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SOME ESTIMATES FOR GENERALIZED COMMUTATORS OF MULTILINEAR CALDERÓN-ZYGMUND OPERATORS

Honghai Liu, Zengyan Si, and Ling Wang

Abstract. Let T be an m-linear Calderón-Zygmund operator. $T_{\vec{b},S}$ is the generalized commutator of T with a class of measurable functions $\{b_i\}_{i=1}^{\infty}$. In this paper, we will give some new estimates for $T_{\vec{b},S}$ when $\{b_i\}_{i=1}^{\infty}$ belongs to Orlicz-type space and Lipschitz space, respectively.

1. Introduction and main results

Let T be a multilinear operator initially defined on the m-fold product of Schwartz spaces and taking values in the space of tempered distributions,

$$T : S(\mathbb{R}^n) \times \cdots \times S(\mathbb{R}^n) \to S'(\mathbb{R}^n).$$

We say that T is an m-linear Calderón-Zygmund operator if it can be extended to a bounded multilinear operator from $L^1 \times \cdots \times L^1$ to $L^{\frac{1}{m},\infty}$, and if there exists a function K, defined off the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, satisfying

$$T(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) \prod_{j=1}^{m} f_j(y_j) d\vec{y}$$

for all $x \notin \bigcap_{j=1}^m \text{supp} f_j$; and there exists, for some $\varepsilon > 0$, a constant A_ε such that

$$|K(x, y_1, \ldots, y_m) - K(x', y_1, \ldots, y_m)| \le \frac{A_\varepsilon |x - x'|^\varepsilon}{(\sum_{j=1}^m |x - y_j|)^{mn+\varepsilon}} \tag{1.1}$$

whenever $|x - x'| \le \frac{1}{2} \max_{1 \le j \le m} |x - y_j|$, and

$$|K(y_0, y_1, \ldots, y_m)| \le \frac{A_\varepsilon}{(\sum_{k,l=0}^m |y_k - y_l|)^{mn}}. \tag{1.2}$$

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Define, whenever it makes sense, the generalized commutator of T with a class of measurable functions $\{b_i\}_{i=1}^{\infty}$ by

$$T_{\vec{b},S}(\vec{f})(x) = \int_{\mathbb{R}^{nm}} \prod_{(i,j)\in S} (b_i(x) - b_i(y_j))K(x, y_1, \ldots, y_m) \prod_{j=1}^{m} f_j(y_j)d\vec{y},$$

where S is any finite subset of $Z^+ \times \{1, \ldots, m\}$. If $S = \emptyset$, we simply denote $T_{\vec{b},\emptyset} = T$. These commutators are reflexible enough to generalize the following three kinds of commutators which were firstly introduced and studied in [10], [19] and [18], respectively.

$$T_{\vec{b}}(\vec{f})(x) = \sum_{i=1}^{m} \big(b_i T(\vec{f})(x) - T(f_1, \ldots, b_i f_i, \ldots, f_m)(x)\big),$$

$$T_{\vec{b}}(f)(x) = \int_{\mathbb{R}^n} [\prod_{j=1}^{m}(b_j(x) - b_j(y))]K(x, y)f(y)dy,$$

$$T_{\prod b}(\vec{f})(x) = \int_{\mathbb{R}^{nm}} \prod_{j=1}^{m}(b_j(x) - b_j(y_j))K(x, y_1, \ldots, y_m)dy_1 \cdots dy_m.$$

Commutators of singular integral operators have been the subject of many recent articles, see [4, 14–17] and the references therein. Boundedness estimates for commutators of singular integral operators with Lipschitz functions on Lebesgue space, homogenous Triebel-Lizorkin space and Lipschitz spaces respectively can be found in [12, 22, 24]. The vector-valued extensions can be found in [20, 21, 23]. For surveys and historical details about this subject, we refer to [1,2,5,6,8,9,11] and references therein. The main purpose of this article is to prove the strong and endpoint estimates for $T_{\vec{b},S}$.

In order to state our main results, let us give some notations first. Following [25], let R be a map from S to the set of positive numbers that are bigger than one. $|S|$ denotes the cardinal number of S. We denote $r_{ij} = R(i, j)$, $\frac{1}{r_j} = \sum_{i:(i,j)\in S} \frac{1}{r_{ij}}$ and $\vec{R}_S = (\frac{1}{r_1}, \ldots, \frac{1}{r_m})$. For $k \in N^+$, $\|K\|_k = \inf\{\frac{A_\varepsilon}{\varepsilon^k} : (1.1)$ and (1.2) hold$\}$ and $\|T\|_k = \|T\|_{L^1\times\cdots\times L^1 \to L^{\frac{1}{m},\infty}} + \inf\{\frac{A_\varepsilon}{\varepsilon^k} : (1.1)$ and (1.2) hold$\}$. Denote $\|T\|_1$ by $\|T\|$. The main results of this paper are as follows.

When $\{b_i\}_{i=1}^{\infty}$ belongs to Orlicz-type space, we get:

Theorem 1.1. *Let $\vec{\omega} \in A_{\vec{p}}$ with $\frac{1}{p} = \sum_{j=1}^{m} \frac{1}{p_j}, 1 < p_j < \infty$. Suppose that $b_i \in Osc_{\exp L^{r_{ij}}}$ for $r_{ij} \geq 1$ $(j = 1, \ldots, m)$. Then, there exists a constant C depending on $\vec{\omega}$ such that*

$$\|T_{\vec{b},S}(\vec{f})\|_{L^p(\upsilon_{\vec{\omega}})} \leq C\|T\|_{|S|+1} \prod_{(i,j)\in S} \|b_i\|_{Osc_{\exp L^{r_{ij}}}} \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\omega_j)}$$

for any bounded and compactly supported functions f_j $(j = 1, \ldots, m)$.

Theorem 1.2. *Let* $\Phi(t) = t(1+\log^+ t)^{\sum_{j=1}^m \frac{1}{r_j}}$, $\Phi_j(t) = t(1+\log^+ t)^{\frac{1}{r_j}}$ *with* $\frac{1}{r_j} = \sum_{(i,j)\in S}\frac{1}{r_{ij}}$ *for* $r_{ij} \geq 1$ $(j = 1,\dots,m)$. *Suppose that* $\vec{\omega} \in A_{\vec{1}}$ *and* $b_i \in Osc_{\exp L^{r_{ij}}}$. *Then, there exists a constant* C *depending on* $\vec{\omega}$ *and* T *such that for any* $t > 0$

$$\upsilon_{\vec{\omega}}\{x \in \mathbb{R}^n : |T_{\vec{b},S}(\vec{f})(x)| > t^m\} \leq C\prod_{j=1}^m\Big(\int_{\mathbb{R}^n}\Phi\big(\frac{|f_j(x)|\prod_{i:(i,j)\in S}\|b_i\|_{Osc_{\exp L^{r_{ij}}}}}{t}\big)\omega_j(x)dx\Big)^{\frac{1}{m}} \tag{1.3}$$

for any bounded and compactly supported functions f_j $(j = 1,\dots,m)$.

Moreover, when $r_{ij} \equiv 1$, *for any* $(i,j) \in S$, *this result is sharp in the sense that it doesn't hold for* $\Phi(t) = t(1+\log^+ t)^\alpha$ *with* $\alpha < \sum_{j=1}^m \frac{1}{r_j}$.

When $\{b_i\}_{i=1}^\infty$ belongs to Lipschitz-type space, we get:

Theorem 1.3. *Let* $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m} - \frac{\sum_{(i,1)\in S}\beta_{i1}}{n} - \cdots - \frac{\sum_{(i,m)\in S}\beta_{im}}{n}$ *for* $1 < q_j < \infty$ *and* $\beta = \sum_{(i,1)\in S}\beta_{i1} + \cdots + \sum_{(i,m)\in S}\beta_{im}$ *such that*

$$\frac{1}{q_1} > \frac{\sum_{(i,1)\in S}\beta_{i1}}{n}, \dots, \frac{1}{q_m} > \frac{\sum_{(i,m)\in S}\beta_{im}}{n}.$$

Suppose that $b_i \in Lip(\beta_{ij})$ *with* $0 < \beta_{ij} < 1$ *for any* $(i,j) \in S$. *Then there exists a constant* C *depending on* $\vec{\omega}$ *such that*

$$\|T_{\vec{b},S}(\vec{f})\|_{L^q} \leq C\prod_{(i,j)\in S}\|b_{ij}\|_{Lip\beta_{ij}}\prod_{j=1}^m\|f_j\|_{L^{q_j}}$$

for any bounded and compactly supported functions f_j $(j = 1,\dots,m)$.

2. Some preliminaries

2.1. Weights

We first recall the definition of multiple weight $A_{\vec{p}}$. Let $1 \leq p_1,\dots,p_m < \infty$, and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\vec{p} = (p_1,\dots,p_m)$. Given $\vec{\omega} = (\omega_1,\dots,\omega_m)$, set $\nu_{\vec{\omega}} = \prod_{j=1}^m \omega_j^{\frac{p}{p_j}}$. We say that a weight $\vec{\omega}$ belongs to the class $A_{\vec{p}}$ if there is a constant C such that

$$\sup_Q\Big(\frac{1}{|Q|}\int_Q\prod_{j=1}^m\omega_j^{\frac{p}{p_j}}\Big)^{\frac{1}{p}}\prod_{j=1}^m\Big(\frac{1}{|Q|}\int_Q\omega_j^{1-p_j'}\Big)^{\frac{1}{p_j'}} < C.$$

When $p_j = 1$, $(\frac{1}{|Q|}\int_Q\omega_j^{1-p_j'})^{\frac{1}{p_j'}}$ is understood as $(\inf_Q\omega_j)^{-1}$. When $m = 1$, this coincides with the classical A_p weight defined in the following way.

Let $\omega(x) \geq 0$ and $\omega(x) \in L^1_{loc}(\mathbb{R}^n)$. We say that ω belongs to A_p for $1 < p < \infty$, if

$$[\omega]_{A_p} := \sup_Q \frac{1}{|Q|}\int_Q \omega(x)dx\bigg(\frac{1}{|Q|}\int_Q \omega(x)^{1-p'}dx\bigg)^{p-1} < \infty,$$

where $\frac{1}{p}+\frac{1}{p'}=1$.

We say that ω belongs to A_1, if there is a constant $C > 0$ such that

$$M\omega(x) \leq C\omega(x), \text{ a.e. } x \in \mathbb{R}^n.$$

2.2. Maximal functions

Throughout the paper, M denotes the Hardy-Littlewood maximal operator. For $\delta > 0, M_\delta$ is defined by

$$M_\delta(f)(x) = \bigg(\sup_{x\in Q}\frac{1}{|Q|}\int_Q |f(y)|^\delta dy\bigg)^{\frac{1}{\delta}}.$$

The Fefferman and Stein sharp maximal function $M^\sharp$ is defined by

$$M^\sharp(f)(x) = \sup_{x\in Q}\inf_{c\in\mathbb{R}}\frac{1}{|Q|}\int_Q |f(y)-c|dy.$$

The new multilinear maximal function $\mathcal{M}$ and $\mathcal{M}_r(\vec{f})(x)$ are defined by

$$\mathcal{M}(\vec{f})(x) = \sup_{x\in Q}\prod_{j=1}^m \frac{1}{|Q|}\int_Q |f_j(y_j)|dy_j$$

and

$$\mathcal{M}_r(\vec{f})(x) = \sup_{x\in Q}\prod_{j=1}^m \bigg(\frac{1}{|Q|}\int_Q |f_j(y_j)|^r dy_j\bigg)^{\frac{1}{r}},$$

where the supremum is taken over all cubes Q containing x.

In a similar way, $\mathcal{M}_{L(\log L)^{\vec{\alpha}}}$ is defined by

$$\mathcal{M}_{L(\log L)^{\vec{\alpha}}}(\vec{f})(x) = \sup_{x\in Q}\prod_{j=1}^m \|f_j\|_{L(\log L)^{\alpha_j},Q},$$

where $\vec{\alpha} = (\alpha_1,\ldots,\alpha_m)$. When $\vec{\alpha} = (0,\ldots,0)$, we write $\mathcal{M}_{L(\log L)^{\vec{0}}}(\vec{f})(x) = \mathcal{M}(\vec{f})(x)$.

2.3. Orlicz space

Let $\Phi : [0,\infty) \to [0,\infty)$ be a Young function, it is convex, increasing, $\Phi(0) = 0$ and $\Phi(t) \to \infty$ as $t \to \infty$, the Φ-norm of a function f over a cube Q is defined by $\|f\|_{\Phi,Q}$. That is

$$\|f\|_{\Phi,Q} = \inf\bigg\{\lambda : \frac{1}{|Q|}\int_Q \Phi\big(\frac{|f(x)|}{\lambda}\big)dx \leq 1\bigg\}.$$

Let Φ_1 and Φ_2 be two Young functions with $\Phi_1(t) \le \Phi_2(t)$, for $t \ge 0$, we have

$$\|f\|_{\Phi_1,Q \le} C\|f\|_{\Phi_1,Q}.$$

The maximal operator $M_\Phi(f)(x)$ is defined by

$$M_\Phi(f)(x) = \sup_{x\in Q} \|f\|_{\Phi,Q},$$

where the supremum is take over all cubes containing x.

When $\Phi(t) = t\log^s(e+t)(s > 0)$, we denote $\|f\|_{\Phi,Q} = \|f\|_{L(\log L)^s,Q}$ and $M_\Phi = M_{L(\log L)^s}$. If $\Phi(t) = e^{t^r} - 1$, we denote $\|f\|_{\Phi,Q} = \|f\|_{\exp L^r,Q}$ and $M_\Phi = M_{\exp L^r}$.

It was shown in [19] that the following generalized Hölder's inequality holds,

$$\frac{1}{|Q|}\int_Q |f_1 \cdots f_m g|dx \le C_m \prod_{j=1}^{m} \|f_j\|_{\exp L^{r_j},Q}\|g\|_{L(\log L)^{\frac{1}{r}},Q}, \tag{2.1}$$

where $r_1, \dots, r_m \ge 1$ and $\frac{1}{r} = \sum_{j=1}^{m} \frac{1}{r_j}$.

For a Young function Φ, the oscillation $Osc_\Phi(f,Q)$ of a function f is defined by $Osc_\Phi(f,Q) = \|f - f_Q\|_{\Phi,Q}$. Also, we define

$$\|f\|_{Osc_\Phi} = \sup_Q\{Osc_\Phi(f,Q)\},$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$.

For $r \ge 1$, we define the space $Osc_{\exp L^r}$ by

$$Osc_{\exp L^r} = \{f \in L^1_{loc}(R^n) : \|f\|_{Osc_{\exp L^r}} < \infty\},$$

where

$$\|f\|_{Osc_{\exp L^r}} = \sup_Q \|f - f_Q\|_{\exp L^r,Q} = \sup_Q \|f - f_Q\|_{e^{t^r}-1,Q},$$

and the supremum is taken over all cubes in $\mathbb{R}^n$. It is easy to see that the space $Osc_{\exp L^r}$ is properly contained in $BMO(\mathbb{R}^n)$ with the norm $\|b\|_{BMO} \le C\|b\|_{Osc_{\exp L^r}}$.

3. Proof of Theorem 1.1 and Theorem 1.2

3.1. Some auxiliary lemmas

Lemma 3.1 ([7]). *For any $0 < p < q < \infty$, there exists a constant C depending on p, q such that for any measurable function f*

$$\|f\|_{L^p(Q,\frac{dx}{|Q|})} \le C\|f\|_{L^{q,\infty}(Q,\frac{dx}{|Q|})}.$$

Lemma 3.2 ([13]). (a) *Let $0 < p < \infty, 0 < \delta < 1$, and let $\omega \in A_\infty$. Then there exists a constant C_n only depending on n,*

$$\|M_\delta f\|_{L^p(\omega)} \le C_n \max\{1,p\}[\omega]_{A_\infty}\|M^\sharp_\delta(f)\|_{L^p(\omega)}$$

for any function f such that the left-hand side is finite.

(b) *Let* $\varphi : (0,\infty) \to (0,\infty)$ *doubling* (*see p.* 550). *Then, there exists a constant* C *depending upon the* A_∞ *constant of* ω *and the doubling condition of* φ *such that*

$$\sup_{\lambda>0} \varphi(\lambda)\omega(y \in \mathbb{R}^n : M_\delta f(y) > \lambda) \le C \sup_{\lambda>0} \varphi(\lambda)\omega(y \in \mathbb{R}^n : M_\delta^\sharp f(y) > \lambda)$$

for any function such that the left-hand side is finite.

Lemma 3.3. *Let* $0 < \delta < \frac{1}{m}$. *Then for any number* δ_0, $\delta < \delta_0 < \infty$, *there exists a constant* C *such that for any bounded and compactly supported functions* f_j $(j = 1, \dots, m)$, *one can obtain*

$$M_\delta^\sharp(T_{\vec{b},S}(\vec{f}))(x) \le C\|T\|_{|S|+1}\mathcal{M}_{L(\log L)^{\vec{R}_S}}(\vec{f})(x) \prod_{(i,j)\in S} \|b_i\|_{Osc_{\exp L^{r_{ij}}}} + C\sum_{D\subset S} M_{\delta_0}(T_{\vec{b},D}(\vec{f}))(x) \prod_{(i,j)\in S\setminus D} \|b_i\|_{BMO}.$$

Proof. For a fixed point $x \in \mathbb{R}^n$, let Q be a cube containing x. Below c_Q denotes a positive constant which will be choosen later. We just need to show that

$$\left(\frac{1}{|Q|}\int_Q \left| |T_{\vec{b},S}(\vec{f})(z)|^\delta - |c_Q|^\delta \right| dz\right)^{\frac{1}{\delta}} \le C\|T\|_{|S|+1}\mathcal{M}_{L(\log L)^{\vec{R}_S}}(\vec{f})(x) \prod_{(i,j)\in S} \|b_i\|_{Osc_{\exp L^{r_{ij}}}} + C\sum_{D\subset S} M_{\delta_0}(T_{\vec{b},D}(\vec{f}))(x) \prod_{(i,j)\in S\setminus D} \|b_i\|_{BMO},$$

where C is independent of x and Q.

Observe that $|\alpha|^\delta - |\beta|^\delta \le |\alpha - \beta|^\delta$ for $0 < \delta < 1$. We need to verify now that

$$\left(\frac{1}{|Q|}\int_Q |T_{\vec{b},S}(\vec{f})(z) - c_Q|^\delta dz\right)^{\frac{1}{\delta}} \le C\|T\|_{|S|+1}\mathcal{M}_{L(\log L)^{\vec{R}_S}}(\vec{f})(x) \prod_{(i,j)\in S} \|b_i\|_{Osc_{\exp L^{r_{ij}}}} + C\sum_{D\subset S} M_{\delta_0}(T_{\vec{b},D}(\vec{f}))(x) \prod_{(i,j)\in S\setminus D} \|b_i\|_{BMO}.$$

Next, we shall use the following identity in [25, Lemma 3.2]:

$$\prod_{(i,j)\in S} (x_{i0} - x_{ij}) = \prod_{(i,j)\in S} (\lambda_i - x_{ij}) + \sum_{D\subset S} (-1)^{|S\setminus D|+1} \prod_{(i,j)\in D} (x_{i0} - x_{ij}) \prod_{(i,j)\in S\setminus D} (x_{i0} - \lambda_i)$$

holds for any constants λ_i, where x_{ij} is a sequence of real numbers for $(i,j) \in S$.

By viewing $b_i(z)$ as x_{i0} and $b_i(y_j)$ as x_{ij} and letting $\lambda_i = (b_i)_Q$ be the average of b_i on Q, we get

$$\begin{aligned}
&\prod_{(i,j)\in S}(b_i(z)-b_i(y_j))\\
=&\prod_{(i,j)\in S}((b_i)_Q-b_i(y_j))\\
&+\sum_{D\subset S}(-1)^{|S\setminus D|+1}\prod_{(i,j)\in D}(b_i(z)-b_i(y_j))\prod_{(i,j)\in S\setminus D}(b_i(z)-(b_i)_Q).
\end{aligned}$$

Then, we have

$$\begin{aligned}
I(z) &= |T_{\vec{b},S}(\vec{f})(z)-c_Q|\\
&=\Big|\int_{\mathbb{R}^{nm}}K(z,\vec{y})\prod_{(i,j)\in S}(b_i(z)-b_i(y_j))\prod_{j=1}^{m}f_j(y_j)d\vec{y}-c_Q\Big|\\
&=\Big|\int_{\mathbb{R}^{nm}}K(z,\vec{y})\prod_{(i,j)\in S}((b_i)_Q-b_i(y_j))\\
&\quad+\sum_{D\subset S}(-1)^{|D^c|+1}\prod_{(i,j)\in S\setminus D}(b_i(z)-(b_i)_Q)\\
&\quad\times\prod_{(i,j)\in D}(b_i(z)-b_i(y_j))\prod_{j=1}^{m}f_j(y_j)d\vec{y}-c_Q\Big|\\
&\le\Big|\int_{\mathbb{R}^{nm}}K(z,\vec{y})\prod_{(i,j)\in S}((b_i)_Q-b_i(y_j))\prod_{j=1}^{m}f_j(y_j)d\vec{y}-c_Q\\
&\quad+\int_{\mathbb{R}^{nm}}K(z,\vec{y})\sum_{D\subset S}(-1)^{|D|^c+1}\prod_{(i,j)\in S\setminus D}(b_i(z)-(b_i)_Q)\\
&\quad\times\prod_{(i,j)\in D}(b_i(z)-b_i(y_j))\prod_{j=1}^{m}f_j(y_j)d\vec{y}\Big|\\
&\le\Big|\int_{\mathbb{R}^{nm}}K(z,\vec{y})\prod_{(i,j)\in S}((b_i)_Q-b_i(y_j))\prod_{j=1}^{m}f_j(y_j)d\vec{y}-c_Q\Big|\\
&\quad+\sum_{D\subset S}\prod_{(i,j)\in S\setminus D}|b_i(z)-(b_i)_Q|\\
&\quad\times\Big|\int_{\mathbb{R}^{nm}}K(z,\vec{y})\prod_{(i,j)\in D}(b_i(z)-b_i(y_j))\prod_{j=1}^{m}f_j(y_j)d\vec{y}\Big|.
\end{aligned}$$

In order to control $I(z)$, we set

$$c_Q=\sum_{\vec{\alpha},\vec{\alpha}\neq\vec{0}}\int_{\mathbb{R}^{nm}}K(x,\vec{y})\prod_{(i,j)\in S}((b_i)_Q-b_i(y_j))\prod_{j=1}^{m}f_j(y_j)d\vec{y};$$

$$T_{\vec{b},D}(\vec{f})(z)=\int_{\mathbb{R}^{nm}}K(z,\vec{y})\prod_{(i,j)\in D}(b_i(z)-b_i(y_j))\prod_{j=1}^{m}f_j(y_j)d\vec{y}.$$

We next split each f_j as $f_j=f_j\chi_Q+f_j\chi_{Q^c}=f_j^0+f_j^\infty$ and write

$$\prod_{j=1}^{m}f_j(y_j)=\vec{f}^{\vec{0}}+\sum_{\vec{\alpha},\vec{\alpha}\neq\vec{0}}\vec{f}^{\vec{\alpha}},$$

where $\vec{\alpha}=(\alpha_1,\dots,\alpha_m)$, $\alpha_i=0$ or ∞, $\vec{f}^{\vec{\alpha}}=\prod_{j=1}^{m}f_j^{\alpha_j}(y_j)$.

Then, we have

$$\begin{aligned}
I(z)\leq&\Big|\int_{\mathbb{R}^{nm}}K(z,\vec{y})\prod_{(i,j)\in S}((b_i)_Q-b_i(y_j))[\prod_{j=1}^{m}f_j^0(y_j)+\sum_{\vec{\alpha},\vec{\alpha}\neq\vec{0}}\prod_{j=1}^{m}f_j^{\alpha_j}(y_j)]d\vec{y}\\
&-\sum_{\vec{\alpha},\vec{\alpha}\neq\vec{0}}\int_{\mathbb{R}^{nm}}K(x,\vec{y})\prod_{(i,j)\in S}((b_i)_Q-b_i(y_j))\prod_{j=1}^{m}f_j^{\alpha_j}(y_j)d\vec{y}\Big|\\
&+\sum_{D\subset S}\prod_{(i,j)\in S\setminus D}|(b_i(z)-(b_i)_Q)|\cdot|K_{\vec{b},D}(\vec{f})(z)|\\
\leq&\Big|\int_{\mathbb{R}^{nm}}K(z,\vec{y})\prod_{(i,j)\in S}((b_i)_Q-b_i(y_j))\prod_{j=1}^{m}f_j^0(y_j)d\vec{y}\Big|\\
&+\Big|\int_{\mathbb{R}^{nm}}K(z,\vec{y})\prod_{(i,j)\in S}((b_i)_Q-b_i(y_j))\sum_{\vec{\alpha},\vec{\alpha}\neq\vec{0}}\prod_{j=1}^{m}f_j^{\alpha_j}(y_j)d\vec{y}\\
&-\sum_{\vec{\alpha},\vec{\alpha}\neq\vec{0}}\int_{\mathbb{R}^{nm}}K(x,\vec{y})\prod_{(i,j)\in S}((b_i)_Q-b_i(y_j))\prod_{j=1}^{m}f_j^{\alpha_j}(y_j)d\vec{y}\Big|\\
&+\sum_{D\subset S}\prod_{(i,j)\in S\setminus D}|(b_i(z)-(b_i)_Q)|\cdot|K_{\vec{b},D}(\vec{f})(z)|\\
\leq&\Big|\int_{\mathbb{R}^{nm}}K(z,\vec{y})\prod_{(i,j)\in S}((b_i)_Q-b_i(y_j))\prod_{j=1}^{m}f_j^0(y_j)d\vec{y}\Big|\\
&+\sum_{\vec{\alpha},\vec{\alpha}\neq\vec{0}}\Big|\int_{\mathbb{R}^{nm}}(K(z,\vec{y})-K(x,\vec{y}))\prod_{(i,j)\in S}((b_i)_Q-b_i(y_j))\prod_{j=1}^{m}f_j^{\alpha_j}(y_j)d\vec{y}\Big|\\
&+\sum_{D\subset S}\prod_{(i,j)\in S\setminus D}|(b_i(z)-(b_i)_Q)|\cdot|T_{\vec{b},D}(\vec{f})(z)|.
\end{aligned}$$

Let

$$I_{\vec{0}}(z)=\int_{\mathbb{R}^{nm}}K(z,\vec{y})\prod_{(i,j)\in S}((b_i)_Q-b_i(y_j))\prod_{j=1}^{m}f_j^0(y_j)d\vec{y}$$

and

$$I_{\vec{\alpha}}(z)=\int_{\mathbb{R}^{nm}}(K(z,\vec{y})-K(x,\vec{y}))\prod_{(i,j)\in S}((b_i)_Q-b_i(y_j))\prod_{j=1}^{m}f_j^{\alpha_j}(y_j)d\vec{y}.$$

Hence, we have

$$I(z)\le I_{\vec{0}}(z)+\sum_{\vec{\alpha},\vec{\alpha}\neq\vec{0}}I_{\vec{\alpha}}(z)+\sum_{D\subset S}\prod_{(i,j)\in S\setminus D}|(b_i(z)-(b_i)_Q)|\cdot|T_{\vec{b},D}(\vec{f})(z)|.$$

It yields that

$$\begin{aligned}(3.1)\qquad &\left(\frac{1}{|Q|}\int_Q|I(z)|^\delta dz\right)^{\frac{1}{\delta}}\\ &\le C\left(\frac{1}{|Q|}\int_Q|I_{\vec{0}}(z)|^\delta(z)dz\right)^{\frac{1}{\delta}}+C\sum_{\vec{\alpha},\vec{\alpha}\neq\vec{0}}\left(\frac{1}{|Q|}\int_Q|I_{\vec{\alpha}}(z)|^\delta(z)dz\right)^{\frac{1}{\delta}}\\ &\quad+C\sum_{D\subset S}\left(\frac{1}{|Q|}\int_Q\prod_{(i,j)\in S\setminus D}|(b_i(z)-(b_i)_Q)|^\delta\cdot|T_{\vec{b},D}(\vec{f})(z)|^\delta dz\right)^{\frac{1}{\delta}}\\ &=C\Big(I_{\vec{0}}+\sum_{\vec{\alpha},\vec{\alpha}\neq\vec{0}}I_{\vec{\alpha}}+\sum_{D\subset S}I_D\Big).\end{aligned}$$

Since $\delta_{ij}\ge 1$, $\delta_0\ge 0$ and $\sum_{(i,j)\in D^c}\frac{1}{\delta_{ij}}+\frac{1}{\delta_0}=\frac{1}{\delta}$, by Hölder's inequality

$$\begin{aligned}(3.2)\quad &I_D\\ &\le\prod_{(i,j)\in S\setminus D}\left(\frac{1}{|Q|}\int_Q|b_i(z)-(b_i)_Q|^{\delta_{ij}}dz\right)^{\frac{1}{\delta_{ij}}}\left(\frac{1}{|Q|}\int_Q|T_{\vec{b},D}(\vec{f})(z)|^{\delta_0}dz\right)^{\frac{1}{\delta_0}}\\ &\le C\prod_{(i,j)\in S\setminus D}\|b_i\|_{BMO}M_{\delta_0}(T_{\vec{b},D}(\vec{f}))(x)).\end{aligned}$$

Observe that T is bounded from $L^1\times\cdots\times L^1$ to $L^{\frac{1}{m},\infty}$ and $0<\delta<\frac{1}{m}$, then, by Lemma 3.1 and the generalized Hölder's inequality (2.1), we get

$$\begin{aligned}(3.3)\quad &I_{\vec{0}}\\ &=\left(\frac{1}{|Q|}\int_Q|T(f_1\prod_{(i,1)\in S}((b_i)_Q-b_i(y_1)),\ldots,f_m\prod_{(i,m)\in S}((b_i)_Q-b_i(y_m))|^\delta dz\right)^{\frac{1}{\delta}}\\ &\le C\|T(f_1\prod_{(i,1)\in S}((b_i)_Q-b_i(y_1)),\ldots,f_m\prod_{(i,m)\in S}((b_i)_Q-b_i(y_m))\|_{L^{\frac{1}{m},\infty}(Q,\frac{dx}{|Q|})}\end{aligned}$$

$$\le \|T\|_{L^1\times\cdots\times L^1\to L^{\frac{1}{m},\infty}} \prod_{j=1}^{m} \frac{1}{|Q|}\int_Q |f_j| \prod_{(i,j)\in S} |(b_i)_Q - b_i(y_j)|dy_j$$
$$\le C\|T\|\mathcal{M}_{L(\log L)^{\vec{R}_S}}(\vec{f})(x) \prod_{(i,j)\in S} \|b_i\|_{Osc_{\exp L^{r_{ij}}}}.$$

We are now in a position to estimate $I_{\vec{\alpha}}$ with $\vec{\alpha}\neq\vec{0}$. Without loss of generality, we assume that $\alpha_{j_1}=\cdots=\alpha_{j_l}=0$ and $\alpha_j=\infty$ if $j\notin j_1,\ldots,j_l$, $0\le l<m$. By (1.1), we obtain

$$I_{\vec{\alpha}} \le A_\varepsilon \prod_{j=j_1,\ldots,j_l} \int_Q |f_j(y_j)| \prod_{(i,j)\in S} |(b_i)_Q - b_i(y_j)|dy_j$$
$$\times \sum_{k=1}^{\infty} \frac{|Q|^{\frac{\varepsilon}{n}}}{(3^k|Q^{\frac{1}{n}}|)^{nm+\varepsilon}} \int_{3^kQ} \prod_{j\notin\{j_1,\ldots,j_l\}} (|f_j(y_j)| \prod_{(i,j)\in S} |(b_i)_Q - b_i(y_j)|)dy_j$$
$$\le A_\varepsilon \sum_{k=1}^{\infty} \frac{1}{3^{k\varepsilon}} \prod_{j=1}^{m} \frac{1}{|3^kQ|}\int_{3^kQ} |f_j(y_j)| \prod_{(i,j)\in S} |(b_i)_Q - b_i(y_j)|dy_j.$$

Applying the generalized Hölder's inequality (2.1), and noting that

$$\sum_{k=1}^{\infty} \frac{k^{|S|}}{3^{k\varepsilon}} \le 2\int_0^\infty \frac{x^{|S|}}{3^{\varepsilon x}}dx = \frac{2}{(\varepsilon\ln 3)^{|S|+1}} \int_1^\infty \frac{(\ln y)^{|S|}}{y^2}dy < \infty,$$

one obtain that

$$I_{\vec{\alpha}} \le A_\varepsilon \sum_{k=1}^{\infty} \frac{1}{3^{k\varepsilon}} \prod_{j=1}^{m} \|f_j\|_{L(\log L)^{\frac{1}{r_j}},3^kQ} \prod_{(i,j)\in S} \|b_i(y_j)-(b_i)_Q\|_{\exp L^{r_{ij}},3^kQ} \tag{3.4}$$
$$\le A_\varepsilon \sum_{k=1}^{\infty} \frac{k^{|S|}}{3^{k\varepsilon}} \prod_{j=1}^{m} \|f_j\|_{L(\log L)^{\frac{1}{r_j}},3^kQ} \prod_{(i,j)\in S} \|b_i(y_j)-(b_i)_{3^kQ}\|_{\exp L^{r_{ij}},3^kQ}$$
$$\le \frac{C}{\varepsilon^{|S|+1}} A_\varepsilon \mathcal{M}_{L(\log L)^{\vec{R}_S}}(\vec{f})(x) \prod_{(i,j)\in S} \|b_i\|_{Osc_{\exp L^{r_{ij}}}}.$$

By (3.1), (3.2), (3.3) and (3.4), Lemma 3.3 is proved. □

By using the arguments in the proof of Theorem 1.1 and Theorem 1.3 in [25], we get the following two lemmas.

Lemma 3.4. *Let $0<p<\infty$, $\omega\in A_\infty$. Then, for any bounded and compactly supported functions f_j $(j=1,\ldots,m)$, we have*

$$\| T_{\vec{b},S}(\vec{f}) \|_{L^p(w)}$$
$$\le C\big(\|T\|[\omega]_{A_\infty}^{|S|} + \|K\|_{|S|+1}\big)[\omega]_{A_\infty} \prod_{(i,j)\in S} \|b_i\|_{Osc_{\exp L^{r_{ij}}}} \|\mathcal{M}_{L(\log)L^{\vec{R}_s}}(\vec{f})\|_{L^p(w)}.$$

We say that φ is doubling if $\varphi(2t)\le C\varphi(t)$ for any $t>0$.

Lemma 3.5. *Suppose $p > 0$ and $\omega \in A_\infty$. Let $\varphi : (0,\infty) \longrightarrow (0,\infty)$ be doubling and $\varphi(t) < C_1 t$ for any $t > 0$. Suppose that $b_i \in Osc_{\exp L^{r_{ij}}}$, $r_{ij} \geq 1$ ($j = 1,\dots,m$). Then, there exists a constant $C > 0$ depending on the A_∞ constant of ω, such that*

$$\begin{aligned}&\sup_{\lambda>0}\varphi(\lambda)\omega\Big\{x \in \mathbb{R}^n : |T_{\vec{b},S}(\vec{f})(x)| > \lambda^m\Big\}\\ &\leq C\sup_{\lambda>0}\varphi(\lambda)\omega\Big\{x \in \mathbb{R}^n : \mathcal{M}_{L(\log L)^{\vec{R}_s}}(\vec{f})(x) > \frac{\lambda^m}{\|T\|_{\|S\|+1}\prod_{(i,j)\in S}\|b_i\|_{Osc_{\exp L^{r_{ij}}}}}\Big\}\end{aligned}$$

for any bounded and compactly supported functions f_j ($j = 1,\dots,m$).

3.2. Proof of Theorem 1.1 and Theorem 1.2

Proof of Theorem 1.1. Now, by Lemma 3.4 and the fact $\nu_{\vec{\omega}}$ is also in A_∞, we get

$$\begin{aligned}&\|T_{\vec{b},S}(\vec{f})\|_{L^p(w)}\\ &\leq C(\|T\|[\omega]_{A_\infty}^{|S|} + \|K\|_{|S|+1})[\omega]_{A_\infty}\prod_{(i,j)\in S}\|b_i\|_{Osc_{\exp L^{r_{ij}}}}\|\mathcal{M}_{L(\log)L^{\vec{R}_s}}(\vec{f})\|_{L^p(w)}\\ &\leq C(\|T\|[\nu_{\vec{\omega}}]_{A_\infty}^{|S|+1} + \|K\|_{|S|+1}[\nu_{\vec{\omega}}]_{A_\infty}^{|S|+1}[\nu_{\vec{\omega}}]_{A_\infty}^{-|S|})\\ &\quad\times\prod_{(i,j)\in S}\|b_i\|_{Osc_{\exp L^{r_{ij}}}}\|\mathcal{M}_{L(\log)L^{\vec{R}_s}}(\vec{f})\|_{L^p(\nu_{\vec{\omega}})}\\ &\leq C(\|T\| + \|K\|_{|S|+1})\prod_{(i,j)\in S}\|b_i\|_{Osc_{\exp L^{r_{ij}}}}\|\mathcal{M}_{L(\log)L^{\vec{R}_s}}(\vec{f})\|_{L^p(\nu_{\vec{\omega}})}.\end{aligned}$$

Thus, we just have to prove

$$\|\mathcal{M}_{L(\log)L^{\vec{R}_s}}(\vec{f})\|_{L^p(\nu_{\vec{\omega}})} \leq C\prod_{j=1}^m\|f_j\|_{L^{p_j}(\omega_j)}.$$

Let $r > 1$, $\Phi(t) = t\log^{\frac{1}{r_j}}(e+t) < t^r$, where $t > 1$. By Generalized Jensen's inequality

$$\|f\|_{L(\log L)^{\frac{1}{r_j}},Q} \leq C\|f\|_{t^r,Q}.$$

It is easy to check that

$$\begin{aligned}\|f\|_{t^r,Q} &= \inf\{\lambda > 0 : \frac{1}{|Q|}\int_Q\left(\frac{|f(y)|}{\lambda}\right)^r dy \leq 1\}\\ &= \inf\{\lambda > 0 : \left(\frac{1}{|Q|}\int_Q|f(y)|^r dy\right)^{\frac{1}{r}} \leq \lambda\}\\ &= \left(\frac{1}{|Q|}\int_Q|f(y)|^r dy\right)^{\frac{1}{r}}.\end{aligned}$$

Therefore, we have

$$\|f\|_{L(\log L)^{\frac{1}{r_j}},Q} \leq C\left(\frac{1}{|Q|}\int_Q |f(y)|^r dy\right)^{\frac{1}{r}}.$$

Thus

$$\begin{aligned}&\|\mathcal{M}_{L(\log)L^{\vec{R}_s}}(\vec{f})\|_{L^p(\nu_{\vec{\omega}})}\\ &\leq C(\|T\|+\|K\|_{|S|+1})[\nu_{\vec{\omega}}]_{A_\infty}^{|S|+1}\prod_{(i,j)\in S}\|b_i\|_{Osc_{\exp L^{r_{ij}}}}\|\mathcal{M}_r(\vec{f})(x)\|_{L^p(\nu_{\vec{\omega}})}.\end{aligned}$$

We need to verify now that

$$\|\mathcal{M}_r(\vec{f})(x)\|_{L^p(\nu_{\vec{\omega}})} \leq C\prod_{j=1}^m \|f_j\|_{L^{p_j}(\omega_j)}.$$

It is equivalent to prove

$$\|\mathcal{M}(\vec{f})(x)\|_{L^{\frac{p}{r}}(\nu_{\vec{\omega}})} \leq C\prod_{j=1}^m \|f_j\|_{L^{\frac{p_j}{r}}(\omega_j)}.$$

By Theorem 3.7 in [10], this is equivalent to show that $\vec{\omega}\in \vec{A}_{\frac{\vec{p}}{r}}$ and we already know that this is true for some $r>1$ because of Lemma 6.1 in [10]. □

Proof of Theorem 1.2. By homogeneity, we only need to prove (1.3) when $t=1$. It is easy to see that $\frac{1}{\Phi(\frac{1}{t})}$ is doubling, and $\frac{1}{\Phi(\frac{1}{t})}\leq Ct$ for some $C>0$.

By Lemma 3.5 and Theorem 1.5 in [25], we have

$$\begin{aligned}&\nu_{\vec{\omega}}\Big\{x\in\mathbb{R}^n : |T_{\vec{b},S}(\vec{f})(x)|>1\Big\}\\ &\leq C\sup_{t>0}\frac{1}{\Phi(\frac{1}{t})}\nu_{\vec{\omega}}\Big\{y\in\mathbb{R}^n : \mathcal{M}_{L(\log L)^{\vec{R}_S}}(\vec{f})(y) > \frac{t^m}{\prod_{(i,j)\in S}\|b_i\|_{Osc_{\exp L^{r_{ij}}}}}\Big\}.\\ &\leq C\sup_{t>0}\frac{1}{\Phi(\frac{1}{t})}\prod_{j=1}^m\bigg(\int_{\mathbb{R}^n}\Phi\Big(\frac{\prod_{(i,j)\in S}\|b_i\|_{Osc_{\exp L^{r_{ij}}}}|f_j(x)|}{t}\Big)\omega_j(x)dx\bigg)^{\frac{1}{m}}\\ &\leq C\sup_{t>0}\frac{1}{\Phi(\frac{1}{t})}\prod_{j=1}^m\bigg(\int_{\mathbb{R}^n}\Phi(\frac{1}{t})\Phi(\prod_{(i,j)\in S}\|b_i\|_{Osc_{\exp L^{r_{ij}}}}|f_j(x)|)\omega_j(x)dx\bigg)^{\frac{1}{m}}\\ &\leq C\sup_{t>0}\prod_{j=1}^m\bigg(\int_{\mathbb{R}^n}\Phi(\prod_{(i,j)\in S}\|b_i\|_{Osc_{\exp L^{r_{ij}}}}|f_j(x)|)\omega_j(x)dx\bigg)^{\frac{1}{m}}.\end{aligned}$$

The sharpness of Theorem 1.2 follows from the celebrated example due to Pérez [18]: take $m=2$, $n=1$, $f_j=\chi_{(0,1)}$, and $b_i(x)=\log|1+x|$. The general case follows in a similar way. □

4. Proof of Theorem 1.3

4.1. Auxiliary results

For $0 < \beta < \frac{n}{r}$, we define

$$M_{r,\beta}f(x) = \sup_{x\in Q}\left(\frac{1}{|Q|^{1-\frac{r\beta}{n}}}\int_Q |f(y)|^r dy\right)^{\frac{1}{r}}.$$

When $\beta = 0$, we denote $M_{r,\beta}$ simply by M_r and if $r < q < \infty$, then we get

$$\|M_r f\|_{L^q} \leq C\|f\|_{L^q}. \tag{4.1}$$

Lemma 4.1 ([3]). *For $0 < \beta < n, 0 < r < p < \frac{n}{\beta}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{n}$, we have*

$$\|M_{r,\beta}f\|_{L^q} \leq C\|f\|_{L^p}.$$

Lemma 4.2 ([15]). (1) *For $0 < \beta < 1$, $1 \leq q < \infty$, we have*

$$\|f\|_{Lip_\beta} \approx \sup_Q \frac{1}{|Q|^{1+\frac{\beta}{n}}}\int_Q |f - f_Q| \approx \sup_Q \frac{1}{|Q|^{\frac{\beta}{n}}}(\frac{1}{|Q|}\int_Q |f - f_Q|^q)^{\frac{1}{q}}.$$

(2) *For $0 < \beta < 1$, $1 \leq p < \infty$, we have*

$$\|f\|_{\dot{F}_p^{\beta,\infty}} \approx \| \sup_Q \frac{1}{|Q|^{1+\frac{\beta}{n}}}\int_Q |f - f_Q| \, \|_{L^p}.$$

Lemma 4.3 ([15]). *Let $b \in Lip_\beta$, $0 < \beta < 1$. For any cubes Q, Q' in $\mathbb{R}^n$ and $Q' \subset Q$, we have*

$$|b_{Q'} - b_Q| \leq C\|b\|_{Lip_\beta}|Q|^{\frac{\beta}{n}}.$$

4.2. A key lemma

Lemma 4.4. *Let $0 < \delta < \frac{1}{m}$ and $1 < p_1, p_2, \ldots, p_m < \infty$. Suppose that $\delta < \delta_0 < \infty$ and $0 < \beta_{ij} < 1$ for any $(i,j) \in S$. Then there exists a constant C such that*

$$\begin{aligned} M_\delta^\sharp(T_{\vec{b},S}(\vec{f}))(x) &\leq C\prod_{(i,j)\in S}\|b_i\|_{Lip\beta_{ij}}\prod_{j=1}^m M_{p_j,\sum_{(i,j)\in S}\beta_{ij}}(f_j)(x) \\ &\quad + C\sum_{D\subset S}\prod_{(i,j)\in S\setminus D}\|b_i\|_{Lip\beta_{ij}} M_{\delta_0,\sum_{(i,j)\in S\setminus D}\beta_{ij}}(T_{\vec{b},D}(\vec{f}))(x) \end{aligned}$$

for any bounded and compactly supported functions f_j $(j = 1, \ldots, m)$.

Proof. We will adopt the idea in Lemma 3.3. We only give the different parts. Lemma 4.4 will be proved if we can show that

$$\begin{aligned} &\left(\frac{1}{|Q|}\int_Q |I(z)|^\delta dz\right)^{\frac{1}{\delta}} \\ &\leq C\prod_{(i,j)\in S}\|b_i\|_{Lip\beta_{ij}}\prod_{j=1}^m M_{p_j,\sum_{(i,j)\in S}\beta_{ij}}(f_j)(x) \end{aligned}$$

$$+ C \sum_{D\subset S} \prod_{(i,j)\in S\setminus D} \|b_i\|_{Lip\beta_{ij}} M_{\delta_0, \sum_{(i,j)\in S\setminus D}\beta_{ij}}(T_{\vec{b},D}(\vec{f}))(x),$$

where, $I(z) = |T_{\vec{b},S}(\vec{f})(z) - c_Q|$.

Let

$$I_{\vec{0}}(z) = \int_{\mathbb{R}^{nm}} K(z,\vec{y}) \prod_{(i,j)\in S} ((b_i)_Q - b_i(y_j)) \prod_{j=1}^{m} f_j^0(y_j) d\vec{y}$$

and

$$I_{\vec{\alpha}}(z) = \int_{\mathbb{R}^{nm}} (K(z,\vec{y}) - K(x,\vec{y})) \prod_{(i,j)\in S} ((b_i)_Q - b_i(y_j)) \prod_{j=1}^{m} f_j^{\alpha_j}(y_j) d\vec{y}.$$

We can control $I(z)$ as

$$I(z) \le I_{\vec{0}}(z) + \sum_{\vec{\alpha},\vec{\alpha}\neq\vec{0}} I_{\vec{\alpha}}(z) + \sum_{D\subset S} \prod_{(i,j)\in S\setminus D} |b_i(z) - (b_i)_Q| \cdot |T_{\vec{b},D}(\vec{f})(z)|.$$

Then, we derive that

$$\begin{aligned} &\left(\frac{1}{|Q|}\int_Q |I(z)|^\delta dz\right)^{\frac{1}{\delta}} \qquad (4.2)\\ &\le C\left(\frac{1}{|Q|}\int_Q |I_{\vec{0}}(z)|^\delta (z) dz\right)^{\frac{1}{\delta}} + C\sum_{\vec{\alpha},\vec{\alpha}\neq\vec{0}} \left(\frac{1}{|Q|}\int_Q |I_{\vec{\alpha}}(z)|^\delta (z) dz\right)^{\frac{1}{\delta}} \\ &\quad + C\sum_{D\subset S}\left(\frac{1}{|Q|}\int_Q \prod_{(i,j)\in S\setminus D} |(b_i(z) - (b_i)_Q)|^\delta \cdot |T_{\vec{b},D}(\vec{f})(z)|^\delta dz\right)^{\frac{1}{\delta}} \\ &= C\left(I_{\vec{0}} + \sum_{\vec{\alpha},\vec{\alpha}\neq\vec{0}} I_{\vec{\alpha}} + \sum_{D\subset S} I_D\right). \end{aligned}$$

Since $\delta_{ij} \ge 1$, $\delta_0 \ge 0$ and $\sum_{(i,j)\in D^c} \frac{1}{\delta_{ij}} + \frac{1}{\delta_0} = \frac{1}{\delta}$, by Hölder's inequality

$$\begin{aligned} &I_D \qquad (4.3)\\ &\le \prod_{(i,j)\in S\setminus D} \left(\frac{1}{|Q|}\int_Q |b_i(z) - (b_i)_Q|^{\delta_{ij}} dz\right)^{\frac{1}{\delta_{ij}}} \left(\frac{1}{|Q|}\int_Q |T_{\vec{b},D}(\vec{f})(z)|^{\delta_0} dz\right)^{\frac{1}{\delta_0}} \\ &\le C \prod_{(i,j)\in S\setminus D} \|b_i\|_{Lip_{\beta_{ij}}} |Q|^{\frac{\beta_{ij}}{n}} \left(\frac{1}{|Q|}\int_Q |T_{\vec{b},D}(\vec{f})(z)|^{\delta_0} dz\right)^{\frac{1}{\delta_0}} \\ &\le C \prod_{(i,j)\in S\setminus D} \|b_i\|_{Lip_{\beta_{ij}}} \left(\frac{1}{|Q|^{1-\frac{\delta_0\sum_{(i,j)\in S\setminus D}\beta_{ij}}{n}}}\int_Q |T_{\vec{b},D}(\vec{f})(z)|^{\delta_0} dz\right)^{\frac{1}{\delta_0}} \\ &\le C \prod_{(i,j)\in S\setminus D} \|b_i\|_{Lip_{\beta_{ij}}} M_{\delta_0, \sum_{(i,j)\in S\setminus D}\beta_{ij}}(T_{\vec{b},D}(\vec{f}))(x). \end{aligned}$$

Using the fact $0<\delta<1/m$, and T is bounded from $L^1\times\cdots\times L^1$ to $L^{\frac{1}{m},\infty}$, together with Lemma 3.1, we get

$$\begin{aligned}
I_{\vec{0}} &= \left(\frac{1}{|Q|}\int_Q |T(f_1\prod_{(i,1)\in S}((b_i)_Q-b_i(y_1)),\ldots,f_m\prod_{(i,m)\in S}((b_i)_Q-b_i(y_m)))|^\delta dz\right)^{\frac{1}{\delta}}\\
&\le C\|T(f_1\prod_{(i,1)\in S}((b_i)_Q-b_i(y_1)),\ldots,f_m\prod_{(i,m)\in S}((b_i)_Q-b_i(y_m)))\|_{L^{\frac{1}{m},\infty}(Q,\frac{dx}{|Q|})}\\
&\le C\|T\|_{L^1\times\cdots\times L^1\to L^{\frac{1}{m},\infty}}\prod_{j=1}^m\frac{1}{|Q|}\int_Q|f_j|\prod_{(i,j)\in S}|(b_i)_Q-b_i(y_j)|dy_j\\
&\le C\prod_{j=1}^m\frac{1}{|Q|}\int_Q|f_j|\prod_{(i,j)\in S}|(b_i)_Q-b_i(y_j)|dy_j.
\end{aligned}$$

Set $\frac{1}{p_j}+\sum_{(i,j)\in S}\frac{1}{p_{ij}}=1$ for $j=1,\ldots,m$. By Hölder's inequality, we have

$$\begin{aligned}
&\prod_{j=1}^m\frac{1}{|Q|}\int_Q|f_j|\prod_{(i,j)\in S}|(b_i)_Q-b_i(y_j)|dy_j \qquad (4.4)\\
&\le C\prod_{j=1}^m\left(\frac{1}{|Q|}\int_Q|f_j|^{p_j}dy_j\right)^{\frac{1}{p_j}}\prod_{(i,j)\in S}\left(\frac{1}{|Q|}\int_Q|(b_i)_Q-b_i(y_j)|^{p_{ij}}dy_j\right)^{\frac{1}{p_{ij}}}\\
&\le C\prod_{j=1}^m\left(\frac{1}{|Q|}\int_Q|f_j|^{p_j}dy_j\right)^{\frac{1}{p_j}}\prod_{(i,j)\in S}\|b_i\|_{Lip_{\beta_{ij}}}|Q|^{\frac{\beta_{ij}}{n}}\\
&\le C\prod_{j=1}^m\left(\frac{1}{|Q|^{1-p_j\frac{\sum_{(i,j)\in S}\beta_{ij}}{n}}}\int_Q|f_j|^{p_j}dy_j\right)^{\frac{1}{p_j}}\prod_{(i,j)\in S}\|b_i\|_{Lip_{\beta_{ij}}}\\
&\le C\prod_{(i,j)\in S}\|b_i\|_{Lip_{\beta_{ij}}}\prod_{j=1}^m M_{p_j,\sum_{(i,j)\in S}\beta_{ij}}(f_j)(x).
\end{aligned}$$

It remain to estimate $I_{\vec{\alpha}}$ with $\vec{\alpha}\neq\vec{0}$. Without loss of generality, we assume that $\alpha_{j_1}=\cdots=\alpha_{j_l}=0$ and $\alpha_j=\infty$ if $j\notin j_1,\ldots,j_l$, $0\le l<m$. We have that

$$\begin{aligned}
I_{\vec{\alpha}} &\le A_\varepsilon\prod_{j=j_1,\ldots,j_l}\int_Q|f_j(y_j)|\prod_{(i,j)\in S}|(b_i)_Q-b_i(y_j)|dy_j\\
&\quad\times\sum_{k=1}^\infty\frac{|Q|^{\frac{\varepsilon}{n}}}{(3^k|Q^{\frac{1}{n}}|)^{nm+\varepsilon}}\int_{3^kQ}\prod_{j\notin\{j_1,\ldots,j_l\}}(|f_j(y_j)|\prod_{(i,j)\in S}|(b_i)_Q-b_i(y_j)|)dy_j\\
&\le A_\varepsilon\sum_{k=1}^\infty\frac{1}{3^{k\varepsilon}}\prod_{j=1}^m\frac{1}{|3^kQ|}\int_{3^kQ}|f_j(y_j)|\prod_{(i,j)\in S}|(b_i)_Q-b_i(y_j)|dy_j.
\end{aligned}$$

Let

$$S_1 = \{(i,j) \in S \mid j = 1\}, \dots, S_m = \{(i,j) \in S \mid j = m\}.$$

And let

$$\frac{1}{p_1} + \frac{1}{p_{11}} + \cdots + \frac{1}{p_{|S_1|1}} = 1, \dots, \frac{1}{p_m} + \frac{1}{p_{1m}} + \cdots + \frac{1}{p_{|S_m|m}} = 1.$$

Then, by Lemma 4.3 and Hölder's inequality, we have

$$\begin{aligned}
(4.5)\quad I_{\vec{\alpha}} &\le A_\varepsilon \sum_{k=1}^{\infty} \frac{1}{3^{k\varepsilon}} \Big(\frac{1}{|3^kQ|} \int_{3^kQ} |f_1(y_1)| \prod_{(i,1)\in S} |b_i(y_1) - (b_i)_{3^kQ} + (b_i)_{3^kQ} - (b_i)_Q| dy_1 \Big) \\
&\quad \cdots \Big(\frac{1}{|3^kQ|} \int_{3^kQ} |f_m(y_m)| \prod_{(i,m)\in S} |b_i(y_m) - (b_i)_{3^kQ} + (b_i)_{3^kQ} - (b_i)_Q| dy_m \Big) \\
&\le A_\varepsilon \sum_{k=1}^{\infty} \frac{1}{3^{k\varepsilon}} \Big(\frac{1}{|3^kQ|} \int_{3^kQ} |f_1(y_1)|^{p_1} dy_1 \Big)^{\frac{1}{p_1}} \\
&\quad \times \Big(\frac{1}{|3^kQ|} \int_{3^kQ} |b_1(y_1) - (b_1)_{3^kQ} + (b_1)_{3^kQ} - (b_1)_Q|^{p_{11}} dy_1 \Big)^{\frac{1}{p_{11}}} \\
&\quad \cdots \Big(\frac{1}{|3^kQ|} \int_{3^kQ} |b_{|S_1|}(y_1) - (b_{|S_1|})_{3^kQ} + (b_{|S_1|})_{3^kQ} - (b_{|S_1|})_Q|^{p_{|S_1|1}} dy_1 \Big)^{\frac{1}{p_{|S_1|1}}} \\
&\quad \cdots \Big(\frac{1}{|3^kQ|} \int_{3^kQ} |f_m(y_m)|^{p_m} dy_m \Big)^{\frac{1}{p_m}} \\
&\quad \times \Big(\frac{1}{|3^kQ|} \int_{3^kQ} |b_1(y_m) - (b_1)_{3^kQ} + (b_1)_{3^kQ} - (b_1)_Q|^{p_{1m}} dy_m \Big)^{\frac{1}{p_{1m}}} \\
&\quad \cdots \Big(\frac{1}{|3^kQ|} \int_{3^kQ} |b_{|S_m|}(y_m) - (b_{|S_m|})_{3^kQ} + (b_{|S_m|})_{3^kQ} - (b_{|S_m|})_Q|^{p_{|S_m|m}} dy_m \Big)^{\frac{1}{p_{|S_1|m}}} \\
&\le A_\varepsilon \prod_{(i,1)\in S} \|b_i\|_{Lip_{\beta_{i1}}} \sum_{k=1}^{\infty} \frac{1}{3^{k\varepsilon}} \Big(\frac{1}{|3^kQ|^{1-p_1 \frac{\sum_{(i,1)\in S} \beta_{i,1}}{n}}} \int_Q |f_1|^{p_1} dy_1 \Big)^{\frac{1}{p_1}} \\
&\quad \cdots \prod_{(i,m)\in S} \|b_i\|_{Lip_{\beta_{im}}} \Big(\frac{1}{|3^kQ|^{1-p_m \frac{\sum_{(i,m)\in S} \beta_{i,m}}{n}}} \int_Q |f_m|^{p_m} dy_m \Big)^{\frac{1}{p_m}} \\
&\le C A_\varepsilon \prod_{(i,j)\in S} \|b_i\|_{Lip_{\beta_{ij}}} \prod_{j=1}^{m} M_{p_j, \sum_{(i,j)\in S} \beta_{ij}}(f_j)(x).
\end{aligned}$$

By (4.2), (4.3), (4.4) and (4.5), Lemma 4.4 is proved. □

4.3. Proof of Theorem 1.3

Proof of Theorem 1.3. We can choose $1 < p_j < q_j$ for $j = 1, \dots, m$. Let $b_i \in L^\infty$ and $f_1, \dots, f_m \in L_c^\infty(\mathbb{R}^n)$. Modifying the argument in [10], one can obtain that $\|M_\delta(T_{\vec{b},S}(\vec{f}))(x)\|_{L^q} < \infty$. Then, by Lemma 3.2, Lemma 4.4, we have

$$\begin{aligned}
&\|T_{\vec{b},S}(\vec{f})(x)\|_{L^q} \\
&\le C\|M_\delta(T_{\vec{b},S}(\vec{f}))(x)\|_{L^q}
\end{aligned}$$

$$\begin{aligned}
&\leq C\|M_\delta^\sharp(T_{\vec{b},S}(\vec{f}))(x)\|_{L^q}\\
&\leq C\prod_{(i,j)\in S}\|b_i\|_{Lip\beta_{ij}}\|M_{p_1,\sum_{(i,1)\in S}\beta_{i1}}(f_1)(x)\cdots M_{p_m,\sum_{(i,m)\in S}\beta_{im}}(f_m)(x)\|_{L^q}\\
&\quad+C\sum_{D\subset S}\prod_{(i,j)\in D^c}\|b_i\|_{Lip\beta_{ij}}\|M_{\delta_0,\sum_{(i,j)\in S\setminus D}\beta_{ij}}(T_{\vec{b},D}(\vec{f}))(x)\|_{L^q}.
\end{aligned}$$

Let

$$\frac{1}{t_1}=\frac{1}{q_1}-\frac{\sum_{(i,1)\in S}\beta_{i1}}{n},\dots,\frac{1}{t_m}=\frac{1}{q_m}-\frac{\sum_{(i,m)\in S}\beta_{im}}{n}$$

and

$$\frac{1}{q'}=\frac{1}{q}+\frac{\sum_{(i,j)\in D^c}\beta_{ij}}{n}=\frac{1}{q_1}+\cdots+\frac{1}{q_m}-\frac{\sum_{(i,j)\in D}\beta_{ij}}{n}.$$

By Lemma 4.1, we have

$$\begin{aligned}
&\|M_\delta^\sharp(T_{\vec{b},S}(\vec{f}))(x)\|_{L^q}\\
&\leq C\prod_{(i,j)\in S}\|b_i\|_{Lip_{\beta_{ij}}}\|M_{p_1,\sum_{(i,1)\in S}\beta_{i1}}(f_1)(x)\|_{L^{t_1}}\cdots\|M_{p_m,\sum_{(i,m)\in S}\beta_{im}}(f_m)(x)\|_{L^{t_m}}\\
&\quad+C\sum_{D\subset S}\prod_{(i,j)\in S\setminus D}\|b_i\|_{Lip\beta_{ij}}\|T_{\vec{b},D}(\vec{f}))(x)\|_{L^{q'}}\\
&\leq C\prod_{(i,j)\in S}\|b_i\|_{Lip_{\beta_{ij}}}\|f_1\|_{L^{q_1}}\cdots\|f_m\|_{L^{q_m}}+C\sum_{D\subset S}\prod_{(i,j)\in S\setminus D}\|b_i\|_{Lip_{\beta_{ij}}}\|T_{\vec{b},D}(\vec{f}))(x)\|_{L^{q'}}.
\end{aligned}$$

Let

$$\frac{1}{t_1'}=\frac{1}{q_1}-\frac{\sum_{(i,1)\in D}\beta_{i1}}{n},\dots,\frac{1}{t_m'}=\frac{1}{q_m}-\frac{\sum_{(i,m)\in D}\beta_{im}}{n}$$

and

$$\frac{1}{q''}=\frac{1}{q'}+\frac{\sum_{(i,j)\in D_1^c}\beta_{ij}}{n}=\frac{1}{q_1}+\cdots+\frac{1}{q_m}-\frac{\sum_{(i,j)\in D_1}\beta_{ij}}{n}.$$

Regarding set D, we can discuss two cases. One case is that set D contains some j but not all j, and the other case is that set D contains all j. If set D contains all j, the same method can be repeated above.

For the first case, D does not contain some j, we have $\sum_{(i,j)\in D}\beta_{ij}=0$ and $t_j'=q_j$. Thus

$$\|M_{p_j,\sum_{(i,j)\in D}\beta_{ij}}(f_j)(x)\|_{L^{t_j'}}=\|M_{p_j}(f_j)(x)\|_{L^{q_j}}.$$

Since $p_j<q_j$, by (4.1), we have

$$\|M_{p_j}(f_j)(x)\|_{L^{q_j}}\leq C\|(f_j)(x)\|_{L^{q_j}}.$$

Then, we have

$$\begin{aligned}
&\|M_\delta^\sharp(T_{\vec{b},D}(\vec{f}))(x)\|_{L^{q'}}\\
&\leq C\prod_{(i,j)\in D}\|b_i\|_{Lip_{\beta_{ij}}}\|M_{p_1,\sum_{(i,1)\in D}\beta_{i1}}(f_1)(x)\cdots M_{p_m,\sum_{(i,m)\in D}\beta_{im}}(f_m)(x)\|_{L^{q'}}
\end{aligned}$$

$$+ C \sum_{D_1 \subset D} \prod_{(i,j)\in D_1^c} \|b_i\|_{Lip_{\beta_{ij}}} \|M_{\delta_1, \sum_{(i,j)\in D_1^c} \beta_{ij}}(T_{\vec{b},D_1}(\vec{f}))(x)\|_{L^{q'}}$$

$$\le C \prod_{(i,j)\in D} \|b_i\|_{Lip_{\beta_{ij}}} \|M_{p_1, \sum_{(i,1)\in D} \beta_{i1}}(f_1)(x)\|_{L^{t_1'}} \cdots \|M_{p_m, \sum_{(i,m)\in D} \beta_{im}}(f_m)(x)\|_{L^{t_m'}}$$

$$+ C \sum_{D_1 \subset D} \prod_{(i,j)\in D_1^c} \|b_i\|_{Lip_{\beta_{ij}}} \|T_{\vec{b},D_1}(\vec{f}))(x)\|_{L^{q''}}$$

$$\le C \prod_{(i,j)\in D} \|b_i\|_{Lip_{\beta_{ij}}} \|f_1\|_{L^{q_1}} \cdots \|f_m\|_{L^{q_m}} + C \sum_{D_1 \subset D} \prod_{(i,j)\in D_1^c} \|b_i\|_{Lip_{\beta_{ij}}} \|T_{\vec{b},D_1}(\vec{f}))(x)\|_{L^{q''}}.$$

We obtain

$$\|M_\delta^\sharp(T_{\vec{b},S}(\vec{f}))(x)\|_{L^q}$$

$$\le C \prod_{(i,j)\in S} \|b_i\|_{Lip_{\beta_{ij}}} \|f_1\|_{L^{q_1}} \cdots \|f_m\|_{L^{q_m}}$$

$$+ C \sum_{D\subset S} \prod_{(i,j)\in D^c} \|b_i\|_{Lip_{\beta_{ij}}} \prod_{(i,j)\in D} \|b_i\|_{Lip_{\beta_{ij}}} \|f_1\|_{L^{q_1}} \cdots \|f_m\|_{L^{q_m}}$$

$$+ C \sum_{D_1 \subset D} \prod_{(i,j)\in D_1^c} \|b_i\|_{Lip_{\beta_{ij}}} \|(T_{\vec{b},D_1}(\vec{f}))(x)\|_{L^{q''}}$$

$$\le C \prod_{(i,j)\in S} \|b_i\|_{Lip_{\beta_{ij}}} \|f_1\|_{L^{q_1}} \cdots \|f_m\|_{L^{q_m}}$$

$$+ C \sum_{D\subset S} \sum_{D_1 \subset D} \prod_{(i,j)\in D_1^c} \|b_i\|_{Lip_{\beta_{ij}}} \prod_{(i,j)\in D^c} \|b_i\|_{Lip_{\beta_{ij}}} \|(T_{\vec{b},D_1}(\vec{f}))(x)\|_{L^{q''}}$$

$$\le C \prod_{(i,j)\in S} \|b_i\|_{Lip_{\beta_{ij}}} \|f_1\|_{L^{q_1}} \cdots \|f_m\|_{L^{q_m}}$$

$$+ C \sum_{D\subset S} \sum_{D_1 \subset D} \prod_{(i,j)\in S\setminus D_1} \|b_i\|_{Lip_{\beta_{ij}}} \|(T_{\vec{b},D_1}(\vec{f}))(x)\|_{L^{q''}}.$$

Let $D = D_0$ for every family of subsets $D \subset S$, every family of subsets $D_{k+1} \subset D_k$, $0 \le k \le |S|-1$, we continue with the above to decompose these subsets until $|D_k| = 0$. We get a strictly proper subset.

Then we will obtain

$$\|M_\delta^\sharp(T_{\vec{b},S}(\vec{f}))(x)\|_{L^q}$$

$$\le C \prod_{(i,j)\in S} \|b_i\|_{Lip_{\beta_{ij}}} \|f_1\|_{L^{q_1}} \cdots \|f_m\|_{L^{q_m}}$$

$$+ C \sum_{D\subset S} \cdots \sum_{D_{|S|-1}\subset D_{|S|-2}} \prod_{(i,j)\in S\setminus D_{|S|-1}} \|b_i\|_{Lip_{\beta_{ij}}} \|(T_{\vec{b},D_{|S|-1}}(\vec{f}))(x)\|_{L^{q^{|S|}}}.$$

$$\le C \prod_{(i,j)\in S} \|b_i\|_{Lip\beta_{ij}} \|f_1\|_{L^{q_1}} \cdots \|f_m\|_{L^{q_m}},$$

since $|D_{|S|-1}| = 0$ and $\frac{1}{q^{|S|}} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$.

Thus, Theorem 1.3 is proved. □

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Honghai Liu
School of Mathematics and Information Science
Henan Polytechnic University
Jiaozuo 454000, P. R. China
Email address: hhliu@hpu.edu.cn

Zengyan Si
School of Mathematics and Information Science
Henan Polytechnic University
Jiaozuo 454000, P. R. China
Email address: zengyan@hpu.edu.cn

Ling Wang
School of Mathematics and Information Science
Henan Polytechnic University
Jiaozuo 454000, P. R. China
Email address: gllkycg66666@163.com