurrence of a after y. There are new occurrence of ab in $\mathtt{P}_2(x')$ due to new b (after $\mathtt{P}_2(y)$). Also notice that for each a in $\mathtt{P}_2(y)$, together with the new b, the contribution of it to $\mathtt{S}_{\mathtt{P}_2(x'),ab}(\mathtt{q})$ is the same as its contribution in $\mathtt{S}_{\mathtt{P}_2(y),a}(\mathtt{q})$. Also there occurs an ab in $\mathtt{P}_2(x')$ after $\mathtt{P}_2(y)$. Here using result by (1) and by the induction hypothesis

$$\begin{aligned}&\mathtt{S}_{\mathtt{P}_2(x'),ab}(\mathtt{q})\\ =\ &\left[|y|_a + |y|_{aa} + |y|_{ba}\right]\mathtt{q}^{|y|} + \left[|y|_b + |y|_{bb} + |y|_{ab}\right]\mathtt{q}^{|y|+1} +\\ &\left[|y|_a + |y|_b + 1\right]\mathtt{q}^{|y|}\\ =\ &\left[(|x'|_a - 1) + (|x'|_{aa} - |x'|_a + 1) + (|x'|_{ba} - |x'|_b)\right]\mathtt{q}^{|x'|-1}\\ &+\left[|x'|_b + |x'|_{bb} + |x'|_{ab}\right]\mathtt{q}^{|x'|} + \left[(|x'|_a - 1) + |x'|_b + 1\right]\mathtt{q}^{|x'|-1}\end{aligned}$$

$$= \quad [|x'|_a + |x'|_{aa} + |x'|_{ba}]\,\mathtt{q}^{|x'|-1} + [|x'|_b + |x'|_{bb} + |x'|_{ab}]\,\mathtt{q}^{|x'|}$$
$$= \quad \left[\frac{1}{2}|x'|_a\,(|x'|_a+1) + |x'|_{ba}\right]\mathtt{q}^{|x'|-1} + \left[\frac{1}{2}|x'|_b\,(|x'|_b+1) + |x'|_{ab}\right]\mathtt{q}^{|x'|}.$$

Case : 8 Consider $x' = yb$. This is similar to *Case* 7.

(5) Here also we argue by induction on length of x to show that $\mathtt{S}_{\mathtt{P}_2(x),bc}(\mathtt{q}) = 1 + 2\mathtt{q} + 3\mathtt{q}^2 + ... + (|x|-1)\mathtt{q}^{|x|-2} + |x|_b\mathtt{q}^{|x|-1}$. The base step holds since the word of length 1 satisfies $\mathtt{S}_{\mathtt{P}_2(x),bc}(\mathtt{q})$. At this induction step there are two cases depending on the last letter.

Case : 9 Consider $x' = ya$. By applying Prouhet string morphism $\mathtt{P}_2$ we get $\mathtt{P}_2(x') = \mathtt{P}_2(y)cab$. For each occurrences of bc in $\mathtt{P}_2(y)$, the contribution of it to $\mathtt{S}_{\mathtt{P}_2(x'),bc}(\mathtt{q})$ is similar to $\mathtt{S}_{\mathtt{P}_2(y),c}(\mathtt{q})$ that is $\sum_{i=1}^{|y|}(|y|+1-i)\mathtt{q}^{|y|-i}$. There are new occurrence of bc in $\mathtt{P}_2(x')$ due to new c (after $\mathtt{P}_2(y)$). However notice that for each b in $\mathtt{P}_2(y)$ (which is corresponds to b in y), together with the new c, the contribution of it to $\mathtt{S}_{\mathtt{P}_2(x'),bc}(\mathtt{q})$ is $(|y|_b)\mathtt{q}^{|y|}$. Now by the induction hypothesis

$$\begin{aligned}
\mathtt{S}_{\mathtt{P}_2(x'),bc}(\mathtt{q}) &= \sum_{i=1}^{|y|}(|y|+1-i)\mathtt{q}^{|y|-i} + (|y|_b)\mathtt{q}^{|y|} \\
&= \sum_{i=1}^{|x'|-1}(|x'|-1+1-i)\mathtt{q}^{|x'|-1-i} + (|x'|_b)\mathtt{q}^{|x'|-1} \\
&= \sum_{i=1}^{|x'|-1}(|x'|-i)\mathtt{q}^{|x'|-1-i} + (|x'|_b)\mathtt{q}^{|x'|-1} \\
&= 1 + 2\mathtt{q} + \cdots + (|x'|-1)\mathtt{q}^{|x'|-2} + |x'|_b\mathtt{q}^{|x'|-1}.
\end{aligned}$$

Case : 10 Consider $x' = yb$. This is similar to *Case* 9.

(6) By using the result for part (4) and (5) and by induction hypothesis, we can prove (6) and hence the proof part is omitted. □

We state the formulae to calculate $\mathtt{q}$-counting scattered subwords of the image of any arrays under Prouhet array morphism.

Theorem 3.9. *Let $X = rows(x_1, x_2, \cdots, x_m)$ be an $m \times n$ array over Σ_2. Then the $\mathtt{q}$-counting of a image array in X under Prouhet array morphism are as follows:*
(i) $S_{P(X),a}(q) = S_{P(X),b}(q) = 2mn\,q^n + mn\,q^{n-1}$
(ii) $S_{P(X),c}(q) = 3m\left[\sum_{i=1}^{n} q^{i-1}\right]$.

Proof. Let $x_1, x_2, \cdots, x_m$ be the words in the consecutive rows of the array X such that $|x_i| = |x_i|_a + |x_i|_b$ which implies $|x_i| = n$. Let $\mathtt{P}(x_i)$ be the image of each row x_i of X under Prouhet array morphism $\mathtt{P}(X)$. Each row x_i in X yields three consecutive rows of words in $\mathtt{P}(X)$ and these are obtained by Definition 3.1 we have $\mathtt{P}(a) = abc$, $\mathtt{P}(b) = bca$, $\mathtt{P}(c) = cab$ and by Defintions 3.5, 3.6 we have $\mathtt{P}_1(a) = bca$, $\mathtt{P}_1(b) = cab$, $\mathtt{P}_1(c) = abc$ and $\mathtt{P}_2(a) = cab$, $\mathtt{P}_2(b) = abc$, $\mathtt{P}_2(c) = bca$. By Lemma 3.2 we have $\mathtt{S}_{\mathtt{P}(x),a}(\mathtt{q}) = |x|_a\mathtt{q}^{|x|} + |x|_b\mathtt{q}^{|x|-1}$ and by Lemmas 3.7, 3.8 we have $\mathtt{S}_{\mathtt{P}_1(x),a}(\mathtt{q}) = |x|_a\mathtt{q}^{|x|-1} + |x|_b\mathtt{q}^{|x|}$ and $\mathtt{S}_{\mathtt{P}_2(x),a}(\mathtt{q}) = |x|\mathtt{q}^{|x|}$ so that

$$\begin{aligned}
\mathtt{S}_{\mathtt{P}(X),a}(\mathtt{q}) &= \sum_{i=1}^{m} \mathtt{S}_{\mathtt{P}(x_i),a}(\mathtt{q}) + \sum_{i=1}^{m} \mathtt{S}_{\mathtt{P}_1(x_i),a}(\mathtt{q}) + \sum_{i=1}^{m} \mathtt{S}_{\mathtt{P}_2(x_i),a}(\mathtt{q}) \\
&= \sum_{i=1}^{m} \left[|x_i|_a\mathtt{q}^{|x_i|} + |x_i|_b\mathtt{q}^{|x_i|-1}\right] + \sum_{i=1}^{m} \left[|x_i|_a\mathtt{q}^{|x_i|-1} + |x_i|_b\mathtt{q}^{|x_i|}\right] + \\
&\quad \sum_{i=1}^{m} \left[|x_i|\mathtt{q}^{|x_i|}\right] \\
&= \sum_{i=1}^{m} \left[|x_i|\mathtt{q}^{|x_i|} + |x_i|\mathtt{q}^{|x_i|} - 1 + |x_i|\mathtt{q}^{|x_i|}\right] \\
&= \sum_{i=1}^{m} \left[2n\mathtt{q}^{n} + n\mathtt{q}^{n-1}\right] \\
&= 2mn\mathtt{q}^{n} + mn\mathtt{q}^{n-1}.
\end{aligned}$$

Similarly we can show that $\mathtt{S}_{\mathtt{P}(X),b}(\mathtt{q}) = 2mn\mathtt{q}^{n} + mn\mathtt{q}^{n-1}$.

$$\begin{aligned}
\mathtt{S}_{\mathtt{P}(X),c}(\mathtt{q}) &= \sum_{i=1}^{m} \mathtt{S}_{\mathtt{P}(x_i),c}(\mathtt{q}) + \sum_{i=1}^{m} \mathtt{S}_{\mathtt{P}_1(x_i),c}(\mathtt{q}) + \sum_{i=1}^{m} \mathtt{S}_{\mathtt{P}_2(x_i),c}(\mathtt{q}) \\
&= \sum_{i=1}^{m} \left[\sum_{i=1}^{|x_i|} \mathtt{q}^{i-1}\right] + \sum_{i=1}^{m} \left[\sum_{i=1}^{|x_i|} \mathtt{q}^{i-1}\right] + \sum_{i=1}^{m} \left[\sum_{i=1}^{|x_i|} \mathtt{q}^{i-1}\right] \\
&= \sum_{i=1}^{m} \left[3\sum_{i=1}^{|x_i|} \mathtt{q}^{i-1}\right] \\
&= 3m \left[\sum_{i=1}^{|x_i|} \mathtt{q}^{i-1}\right].
\end{aligned}$$

□

Theorem 3.10. *Let $X = rows(x_1, x_2, \cdots, x_m)$ be an $m \times n$ array over Σ_2. Then the $\mathtt{q}$-counting ab of an image array in X under Prouhet array morphism is as follows:*

$$\mathtt{S}_{\mathtt{P}(X),ab}(\mathtt{q}) = \left[\frac{mn}{2}(n+1) + \sum_{i=1}^{m} |x_i|_{ab}\right] \mathtt{q}^{n} + \left[mn^2 - 2\sum_{i=1}^{m} |x_i|_{ab}\right] \mathtt{q}^{n-1}$$

$$+\left[\sum_{i=1}^{m}|x_i|_{ba}\right]\mathtt{q}^{n-2}.$$

Proof. Let $x_1, x_2, \cdots, x_m$ be the words in the consecutive rows of the array X such that $|x_i| = |x_i|_a + |x_i|_b$ which implies $|x_i| = n$. Let $\mathtt{P}(x_i)$ be the image of each row x_i of X under Prouhet array morphism $\mathtt{P}(X)$. Each row x_i in X yields three consecutive rows of words in $\mathtt{P}(X)$ and these are obtained by Definition 3.1 we have $\mathtt{P}(a) = abc$, $\mathtt{P}(b) = bca$, $\mathtt{P}(c) = cab$ and by Defintions 3.5, 3.6 we have $\mathtt{P}_1(a) = bca$, $\mathtt{P}_1(b) = cab$, $\mathtt{P}_1(c) = abc$ and $\mathtt{P}_2(a) = cab$, $\mathtt{P}_2(b) = abc$, $\mathtt{P}_2(c) = bca$. By Lemmas 3.2, 3.7, 3.8 we have

$$\begin{aligned}
&\mathtt{S}_{\mathtt{P}(X),ab}(\mathtt{q})\\
&=\sum_{i=1}^{m}\mathtt{S}_{\mathtt{P}(x_i),ab}(\mathtt{q})+\sum_{i=1}^{m}\mathtt{S}_{\mathtt{P}_1(x_i),ab}(\mathtt{q})+\sum_{i=1}^{m}\mathtt{S}_{\mathtt{P}_2(x_i),ab}(\mathtt{q})\\
&=\sum_{i=1}^{m}\left[\left[\frac{1}{2}|x_i|_a\left(|x_i|_a+1\right)+|x_i|_{ab}\right]\mathtt{q}^{|x_i|}+\left[\frac{1}{2}|x_i|_b\left(|x_i|_b-1\right)+|x_i|_{ba}\right]\mathtt{q}^{|x_i|-1}\right]\\
&+\sum_{i=1}^{m}\left[|x_i|_{ba}\mathtt{q}^{|x_i|}+\left[|x_i|_b+\frac{1}{2}|x_i|_a\left(|x_i|_a-1\right)+\frac{1}{2}|x_i|_b(|x_i|_b-1)\right]\mathtt{q}^{|x_i|-1}+|x_i|_{ab}\mathtt{q}^{|x_i|-2}\right]\\
&+\sum_{i=1}^{m}\left[\left[\frac{1}{2}|x_i|_a\left(|x_i|_a+1\right)+|x_i|_{ba}\right]\mathtt{q}^{|x_i|-1}+\left[\frac{1}{2}|x_i|_b\left(|x_i|_b+1\right)+|x_i|_{ab}\right]\mathtt{q}^{|x_i|}\right]\\
&=\sum_{i=1}^{m}\left[\frac{1}{2}|x_i|_a\left(|x_i|_a+1\right)+|x_i|_{ab}+|x_i|_{ba}+\frac{1}{2}|x_i|_b\left(|x_i|_b+1\right)+|x_i|_{ab}\right]\mathtt{q}^{|x_i|}\\
&+\sum_{i=1}^{m}\left[\frac{1}{2}|x_i|_b\left(|x_i|_b-1\right)+|x_i|_{ba}+|x_i|_b+\frac{1}{2}|x_i|_a\left(|x_i|_a-1\right)+\frac{1}{2}|x_i|_b(|x_i|_b-1)\right.\\
&\left.+\frac{1}{2}|x_i|_a\left(|x_i|_a+1\right)+|x_i|_{ba}\right]\mathtt{q}^{|x_i|-1}+\sum_{i=1}^{m}\left[|x_i|_{ab}\right]\mathtt{q}^{|x_i|-2}
\end{aligned}$$

Since $|x_i|_{ab}+|x_i|_{ba}=|x_i|_a\cdot|x_i|_b$ we have

$$\begin{aligned}
&=\sum_{i=1}^{m}\left[\frac{1}{2}\left[\left(|x_i|_a+|x_i|_b\right)^2-2|x_i|_a|x_i|_b+\left(|x_i|_a+|x_i|_b\right)\right]+|x_i|_{ab}+|x_i|_a\cdot|x_i|_b\right]\mathtt{q}^{|x_i|}\\
&+\sum_{i=1}^{m}\left[|x_i|_a^2+|x_i|_b^2+2|x_i|_{ba}\right]\mathtt{q}^{|x_i|-1}+\sum_{i=1}^{m}\left[|x_i|_{ab}\right]\mathtt{q}^{|x_i|-2}\\
&=\sum_{i=1}^{m}\left[\frac{1}{2}\left(|x_i|_a+|x_i|_b\right)^2+\frac{1}{2}\left(|x_i|_a+|x_i|_b\right)+|x_i|_{ab}\right]\mathtt{q}^{|x_i|}\\
&+\sum_{i=1}^{m}\left[\left(|x_i|_a+|x_i|_b\right)^2-2|x_i|_a|x_i|_b+2|x_i|_a|x_i|_b-2|x_i|_{ab}\right]\mathtt{q}^{|x_i|-1}+\sum_{i=1}^{m}\left[|x_i|_{ab}\right]\mathtt{q}^{|x_i|-2}\\
&=\left[\frac{mn}{2}\left(n+1\right)+\sum_{i=1}^{m}|x_i|_{ab}\right]\mathtt{q}^{n}+\left[mn^2-2\sum_{i=1}^{m}|x_i|_{ab}\right]\mathtt{q}^{n-1}+\sum_{i=1}^{m}\left[|x_i|_{ab}\right]\mathtt{q}^{n-2}.
\end{aligned}$$

□

Theorem 3.11. *Let $X = rows(x_1, x_2, \cdots, x_m)$ be an $m \times n$ array over Σ_2. Then the $\mathtt{q}$-counting bc of an image array in X under Prouhet array morphism is $S_{P(X),bc}(\mathtt{q}) = 3m\left[1 + 2\mathtt{q} + 3\mathtt{q}^2 + \cdots + (n-1)\,\mathtt{q}^{n-2}\right] + 2mn\mathtt{q}^{n-1}$.*

Proof. Let $x_1, x_2, \cdots, x_m$ be the words in the consecutive rows of the array X such that $|x_i| = |x_i|_a + |x_i|_b$ which implies $|x_i| = n$. Let $\mathtt{P}(x_i)$ be the image of each row x_i of X under Prouhet array morphism $\mathtt{P}(X)$. Each row x_i in X yields three consecutive rows of words in $\mathtt{P}(X)$ and these are obtained by Definition 3.1 we have $\mathtt{P}(a) = abc$, $\mathtt{P}(b) = bca$, $\mathtt{P}(c) = cab$ and by Defintions 3.5, 3.6 we have $\mathtt{P}_1(a) = bca$, $\mathtt{P}_1(b) = cab$, $\mathtt{P}_1(c) = abc$ and $\mathtt{P}_2(a) = cab$, $\mathtt{P}_2(b) = abc$, $\mathtt{P}_2(c) = bca$. By Lemmas 3.2, 3.7 and 3.8 we have

$$\begin{aligned}
\mathtt{S}_{\mathtt{P}(X),bc}(\mathtt{q}) &= \sum_{i=1}^{m} \mathtt{S}_{\mathtt{P}(x_i),bc}(\mathtt{q}) + \sum_{i=1}^{m} \mathtt{S}_{\mathtt{P}_1(x_i),bc}(\mathtt{q}) + \sum_{i=1}^{m} \mathtt{S}_{\mathtt{P}_2(x_i),bc}(\mathtt{q}) \\
&= \sum_{i=1}^{m} \left[1 + 2\mathtt{q} + 3\mathtt{q}^2 + ... + (|x_i|)\mathtt{q}^{|x_i|-1}\right] \\
&+ \sum_{i=1}^{m} \left[1 + 2\mathtt{q} + 3\mathtt{q}^2 + ... + (|x_i|-1)\mathtt{q}^{|x_i|-2} + |x_i|_a\mathtt{q}^{|x_i|-1}\right] \\
&+ \sum_{i=1}^{m} \left[1 + 2\mathtt{q} + 3\mathtt{q}^2 + ... + (|x_i|-1)\mathtt{q}^{|x_i|-2} + |x_i|_b\mathtt{q}^{|x_i|-1}\right] \\
&= 3\sum_{i=1}^{m} \left[1 + 2\mathtt{q} + 3\mathtt{q}^2 + ... + (|x_i|-1)\mathtt{q}^{|x_i|-2}\right] \\
&+ \sum_{i=1}^{m} \left[|x_i|\right]\mathtt{q}^{|x_i|-1} + \sum_{i=1}^{m} \left[|x_i|_a + |x_i|_b\right]\mathtt{q}^{|x_i|-1} \\
&= 3m\left[1 + 2\mathtt{q} + 3\mathtt{q}^2 + ... + (n-1)\mathtt{q}^{n-2}\right] + 2mn\mathtt{q}^{n-1}
\end{aligned}$$

□

Theorem 3.12. *Let $X = rows(x_1, x_2, \cdots, x_m)$ be an $m \times n$ array over Σ_2. Then the $\mathtt{q}$-counting abc of an image array in X under Prouhet array morphism is as follows:*

$$\begin{aligned}
S_{P(X),abc}(\mathtt{q}) = &\left[\frac{3m}{2}(n-1)(n-2)\right]\mathtt{q}^{n-3} + \left[\frac{3mn}{2}(n-1) - \sum_{i=1}^{m} |x_i|_{ab}\right]\mathtt{q}^{n-2} \\
&+ \left[\frac{mn}{2}(n+1) + \sum_{i=1}^{m} |x_i|_{ab}\right]\mathtt{q}^{n-1}.
\end{aligned}$$

Proof. Let $x_1, x_2, \cdots, x_m$ be the words in the consecutive rows of the array X such that $|x_i| = |x_i|_a + |x_i|_b$ which implies $|x_i| = n$. Let $\mathtt{P}(x_i)$ be the image of each row x_i of X under Prouhet array morphism $\mathtt{P}(X)$. Each row x_i in X yields three consecutive rows of words in $\mathtt{P}(X)$ and these are obtained by Definition 3.1 we have $\mathtt{P}(a) = abc$, $\mathtt{P}(b) = bca$, $\mathtt{P}(c) = cab$ and by Defintions 3.5, 3.6 we have $\mathtt{P}_1(a) = bca$, $\mathtt{P}_1(b) = cab$, $\mathtt{P}_1(c) = abc$ and $\mathtt{P}_2(a) = cab$, $\mathtt{P}_2(b) = abc$, $\mathtt{P}_2(c) = bca$. By Lemma 3.2, 3.7, *and* 3.8, we have

$$
\begin{aligned}
&\mathtt{S}_{\mathtt{P}(X),abc}(\mathtt{q}) \\
&= \sum_{i=1}^{m} \mathtt{S}_{\mathtt{P}(x_i),abc}(\mathtt{q}) + \sum_{i=1}^{m} \mathtt{S}_{\mathtt{P}_1(x_i),abc}(\mathtt{q}) + \sum_{i=1}^{m} \mathtt{S}_{\mathtt{P}_2(x_i),abc}(\mathtt{q}) \\
&= \sum_{i=1}^{m} \left[1 + 3\mathtt{q} + 6\mathtt{q}^2 + ... + \left(\frac{1}{2}|x_i|(|x_i| - 1) \right) \mathtt{q}^{|x_i|-2} + \left(\frac{1}{2}|x_i|_a(|x_i|_a + 1) + |x_i|_{ab} \right) \mathtt{q}^{|x_i|-1} \right] \\
&+ \sum_{i=1}^{m} \left[1 + 3\mathtt{q} + 6\mathtt{q}^2 + ... + \left(\frac{1}{2}(|x_i| - 1)(|x_i| - 2) \right) \mathtt{q}^{|x_i|-3} \right. \\
&\left. + \left(\frac{1}{2}(|x_i|)(|x_i| - 1) - |x_i|_{ab} \right) \mathtt{q}^{|x_i|-2} + |x_i|_{ba} \mathtt{q}^{|x_i|-1} \right] \\
&+ \sum_{i=1}^{m} \left[1 + 3\mathtt{q} + 6\mathtt{q}^2 + ... + \left(\frac{1}{2}|x_i|(|x_i| - 1) \right) \mathtt{q}^{|x_i|-2} + \left(\frac{1}{2}|x_i|_b(|x_i|_b + 1) + |x_i|_{ab} \right) \mathtt{q}^{|x_i|-1} \right] \\
&= 3\sum_{i=1}^{m} \left[\frac{1}{2}(|x_i| - 1)(|x_i| - 2) \right] \mathtt{q}^{|x_i|-3} + \sum_{i=1}^{m} \left[\frac{3}{2}|x_i|^2 - \frac{3}{2}|x_i| - |x_i|_{ab} \right] \mathtt{q}^{|x_i|-2} \\
&+ \sum_{i=1}^{m} \left[\frac{1}{2} \left(|x_i|_a + |x_i|_b \right)^2 + \frac{1}{2} \left(|x_i|_a + |x_i|_b \right) + |x_i|_{ab} \right] \mathtt{q}^{|x_i|-1} \\
&= \left[\frac{3m}{2}(n - 1)(n - 2) \right] \mathtt{q}^{n-3} + \left[\frac{3mn}{2}(n - 1) - \sum_{i=1}^{m} |x_i|_{ab} \right] \mathtt{q}^{n-2} \\
&+ \left[\frac{mn}{2}(n + 1) + \sum_{i=1}^{m} |x_i|_{ab} \right] \mathtt{q}^{n-1}.
\end{aligned}
$$

□

Definition 3.13. [7] Two words $x, y \in \Sigma_k^*$ are said to satisfy $\mathtt{q}-$weak ratio property denoted by $x \sim_{qwr} y$ if $|x|_{a_i} = t|y|_{a_i}$, for all $a_i \in \Sigma_k$, with t a nonzero rational constant and for each $1 \le i \le k - 1$, either one of (C_1) or (C_2) holds true, where (C_1) and $(C_2)$are as follows:
$(C_1) : |x|_{a_i} = |x|_{a_{i+1}}$ and $|y|_{a_i} = |y|_{a_{i+1}}$
$(C_2) : \dfrac{\mathtt{S}_{x,a_i}(\mathtt{q})}{\mathtt{S}_{y,a_i}(\mathtt{q})} = \dfrac{\mathtt{q}^{|x|_{a_{i+1}}} - \mathtt{q}^{|x|_{a_i}}}{\mathtt{q}^{|y|_{a_{i+1}}} - \mathtt{q}^{|y|_{a_i}}}$, where $q \ne 0$ and $|x|_{a_i} \ne |x|_{a_{i+1}}$, $|y|_{a_i} \ne |y|_{a_{i+1}}$.

Theorem 3.14. [13] *For nonempty arrays X, Y over Σ_2 we have $P(X) \sim_{wr} P(Y)$ whenever $X \sim_{wr} Y$, where P is the Prouhet array morphism.*

Theorem 3.15. *If the nonempty arrays X, Y over Σ_2 satisfy $\mathtt{q}-$weak ratio property (i.e $X \sim_{qwr} Y$) then their images under Prouhet array morphism $P(X)$ and $P(Y)$ also satisfy $\mathtt{q}-$weak ratio property (i.e $P(X) \sim_{qwr} P(Y)$).*

Proof. Consider X and Y be the arrays of sizes $m \times n$ and $h \times v$ respectively. If $X \sim_{\mathtt{q}wr} Y$ then $\frac{|X|_a}{|Y|_a} = \frac{|X|_b}{|Y|_b} = \beta$ for some $\beta \neq 0$ and $|X|_a = |X|_b$ and $|Y|_a = |Y|_b$. Let $x_1, x_2, \cdots, x_m$ be the words in the consecutive rows of X such that $|x_i|_a = p_i$, $|x_i|_b = q_i$ and $|x_i| = n$ for some $1 \leq i \leq m$. Therefore $|X|_a = \sum_{i=1}^{m} |x_i|_a = \sum_{i=1}^{m} p_i$ and $|X|_b = \sum_{i=1}^{m} |x_i|_b = \sum_{i=1}^{m} q_i$. Let $y_1, y_2, \cdots, y_h$ be the words in the consecutive rows of Y such that $|y_i|_a = r_i$, $|y_i|_b = s_i$ and $|y_i| = v$ for some $1 \leq i \leq h$. Therefore $|Y|_a = \sum_{i=1}^{h} |y_i|_a = \sum_{i=1}^{h} r_i$ and $|Y|_b = \sum_{i=1}^{h} |y_i|_b = \sum_{i=1}^{h} s_i$. By Theorem 3.14, we have $\mathtt{P}(X) \sim_{wr} \mathtt{P}(Y)$. It is enough to prove that $|\mathtt{P}(X)|_a = |\mathtt{P}(X)|_b$ and $|\mathtt{P}(Y)|_a = |\mathtt{P}(Y)|_b$. By Theorem 3.9, it is also shown that $|\mathtt{P}(X)|_a = |\mathtt{P}(X)|_b = 3mn$ and $|\mathtt{P}(Y)|_a = |\mathtt{P}(Y)|_b = 3hv$ when $\mathtt{q} = 1$. Therefore we have $\mathtt{P}(X) \sim_{\mathtt{q}wr} \mathtt{P}(Y)$. □

Theorem 3.16. [4] *Let the column concatenation (row concatenation) of two arrays X, Y over Σ_2 such that i^{th} row of X and i^{th} row of Y satisfy $\mathtt{q}-$weak ratio property then their Parikh $\mathtt{q}-$matrix commute.*

Now we extend Theorem 3.16 for Prouhet array morphism as follows.

Theorem 3.17. *Let the column concatenation (row concatenation) of two arrays X, Y over Σ_2 such that their Parikh $\mathtt{q}-$matrix commute then the Parikh $\mathtt{q}-$matrix for the images $\mathtt{P}(X)$ and $\mathtt{P}(Y)$ under Prouhet array morphism are also commute.*

Proof. Let $X = \text{rows}(x_1, x_2, \cdots, x_m)$. By applying Prouhet array morphism on x_i $(1 \leq i \leq m)$ yields three consecutive rows of words in $\mathtt{P}(X)$. Let we call these rows u_i, v_i, w_i $(1 \leq i \leq m)$. Then $\mathtt{P}(X)$ is an array with $3m$ rows that is $\mathtt{P}(X) = \text{rows}(u_{11}, v_{11}, w_{11}, u_{21}, v_{21}, w_{21}, \cdots, u_{i1}, v_{i1}, w_{i1}, \cdots, u_{m1}, v_{m1}, w_{m1})$. Similarly since $Y = \text{rows}(y_1, y_2, \cdots, y_m)$ we have $\mathtt{P}(Y) = \text{rows}(u_{12}, v_{12}, w_{12}, u_{22}, v_{22}, w_{22}, \cdots, u_{i2}, v_{i2}, w_{i2}, \cdots, u_{m2}, v_{m2}, w_{m2})$ where u_{i2}, v_{i2}, w_{i2} are the rows of words by applying Prouhet array morphism on y_i $(1 \leq i \leq m)$.In order to prove that Parikh $\mathtt{q}-$matrix for the images $\mathtt{P}(X)$ and $\mathtt{P}(Y)$ under Prouhet array morphism are commute, it is enough to show that $u_{i1} \sim_{qwr} u_{i2}$, $v_{i1} \sim_{qwr} v_{i2}$ and $w_{i1} \sim_{qwr} w_{i2}$. By Theorem 3.14 we have shown that $u_{i1} \sim_{qwr} u_{i2}$, $v_{i1} \sim_{qwr} v_{i2}$ and $w_{i1} \sim_{qwr} w_{i2}$. Therefore by using Theorem 3.16, we get

$$\begin{aligned} R_q(X \circ Y) &= \psi_q(u_{11}u_{12}) \oplus \psi_q(v_{11}v_{12}) \oplus \psi_q(w_{11}w_{12}) \oplus \cdots \oplus \psi_q(u_{i1}u_{i2}) \oplus \\ &\quad \psi_q(v_{i1}v_{i2}) \oplus \psi_q(w_{i1}w_{i2}) \oplus \cdots \oplus \psi_q(u_{m1}u_{m2}) \oplus \psi_q(v_{m1}v_{m2}) \oplus \psi_q(w_{m1}w_{m2}) \\ &= \psi_q(u_{12}u_{11}) \oplus \psi_q(v_{12}v_{11}) \oplus \psi_q(w_{12}w_{11}) \oplus \cdots \oplus \psi_q(u_{i2}u_{i1}) \oplus \\ &\quad \psi_q(v_{i2}v_{i1}) \oplus \psi_q(w_{i2}w_{i1}) \oplus \cdots \oplus \psi_q(u_{m2}u_{m1}) \oplus \psi_q(v_{m2}v_{m1}) \oplus \psi_q(w_{m2}w_{m1}) \\ &= R_q(Y \circ X). \end{aligned}$$

Similarly we can show that $C_q(X \circ Y) = C_q(Y \circ X)$. Therefore Parikh $\mathtt{q}-$matrix for the images $\mathtt{P}(X)$ and $\mathtt{P}(Y)$ under Prouhet array morphism are commute. □

4. Conclusion

We have consider Prouhet array morphism and developed formulae for computing $\mathtt{q}$-counting scattered subwords of the image of any arrays under this array morphism. Also derived certain properties related to q-counting scattered subwords and Parikh q−matrices of the image arrays under this morphism. In future work, it will be interesting to construct q-equivalent arrays whose images under Prouhet array morphism are not q-equivalent and also to analyze the behavior of Parikh q−matrices of picture arrays under $s-$shuffle operator.

Conflicts of interest : The authors declare no conflict of interest.

Data availability : Not applicable

Acknowledgments : We would like to thank the unknown referees for their comments and suggestions on the manuscript in improving from an earlier version. The authors K. Janaki and R. Arulprakasam are very much thankful to the management, SRM Institute of Science and Technology for their continuous support and encouragement.

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