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SOME PROPERTIES OF DEGENERATE q-POLY-TANGENT POLYNOMIALS

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Abstract. In this paper, we give explicit identities for the degenerate q-poly-tangent numbers and polynomials. Finally, we obtain the relation of degenerate q-poly-tangent polynomials and Stirling numbers of the first kind and Stirling numbers of the second kind.

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1. Introduction

Many mathematicians have studied in the area of the tangent numbers and polynomials, poly-Bernoulli numbers and polynomials, poly-Euler numbers and polynomials and special polynomials(see [1-12]). In this paper, we construct degenerate q-poly-tangent polynomials and study some properties of the degenerate q-poly-tangent polynomials. We introduce the tangent polynomials $T_n(x)$ as follows:

$$\left(\frac{2}{e^{2t}+1}\right)e^{xt}=\sum_{n=0}^{\infty}T_n(x)\frac{t^n}{n!}.$$

In the special case, $x=0$, $T_n(0)=T_n$ are called the n-th tangent numbers(see [8]). We remember that the classical Stirling numbers of the first kind $S_1(n,k)$ and $S_2(n,k)$ are defined by the relations(see [12])

$$(x)_n=\sum_{k=0}^{n}S_1(n,k)x^k \text{ and } x^n=\sum_{k=0}^{n}S_2(n,k)(x)_k,$$

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respectively. Here $(x)_n = x(x-1)\cdots(x-n+1)$ denotes the falling factorial polynomial of order n. The numbers $S_2(n,m)$ also admit a representation in terms of a generating function

$$(e^t-1)^m = m!\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!}.$$

We also have

$$m!\sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!} = (\log(1+t))^m.$$

We also need the binomial theorem: for a variable x,

$$\frac{1}{(1-t)^c} = \sum_{n=0}^{\infty}\binom{c+n-1}{n}t^n.$$

The generalized falling factorial $(x|\lambda)_n$ with increment λ is defined by

$$(x|\lambda)_n = \prod_{k=0}^{n-1}(x-\lambda k)$$

with the convention $(x\,|\,\lambda)_0 = 1$. We also need the binomial theorem: for a variable x,

$$(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty}(x|\lambda)_n\frac{t^n}{n!}.$$

The degenerate tangent polynomials $T_n(x;\lambda)$ were introduced by Ryoo [9] by using the following generating function

$$\frac{2}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} T_n(x;\lambda)\frac{t^n}{n!}.$$

The degenerate poly-tangent polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$ were introduced by Ryoo and Agarwal [1, 6] by using the following generating function

$$\frac{2\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!}, \quad (k\in\mathbb{Z}),$$

where

$$\mathrm{Li}_k(t) = \sum_{n=1}^{\infty}\frac{t^n}{n^k}$$

is the kth polylogarithm function. When $x=0$, $\mathcal{T}_n^{(k)}(0,\lambda) = \mathcal{T}_n^{(k)}(\lambda)$ are called the degenerate poly-tangent numbers. Upon setting $k=1$, we have

$$\mathcal{T}_n^{(1)}(x,\lambda) = nT_{n-1}(x;\lambda) \text{ for } n\geq 1.$$

2. Explicit identities for degenerate q-poly-tangent polynomials

In this section, we introduce degenerate q-poly-tangent polynomials. Also, we show a diagram to confirm the structure. In addition, we explore some properties related to degenerate q-poly-tangent polynomials, including addition formula and explicit formula.

Definition 2.1. For any integer k and $0 < q < 1$, degenerate q-poly-tangent polynomials $T_{n,q}^{(k)}(x;\lambda)$ are defined as the following generating function

$$\frac{2Li_{k,q}(1-e^{-t})}{t\left((1+\lambda t)^{\frac{2}{\lambda}}+1\right)}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} T_{n,q}^{(k)}(x;\lambda)\frac{t^n}{n!},$$

where $Li_{k,q}(t) = \displaystyle\sum_{n=1}^{\infty}\frac{t^n}{[n]_q^k}$ is k-th q-analogue of polylogarithm function.

$T_{n,q}^{(k)}(\lambda) = T_{n,q}^{(k)}(0;\lambda)$ are called degenerate q-poly-tangent numbers when $x = 0$. If we set $k = 1$ in Definition 2.1, then the degenerate q-poly-tangent polynomials are reduced to classical tangent polynomials as $q \to 1$ and $\lambda \to 0$ because of $\lim_{q\to 1} Li_{1,q}(1-e^{-t}) = t$ and $\lim_{\lambda\to 0}(1+\lambda t)^{\frac{1}{\lambda}} = e^t$. That is,

$$\lim_{\substack{q\to 1\\ \lambda\to 0}} T_{n,q}^{(1)}(x;\lambda) = T_n(x).$$

Theorem 2.2. For any integer k and a nonnegative integer n and m, we get

$$T_{n,q}^{(k)}(mx;a) = \sum_{l=0}^{n}\binom{n}{l}T_{l,q}^{(k)}(\lambda)\,m^{n-l}\left(x\,|\,\frac{\lambda}{m}\right)_{n-l}.$$

Proof. From Definition 2.1, we have

$$\begin{aligned}\sum_{n=0}^{\infty} T_{n,q}^{(k)}(mx;\lambda)\frac{t^n}{n!} &= \frac{2Li_{k,q}(1-e^{-t})}{t\left((1+\lambda t)^{\frac{2}{\lambda}}+1\right)}(1+\lambda t)^{\frac{mx}{\lambda}}\\ &= \left(\sum_{n=0}^{\infty}T_{n,q}^{(k)}(\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\left(mx\,|\,\lambda\right)_n\frac{t^n}{n!}\right) \qquad (1)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}T_{l,q}^{(k)}(\lambda)\,m^{n-l}\left(x\,|\,\frac{\lambda}{m}\right)_{n-l}\right)\frac{t^n}{n!}.\end{aligned}$$

Therefore, we finish the proof of Theorem 2.2 by comparing the coefficients of $\frac{t^n}{n!}$. □

If $m = 1$ in Theorem 2.2, then we get the following corollary.

Corollary 2.3. For any integer k and a nonnegative integer n, we have

$$T_{n,q}^{(k)}(x;\lambda) = \sum_{l=0}^{n}\binom{n}{l}T_{l,q}^{(k)}(\lambda)\left(x\,|\,\lambda\right)_{n-l}.$$

Theorem 2.4. For any integer k and a nonnegative integer n and m, we obtain

$$T_{n,q}^{(k)}(mx;\lambda)=\sum_{l=0}^{n}\binom{n}{l}T_{l,q}^{(k)}(x;\lambda)\,(m-1)^{n-l}\Big(x\,|\,\frac{\lambda}{m-1}\Big)_{n-l}.$$

Proof. By utlizing Definition 2.1, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}T_{n,q}^{(k)}(mx;\lambda)\frac{t^n}{n!}\\
&=\frac{2Li_{k,q}(1-e^{-t})}{t\Big((1+\lambda t)^{\frac{2}{\lambda}}+1\Big)}(1+\lambda t)^{\frac{mx}{\lambda}}\\
&=\left(\sum_{n=0}^{\infty}T_{n,q}^{(k)}(x;\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\big((m-1)x\,|\,\lambda\big)_n\frac{t^n}{n!}\right)\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}T_{l,q}^{(k)}(x;\lambda)\,(m-1)^{n-l}\Big(x\,|\,\frac{\lambda}{m-1}\Big)_{n-l}\right)\frac{t^n}{n!}.
\end{aligned}\tag{2}$$

Therefore, we end the proof by comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation (2). □

As a result of Theorem 2.2 and Theorem 2.4, $T_{n,q}^{(k)}(mx;\lambda)$ can be presented as degenerate q-poly-tangent numbers and degenerate q-poly-tangent polynomials, respectively.

Theorem 2.5. For any integer k and a nonnegative integer n, we get

$$T_{n,q}^{(k)}(x+y;\lambda)=\sum_{l=0}^{n}\binom{n}{l}T_{l,q}^{(k)}(x;\lambda)\,\big(y\,|\,\lambda\big)_{n-l}.$$

Proof. Proof is omitted since it is a similar method of Theorem 2.4. □

Theorem 2.6. For any integer k and a positive integer n, we have

$$T_{n,q}^{(k)}(x+1;\lambda)-T_{n,q}^{(k)}(x;\lambda)=\sum_{l=0}^{n-1}\binom{n}{l}T_{l,q}^{(k)}(x;\lambda)\big(1\,|\,\lambda\big)_{n-l}.$$

Proof. By using Definition 2.1, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} T_{n,q}^{(k)}(x+1;\lambda)\frac{t^n}{n!} - \sum_{n=0}^{\infty} T_{n,q}^{(k)}(x;\lambda)\frac{t^n}{n!} \\
&= \frac{2Li_{k,q}(1-e^{-t})}{t\left((1+\lambda t)^{\frac{2}{\lambda}}+1\right)}(1+\lambda t)^{\frac{x}{\lambda}}\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right) \\
&= \left(\sum_{n=0}^{\infty} T_{n,q}^{(k)}(x;\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\left(1\,|\,\lambda\right)_n\frac{t^n}{n!}-1\right) \\
&= \sum_{n=1}^{\infty}\left(\sum_{l=0}^{n-1}\binom{n}{l}T_{l,q}^{(k)}(x;\lambda)\left(1\,|\,\lambda\right)_{n-l}\right)\frac{t^n}{n!}.
\end{aligned} \tag{3}$$

Then we compare the coefficients of $\frac{t^n}{n!}$ for $n \geq 1$. The reason both sides of the above equation (3) can be compared the coefficients is that $T_{0,q}^{(k)}(x+1;\lambda) - T_{0,q}^{(k)}(x;\lambda) = 0$. Thus, the proof is done. □

Theorem 2.7. For any integer k and a nonnegative integer n, we get

$$nT_{n-1,q}^{(k)}(x;\lambda) = \sum_{l=0}^{\infty}\sum_{i=0}^{l+1}\sum_{m=0}^{n}\binom{l+1}{i}\binom{n}{m}\frac{(-1)^{i+m}i^m}{[l+1]_q^k}T_{n-m}(x;\lambda),$$

where $T_n(x;\lambda)$ is degenerate tangent polynomials.

Proof. By using Definition 2.1, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} T_{n,q}^{(k)}(x;\lambda)\frac{t^n}{n!} \\
&= \frac{2Li_{k,q}(1-e^{-t})}{t\left((1+\lambda t)^{\frac{2}{\lambda}}+1\right)}(1+\lambda t)^{\frac{x}{\lambda}} \\
&= \frac{1}{t}\sum_{l=1}^{\infty}\frac{(1-e^{-t})^l}{[l]_q^k}\frac{2}{(1+\lambda t)^{\frac{2}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} \\
&= \frac{1}{t}\left(\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{i=0}^{l+1}\binom{l+1}{i}\frac{(-1)^{i+n}i^n}{[l+1]_q^k}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}T_n(x;\lambda)\frac{t^n}{n!}\right) \\
&= \frac{1}{t}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{i=0}^{l+1}\sum_{m=0}^{n}\binom{l+1}{i}\binom{n}{m}\frac{(-1)^{i+m}i^m}{[l+1]_q^k}T_{n-m}(x;\lambda)\right)\frac{t^n}{n!}.
\end{aligned} \tag{4}$$

Because of the identity $\sum_{n=0}^{\infty} T_{n,q}^{(k)}(x;\lambda)\frac{t^{n+1}}{n!} = \sum_{n=0}^{\infty} nT_{n-1,q}^{(k)}(x;\lambda)\frac{t^n}{n!}$, we multiply both sides of the above equation (4) by t and compare the coefficients of $\frac{t^n}{n!}$. Hence, we end the proof. □

Theorem 2.8. For any integer k and a positive integer n, we obtain

$$\begin{aligned} &nT^{(k)}_{n-1,q}(x;\lambda) \\ &= 2\sum_{l=0}^{\infty}\sum_{j=0}^{l}\sum_{i=0}^{j+1}\sum_{m=0}^{n}\binom{j+1}{i}\binom{n}{m}\frac{(-1)^{l-j+i+m}i^m}{[j+1]_q^k}\big(2l-2j+x\,|\,\lambda\big)_{n-m}. \end{aligned}$$

Proof. From Definition 2.1, we have

$$\begin{aligned} &\sum_{n=0}^{\infty}T^{(k)}_{n,q}(x;\lambda)\frac{t^n}{n!} \\ &= \frac{2Li_{k,q}(1-e^{-t})}{t\left((1+\lambda t)^{\frac{2}{\lambda}}+1\right)}(1+\lambda t)^{\frac{x}{\lambda}} \\ &= \frac{2}{t}\left(\sum_{l=0}^{\infty}\frac{(1-e^{-t})^{l+1}}{[l+1]_q^k}\right)\left(\sum_{j=0}^{\infty}(-1)^j(1+\lambda t)^{\frac{2j+x}{\lambda}}\right) \\ &= \frac{2}{t}\sum_{l=0}^{\infty}\sum_{j=0}^{l}\frac{(1-e^{-t})^{j+1}}{[j+1]_q^k}(-1)^{l-j}(1+\lambda t)^{\frac{2l-2j+x}{\lambda}} \\ &= \frac{2}{t}\sum_{l=0}^{\infty}\sum_{j=0}^{l}\frac{(-1)^{l-j}}{[j+1]_q^k}\sum_{i=0}^{j+1}\binom{j+1}{i}(-1)^i \\ &\quad\times\left(\sum_{m=0}^{\infty}(-1)^m i^m\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\big(2l-2j+x\,|\,\lambda\big)_n\frac{t^n}{n!}\right) \\ &= \frac{2}{t}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{j=0}^{l}\sum_{i=0}^{j+1}\sum_{m=0}^{n}\binom{j+1}{i}\binom{n}{m}\right. \\ &\quad\left.\times\frac{(-1)^{l-j+i+m}i^m}{[j+1]_q^k}\big(2l-2j+x\,|\,\lambda\big)_{n-m}\right)\frac{t^n}{n}. \end{aligned} \tag{5}$$

If we multiply both sides of the above equation (5) by t, then we can compare the coefficients. The reason is that $\sum_{n=0}^{\infty}T^{(k)}_{n,q}(x;\lambda)\frac{t^{n+1}}{n!} = \sum_{n=0}^{\infty}nT^{(k)}_{n-1,q}(x;\lambda)\frac{t^n}{n!}$. Therefore, the proof is done. □

3. Relation between degenerate q-poly-tangent polynomials and Stirling numbers of the first kind and Stirling numbers of the second kind

In this section, we obtain the relation of degenerate q-poly-tangent polynomials and Stirling numbers of the first kind and Stirling numbers of the second kind.

Theorem 3.1. For any integer k and a nonnegative integer n, we get

$$T^{(k)}_{n,q}(x;\lambda) = \sum_{m=0}^{l}\sum_{l=0}^{n}\binom{n}{l}T^{(k)}_{n-l,q}(\lambda)x^m\lambda^{l-m}S_1(l,m),$$

where $S_1(l,m)$ is Stirling numbers of the first kind.

Proof. By using Definition 2.1, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}T^{(k)}_{n,q}(x;\lambda)\frac{t^n}{n!}\\
&=\frac{2Li_{k,q}(1-e^{-t})}{t\left((1+\lambda t)^{\frac{2}{\lambda}}+1\right)}(1+\lambda t)^{\frac{x}{\lambda}}\\
&=\left(\sum_{n=0}^{\infty}T^{(k)}_{n,q}(\lambda)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}\left(\frac{x}{\lambda}\right)^m\frac{(\log(1+\lambda t))^m}{m!}\right)\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}T^{(k)}_{n-l,q}(\lambda)x^m\lambda^{l-m}S_1(l,m)\right)\frac{t^n}{n!}
\end{aligned}\tag{6}$$

In equation (6), the reason equation $\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}S_1(n,l)=\sum_{n=0}^{\infty}\sum_{l=0}^{n}S_1(n,l)$ can be satisfied is that $S_1(n,l)=0$ when $n<l$. Thus, the proof is done by comparing the coefficients of $\frac{t^n}{n!}$. □

Theorem 3.2. For any integer k and a nonnegative integer n, we get

$$T^{(k)}_{n,q}(x;\lambda) = \sum_{l=0}^{n}\sum_{m=1}^{l+1}\binom{n}{l}\frac{(-1)^{l+m+1}m!}{[m]_q^k}\frac{S_2(l+1,m)}{l+1}T_{n-l}(x;\lambda),$$

where $T_n(x;\lambda)$ is degenerate tangent polynomials.

Proof. By utilizing Definition 2.1, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}T^{(k)}_{n,q}(x;\lambda)\frac{t^n}{n!}\\
&=\frac{1}{t}\sum_{m=1}^{\infty}\frac{(-1)^m m!}{[m]_q^k}\frac{(e^{-t}-1)^m}{m!}\frac{2}{(1+\lambda t)^{\frac{2}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}}\\
&=\frac{1}{t}\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{(-1)^{n+m}\,m!}{[m]_q^k}S_2(n,m)\frac{t^n}{n!}\frac{2}{(1+\lambda t)^{\frac{2}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}}\\
&=\left(\sum_{n=0}^{\infty}\sum_{m=1}^{n+1}\frac{(-1)^{n+m+1}m!}{[m]_q^k}\frac{S_2(n+1,m)}{n+1}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}T_n(x;\lambda)\frac{t^n}{n!}\right)\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=1}^{l+1}\binom{n}{l}\frac{(-1)^{l+m+1}m!}{[m]_q^k}\frac{S_2(l+1,m)}{l+1}T_{n-l}(x;\lambda)\right)\frac{t^n}{n!}.
\end{aligned}\tag{7}$$

In equation (7), the reason equation

$$\sum_{n=0}^{\infty}\sum_{l=1}^{\infty} S_2(n,l) = \sum_{n=1}^{\infty}\sum_{l=1}^{n} S_2(n,l)$$

can be satisfied is that $S_2(n,l) = 0$ when $n < l$. Thus, the proof is done by comparing the coefficients of $\frac{t^n}{n!}$. □

Theorem 3.3. For any integer k and a positive integer n, we obtain

$$\begin{aligned}
&T_{n,q}^{(k)}(x;\lambda)\\
&= 2\sum_{j=0}^{\infty}\sum_{l=0}^{n}\sum_{m=1}^{l+1}\binom{n}{l}\frac{(-1)^{l+m+j+1}m!}{[m]_q^k}\frac{S_2(l+1,m)}{l+1}\big(2l-2j+x\,|\,\lambda\big)_{n-l}.
\end{aligned}$$

Proof. From Definition 2.1, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} T_{n,q}^{(k)}(x;\lambda)\frac{t^n}{n!}\\
&= \frac{2}{t}\left(\sum_{l=0}^{\infty}\frac{(1-e^{-t})^{l+1}}{[l+1]_q^k}\right)\left(\sum_{j=0}^{\infty}(-1)^j(1+\lambda t)^{\frac{2j+x}{\lambda}}\right)\\
&= \frac{2}{t}\left(\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{(-1)^{n+m}\,m!}{[m]_q^k}S_2(n,m)\frac{t^n}{n!}\right)\\
&\quad\times\left(\sum_{j=0}^{\infty}(-1)^j\sum_{n=0}^{\infty}\big(2l-2j+x\,|\,\lambda\big)_n\frac{t^n}{n!}\right)\\
&= \frac{2}{t}\sum_{n=0}^{\infty}\left(\sum_{j=0}^{\infty}\sum_{l=0}^{n}\sum_{m=1}^{l+1}\binom{n}{l}\frac{(-1)^{l+m+j+1}m!}{[m]_q^k}\right.\\
&\quad\left.\times\frac{S_2(l+1,m)}{l+1}\big(2l-2j+x\,|\,\lambda\big)_{n-l}\right)\frac{t^n}{n!}.
\end{aligned}\tag{8}$$

Thus, we finish the proof by comparing the coefficients of $\frac{t^n}{n!}$. □

Theorem 3.4. For any integer k and a nonnegative integer n, we obtain

$$\begin{aligned}
&T_{n-1,q}^{(k)}(x+2;\lambda) + T_{n-1,q}^{(k)}(x;\lambda)\\
&= \frac{2}{n}\sum_{l=0}^{n}\sum_{m=1}^{l+1}\binom{n}{l}\frac{(-1)^{l+m+1}m!}{[m]_q^k}\frac{S_2(l+1,m)}{l+1}(x\,|\,\lambda)_{n-l}.
\end{aligned}$$

Proof. By using Definition 2.1, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} T_{n,q}^{(k)}(x+2;\lambda)\frac{t^n}{n!} + \sum_{n=0}^{\infty} T_{n,q}^{(k)}(x;\lambda)\frac{t^n}{n!} \\
&= \frac{2Li_{k,q}(1-e^{-t})}{t\left((1+\lambda t)^{\frac{2}{\lambda}}+1\right)}(1+\lambda t)^{\frac{x}{\lambda}}\left((1+\lambda t)^{\frac{2}{\lambda}}+1\right) \\
&= \frac{2}{t}\sum_{m=1}^{\infty}\frac{(-1)^m\, m!}{[m]_q^k}\frac{(e^{-t}-1)^m}{m!}\sum_{n=0}^{\infty}(x\,|\,\lambda)_n\frac{t^n}{n!} \qquad (9)\\
&= \frac{2}{t}\left(\sum_{n=0}^{\infty}\sum_{m=1}^{n+1}\frac{(-1)^{n+m+1}m!}{[m]_q^k}S_2(n+1,m)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(x\,|\,\lambda)_n\frac{t^n}{n!}\right) \\
&= \frac{2}{t}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=1}^{l+1}\binom{n}{l}\frac{(-1)^{l+m+1}m!}{[m]_q^k}\frac{S_2(l+1,m)}{l+1}(x\,|\,\lambda)_{n-l}\right)\frac{t^n}{n!}.
\end{aligned}$$

Let us multiply both sides of the above equation (9) by t. Then we can compare the coefficients of $\frac{t^n}{n!}$ because of the identity $\sum_{n=0}^{\infty} T_{n,q}^{(k)}(x+2;\lambda)\frac{t^{n+1}}{n!} + \sum_{n=0}^{\infty} T_{n,q}^{(k)}(x;\lambda)\frac{t^{n+1}}{n!} = \sum_{n=0}^{\infty} nT_{n-1,q}^{(k)}(x+2;\lambda)\frac{t^n}{n!} + \sum_{n=0}^{\infty} nT_{n-1,q}^{(k)}(x;\lambda)\frac{t^n}{n!}$. Hence, we end the proof. □

4. Zeros of the degenerate q-poly-tangent polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the degenerate q-poly-tangent polynomials $T_{n,q}^{(k)}(x,\lambda)$. The degenerate q-poly-tangent polynomials $T_{n,q}^{(k)}(x,\lambda)$ can be determined explicitly. A few of them are

$$T_{1,q}^{(k)}(x,\lambda) = \frac{3}{2} + \left(\frac{1-q^2}{1-q}\right)^{-k} + x,$$

$$\begin{aligned}
T_{2,q}^{(k)}(x,\lambda) = {}& \frac{4}{3} - 4\left(\frac{1-q^2}{1-q}\right)^{-k} + 2\left(\frac{1-q^3}{1-q}\right)^{-k} - 3x + 2\left(\frac{1-q^2}{1-q}\right)^{-k} x \\
&+ x^2 + \lambda - x\lambda,
\end{aligned}$$

$$\begin{aligned}
T_{3,q}^{(k)}(x,\lambda) = {}& \frac{3}{4} + \frac{19}{2}\left(\frac{1-q^2}{1-q}\right)^{-k} - 15\left(\frac{1-q^3}{1-q}\right)^{-k} + 6\left(\frac{1-q^4}{1-q}\right)^{-k} + 4x \\
&- 12\left(\frac{1-q^2}{1-q}\right)^{-k} x + 6\left(\frac{1-q^3}{1-q}\right)^{-k} x - \frac{9x^2}{2} + 3\left(\frac{1-q^2}{1-q}\right)^{-k} x^2 + x^3 \\
&- \frac{3\lambda}{2} + 3\left(\frac{1-q^2}{1-q}\right)^{-k}\lambda + \frac{15x\lambda}{2} - 3\left(\frac{1-q^2}{1-q}\right)^{-k} x\lambda - 3x^2\lambda - 2\lambda^2 + 2x\lambda^2
\end{aligned}$$

We investigate the beautiful zeros of the degenerate q-poly-tangent polynomials $T_{n,q}^{(k)}(x,\lambda)$ by using a computer. We plot the zeros of the degenerate q-poly-tangent polynomials $T_{n,q}^{(k)}(x,\lambda)$ for $n = 30$(Figure 1). In Figure 1(top-

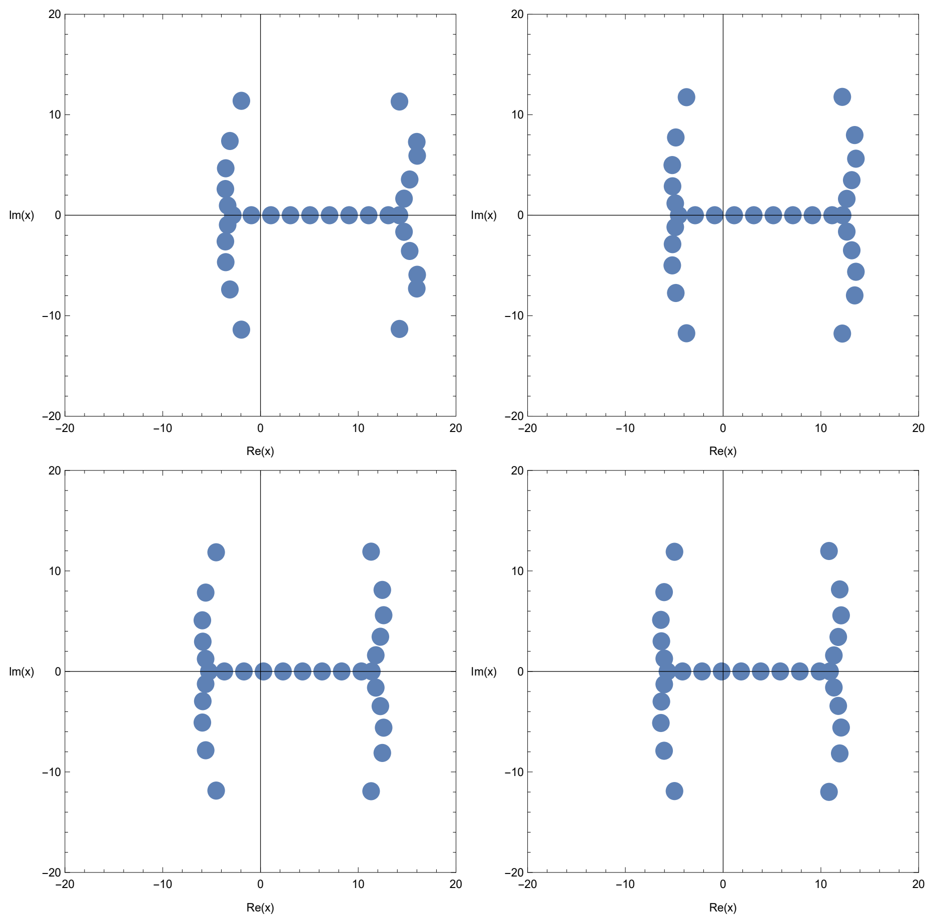

FIGURE 1. Zeros of $T_n^{(k,S)}(x,y)$

left), we choose $n = 30, k = 3, q = \frac{1}{2}$ and $\lambda = \frac{1}{3}$. In Figure 1(top-right), we choose $n = 30, k = 3, q = \frac{1}{2}$ and $\lambda = \frac{1}{5}$. In Figure 1(bottom-left), we choose $n = 30, k = 3, q = \frac{1}{2}$ and $\lambda = \frac{1}{7}$. In Figure 1(bottom-right), we choose $n = 30, k = 3, q = \frac{1}{2}$ and $\lambda = \frac{1}{9}$.

Stacks of zeros of $T_{n,q}^{(k)}(x,\lambda)$ for $1 \leq n \leq 30$ from a 3-D structure are presented(Figure 2). In Figure 2(top-left), we choose $k=3, q=\frac{1}{2}$ and $\lambda=\frac{1}{3}$. In

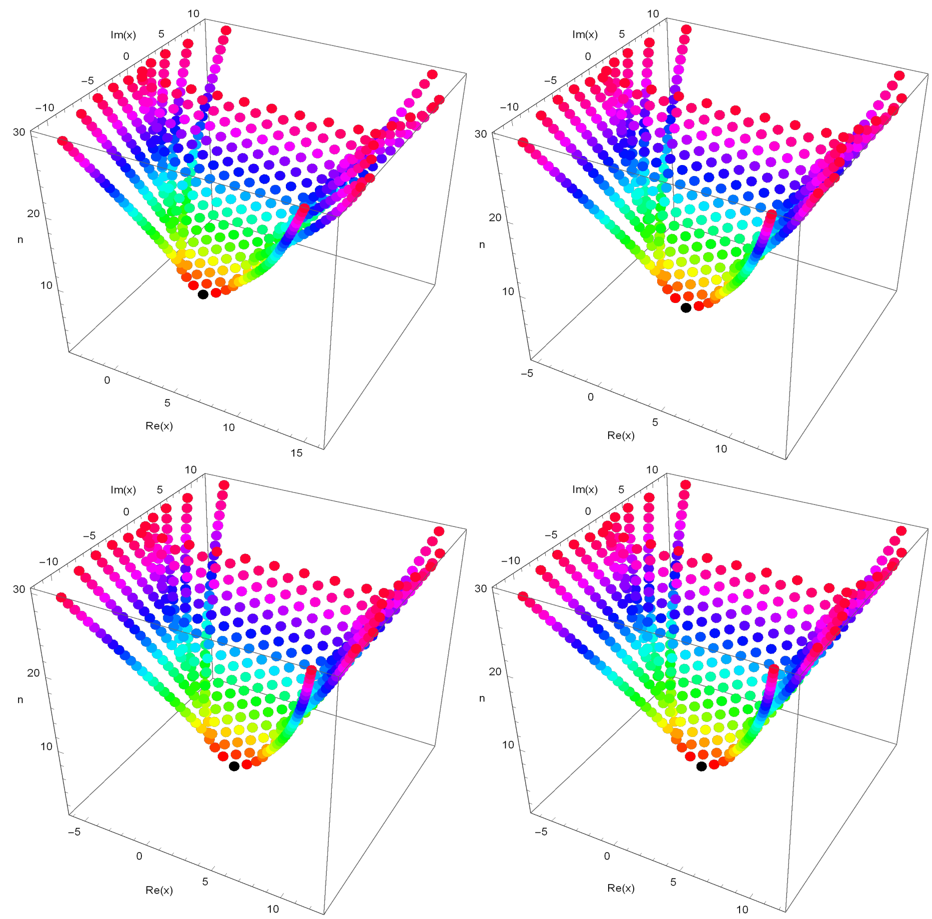

FIGURE 2. Stacks of zeros of $T_{n,q}^{(k)}(x,\lambda)$ for $1 \leq n \leq 30$

Figure 2(top-right), we choose $k=3, q=\frac{1}{2}$ and $\lambda=\frac{1}{5}$. In Figure 2(bottom-left), we choose $k=3, q=\frac{1}{2}$ and $\lambda=\frac{1}{7}$. In Figure 2(bottom-right), we choose $k=3, q=\frac{1}{2}$ and $\lambda=\frac{1}{9}$.

Next, we calculated an approximate solution satisfying poly-sine tangent polynomials $T^{(k)}_{n,q}(x,\lambda) = 0$ for $x \in \mathbb{R}$. The results are given in Table 1.

Table 1. Approximate solutions of $T^{(3)}_{n,\frac{1}{2}}(x, \frac{1}{3}) = 0$

degree n	x
1	1.2037
2	0.35881, 2.3819
3	−0.24208, 1.5759, 3.2772
4	−0.62996, 0.72888, 2.7958, 3.9201
5	−0.75295, −0.11049, 1.8993
6	1.0659, 3.0533
7	0.23279, 2.2327, 4.1407
8	−0.55467, 1.3994, 3.4004, 5.0542
9	−1.0703, 0.56623, 2.5659, 4.5946, 5.6959
10	−1.3236, −0.26864, 1.7328, 3.7322
11	0.89956, 2.8995, 4.8948
12	0.066231, 2.0662, 4.0664, 6.0156

Conflicts of interest : The author declares no conflict of interest.

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