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ON NUCLEARITY OF SEMIGROUP CROSSED PRODUCTS

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Abstract. In this paper, we study nuclearity of semigroup crossed products for quasi-lattice ordered groups. We show the relationships among nuclearity of the semigroup crossed product, amenability of the quasi-lattice ordered group and nuclearity of the underlying C^*-algebra.

1. Introduction

The crossed product of a noncommutative dynamical system is one of the most important constructions in operator algebra theory. It is natural to try to extend this construction to algebraic structures that are even more basic than groups, namely semigroups (see [5], [9] and [12]). In [12], Murphy introduced the concept of the full crossed product of a C^*-algebra A by the semigroup of automorphisms. However, Murphy's construction leads to very complicated C^*-algebras. It turns out that the full semigroup C^*-algebra introduced by Murphy is too large and not fit for studying amenability. For example, the full semigroup C^*-algebra of $\mathbb{N} \times \mathbb{N}$ in the sense of Murphy is not nuclear (see [13, Theorem 6.2]). Hence, Li gave some new constructions of the full semigroup crossed product of a unital C^*-algebra A by a left-cancellative semigroup P in [9] and [10].

Moreover, nuclearity is an important approximation property of C^*-algebras, which is closely related to the amenability of groups (see [2] and [8]). In [9], Li studied semigroup C^*-algebras for left cancellative semigroups and showed how left amenability of semigroups can be expressed in analogy to the group case. Soon after, Li characterized nuclearity of semigroup C^*-algebras in terms of faithfulness of left regular representations and amenability of group actions (see [10]).

In Section 2, we recall constructions of the full semigroup crossed product and the reduced semigroup crossed product. In Section 3, we study nuclearity

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of semigroup crossed products for quasi-lattice ordered groups. In particular, we obtain our main results in Theorem 3.2 and Theorem 3.7.

Theorem 1.1. *Suppose that ω is a state on D_r such that $\omega(E_P) = 1$. If G is amenable, then the following statements are equivalent.*

1. *$A \rtimes_\alpha P$ is nuclear.*
2. *$A \rtimes_{\alpha,r} P$ is nuclear.*
3. *A is nuclear.*

Theorem 1.2. *Suppose that ω is a state on D_r such that $\omega(E_X) = 1$ for all $X \in J$, $X \neq \emptyset$, and A has an α-invariant state τ. If $A \rtimes_{\alpha,r} P$ is nuclear, then (G, P) has approximation property for positive definite functions.*

In Section 4, we focus on the case of lattice ordered groups. In particular, we have the following (see Corollary 4.10).

Corollary 1.3. *Assume that (G, G^+) is countable. Then the following statements are equivalent.*

1. *$C^*(G^+)$ is nuclear.*
2. *$C_r^*(G^+)$ is nuclear.*
3. *G is amenable.*

2. Semigroup crossed product

In this paper, a C^*-dynamical system will refer to a triple (A, M, α), where A is a unital C^*-algebra, M is a left-cancellative monoid, and α is a homomorphism from M to the group $\mathrm{Aut}(A)$ of automorphisms on A.

Let B be a unital C^*-algebra, a covariant homomorphism from (A, M, α) to B is a pair (φ, W), where $\varphi : A \to B$ is a $*$-homomorphism and $W : M \to B$ is an isometric homomorphism, such that

$$\varphi(\alpha_s(a))W_s = W_s\varphi(a)$$

for all $s \in M, a \in A$. If B is the algebra $B(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$, we call $(W, \varphi, \mathcal{H})$ a covariant representation.

We now turn to the construction of the full semigroup C^*-algebra introduced by Li. Given an element $s \in M$ and a subset $X \subseteq M$, we define

$$sX = \{sx : x \in X\} \quad \text{and} \quad s^{-1}X = \{y \in M : sy \in X\}.$$

If M is a subsemigroup of a group G, then we can also translate a subset X by a group element g. We denote the translation by $g \cdot X = \{gx \,|\, x \in X\}$.

A right ideal of M is a subset X of M which is closed under right multiplication. Let J be the smallest family of right ideals of M that contains M and $\emptyset$, and is closed under left multiplication and taking pre-images under left multiplication. In fact, it follows from [9] that

$$J = \{s_1^{-1}t_1 \cdots s_m^{-1}t_m M : m \geq 1,\ s_i, t_i \in M\} \cup \emptyset.$$

We call the elements in J constructible right ideals.

Definition ([9, Definition 2.2])**.** The full semigroup C^*-algebra $C^*(M)$ of M is the universal C^*-algebra generated by isometries $\{v_s : s \in M\}$ and projections $\{e_X : X \in J\}$ satisfying the following relations:

$$I.(1)\ v_{st} = v_s v_t, \quad I.(2)\ v_s e_X v_s^* = e_{sX},$$
$$II.(1)\ e_M = 1, \quad II.(2)\ e_\emptyset = 0, \quad II.(3)\ e_{X\cap Y} = e_X e_Y$$

for all $s, t \in M$ and $X, Y \in J$.

Li also introduced a new construction of the full semigroup crossed product in [9]. The full semigroup crossed product of A by M with respect to the action α is the unital C^*-algebra $A \rtimes_\alpha M$ with two unital $*$-homomorphisms $\iota_A : A \to A \rtimes_\alpha M$ and $\iota_M : C^*(M) \to A \rtimes_\alpha M$ satisfying

$$\iota_A(\alpha_p(a))\iota_M(v_p) = \iota_M(v_p)\iota_A(a)$$

for all $a \in A$ and $p \in M$, which has the following universal property: if D is a unital C^*-algebra and $\varphi_A : A \to D$, $\varphi_M : C^*(M) \to D$ are unital $*$-homomorphisms satisfying the covariance relation

$$\varphi_A(\alpha_p(a))\varphi_M(v_p) = \varphi_M(v_p)\varphi_A(a)$$

for all $a \in A$ and $p \in M$, there exists a unique $*$-homomorphism $\varphi_A \times \varphi_M : A \rtimes_\alpha M \to D$ such that

$$(\varphi_A \times \varphi_M) \circ \iota_A = \varphi_A \text{ and } (\varphi_A \times \varphi_M) \circ \iota_M = \varphi_M.$$

Let $(\pi, \mathcal{H})$ be a faithful representation of A and λ be the regular isometric representation of M on $\ell^2(M)$. For $a \in A$, we define $\overline{\pi}(a) \in B(\ell^2(M, \mathcal{H}))$ as follows:

$$(\overline{\pi}(a)f)(s) = \pi(\alpha_s^{-1}(a))f(s)$$

for all $f \in \ell^2(M, \mathcal{H})$ and $s \in M$. Then $(\overline{\pi}, \mathrm{id}_{\mathcal{H}} \otimes \lambda)$ is a covariant representation, that is called a regular representation. By the universal property, there exists a unique $*$-homomorphism $\lambda_{(A,M,\alpha)}$ from $A \rtimes_\alpha M$ into $B(\ell^2(M, \mathcal{H}))$. We call $\lambda_{(A,M,\alpha)}(A \rtimes_\alpha M)$ the reduced semigroup crossed product of (A, M, α), and denote it by $A \rtimes_{\alpha,r} M$. We identify $\mathrm{id}_{\mathcal{H}} \otimes \lambda$ with λ, and regard A as a C^*-subalgebra of $A \rtimes_{\alpha,r} M$. By the covariance relation, $A \rtimes_{\alpha,r} M$ is the closure of

$$\mathrm{span}\{a\lambda_{s_1}\lambda_{t_1}^* \cdots \lambda_{s_n}\lambda_{t_n}^* : n \in \mathbb{N},\ a \in A,\ s_i, t_i \in M\}.$$

In fact, the reduced semigroup crossed product does not depend on the choice of the faithful representation $(\pi, \mathcal{H})$. If $A = \mathbb{C}$, we call $C_r^*(M) := \mathbb{C} \rtimes_{\alpha,r} M$ the reduced semigroup C^*-algebra of M. In particular, $C_r^*(M)$ is the C^*-algebra generated by the regular isometric representation λ of M on $\ell^2(M)$.

3. Main results

In this section, (G, P) is a quasi-lattice ordered group that acts on a unital C^*-algebra A through an action α.

Definition. A quasi-lattice ordered group is a pair (G, P) consisting of a subsemigroup P of a discrete group G such that

1. $P \cap P^{-1} = \{e\}$, where e is the unit of G;
2. for all $g \in G$, the intersection $P \cap (g \cdot P)$ is either empty or of the form pP for some $p \in P$.

Hence, there is a partial order on G defined by $s \leq t$ if $s^{-1}t \in P$. Let J_P^G be the smallest family of subsets of G which contains J and which is closed under left translations by group elements ($Y \in J_P^G$, $g \in G \Rightarrow g \cdot Y \in J_P^G$) and finite intersections. In fact, $J_P^G = \{g \cdot P : g \in G\} \cup \{\emptyset\}$.

For a subset X of P, we write 1_X for the characteristic function of X defined on P. Let $E_X \in B(\ell^2(P))$ be the multiplication operator corresponding to 1_X and $D_r = C^*(\{E_X : X \in J\}) \subseteq B(\ell^2(P))$. Hence, D_r is an abelian C^*-subalgebra of $C_r^*(P)$. Let $U : G \to B(\ell^2(G))$ be the left regular representation of G. The group G acts on $\ell^\infty(G)$ by the left translation action β_G. Let D_P^G be the smallest C^*-subalgebra of $\ell^\infty(G) \subseteq B(\ell^2(G))$ which is β_G-invariant and contains E_P. Note that $D_r = E_P D_P^G$.

We define the operator $a_{(\alpha)} \in B(\mathcal{H} \otimes \ell^2(G))$ by

$$a_{(\alpha)}(\xi \otimes \delta_s) = (\alpha_s^{-1}(a)(\xi)) \otimes \delta_s$$

for all $\xi \in \mathcal{H}$ and $s \in G$. It follows from [10, Lemma 3.6] that there exists a faithful representation $\hat{\pi}$ of $(A \otimes D_P^G) \rtimes_{\alpha\otimes\beta_G, r} G$ on $\mathcal{H} \otimes \ell^2(G)$ defined by

$$\hat{\pi}((a \otimes d)g) = a_{(\alpha)}(I_{\mathcal{H}} \otimes d)(I_{\mathcal{H}} \otimes U_g)$$

for all $a \in A$, $d \in D_P^G$ and $g \in G$, where $I_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$. We denote the image of $(A \otimes D_P^G) \rtimes_{\alpha\otimes\beta_G, r} G$ under the representation $\hat{\pi}$ by $A \rtimes_{\alpha, r} (P \subseteq G)$. Hence, we identify $(A \otimes D_P^G) \rtimes_{\alpha\otimes\beta_G, r} G$ with $A \rtimes_{\alpha, r} (P \subseteq G)$. It follows from [10, Lemma 3.9] that

$$A \rtimes_{\alpha, r} P \cong (I_{\mathcal{H}} \otimes E_P)(A \rtimes_{\alpha, r} (P \subseteq G))(I_{\mathcal{H}} \otimes E_P).$$

From now on, we do not distinguish between the space $\mathcal{H} \otimes \ell^2(P)$ and the subspace $(I_{\mathcal{H}} \otimes E_P)(\mathcal{H} \otimes \ell^2(G))(I_{\mathcal{H}} \otimes E_P)$. In this way, the element $a \in A \rtimes_{\alpha, r} P$ is the same as $(I_{\mathcal{H}} \otimes E_P)a_{(\alpha)}(I_{\mathcal{H}} \otimes E_P)$, the element $\lambda_s \in A \rtimes_{\alpha, r} P$ is nothing else but $(I_{\mathcal{H}} \otimes E_P)(I_{\mathcal{H}} \otimes U_s)(I_{\mathcal{H}} \otimes E_P)$. For the sake of simplicity, we denote $(I_{\mathcal{H}} \otimes E_P)(A \rtimes_{\alpha, r} (P \subseteq G))(I_{\mathcal{H}} \otimes E_P)$ by B and let

$$B_0 = \operatorname{span}\{(I_{\mathcal{H}} \otimes E_P)a_{(\alpha)}(I_{\mathcal{H}} \otimes E_X)(I_{\mathcal{H}} \otimes U_g)(I_{\mathcal{H}} \otimes E_P) : a \in A, g \in G, X \in J_P^G\}.$$

Hence, B_0 is dense in B. Through a routine computation, we have the following result.

Lemma 3.1. (a) *Let $\bar{\mathcal{E}}$ be the canonical faithful conditional expectation from $(A \otimes D_P^G) \rtimes_{\alpha\otimes\beta_G, r} G$ to $A \otimes D_P^G$. Then $\bar{\mathcal{E}}|_B$ is a conditional expectation from B to $A \otimes D_r$.*

(b) *If ω is a state on D_r such that $\omega(E_P) = 1$, then* $\mathrm{id}_A \otimes \omega$ *is a conditional expectation from $A \otimes D_r$ to A, where* id_A *is the identity map on A.*

In fact, $\bar{\mathcal{E}}|_B$ is the canonical faithful conditional expectation from $A \rtimes_{\alpha,r} P$ to $A \otimes D_r$. The above results suggest that $\mathcal{E} = (\mathrm{id}_A \otimes \omega) \circ \bar{\mathcal{E}}|_B$ is a conditional expectation from $A \rtimes_{\alpha,r} P$ to A such that

$$\mathcal{E}((I_{\mathcal{H}} \otimes E_P)a_{(\alpha)}(I_{\mathcal{H}} \otimes E_X)(I_{\mathcal{H}} \otimes U_g)(I_{\mathcal{H}} \otimes E_P)) = \begin{cases} \omega(E_{P\cap X})a, & g = e; \\ 0, & g \neq e; \end{cases}$$

for all $a \in A$ and $X \in J_P^G$.

Theorem 3.2. *Suppose that ω is a state on D_r such that $\omega(E_P) = 1$. Consider the following statements.*

1. *$A \rtimes_\alpha P$ is nuclear.*
2. *$A \rtimes_{\alpha,r} P$ is nuclear.*
3. *A is nuclear.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3). If G is amenable, then they are equivalent.

Proof. (1) $\Rightarrow$ (2) It follows from the fact that a quotient of a nuclear C^*-algebra is nuclear (see [1, IV 3.1.13]).

(2) $\Rightarrow$ (3) It follows from [2, Exercise 2.3.3].

Assume that G is amenable. [10, Theorem 5.24] shows that $A \rtimes_\alpha P \cong A \rtimes_{\alpha,r} P$. Since D_P^G is abelian, it follows from [16, Proposition 2.1.2] that $(A \otimes D_P^G) \rtimes_{\alpha\otimes\beta_G,r} G$ is nuclear. Hence, $A \rtimes_\alpha P$ is nuclear. □

Remark 3.3. Assume that M is a cancellative, countable, right amenable semigroup with an action α on a nuclear C^*-algebra A. It follows from [9, Lemma 2.15] and the dilation theory for semigroup crossed products by endomorphisms in [6] that $A \rtimes_\alpha M$ is nuclear.

Lemma 3.4. *Suppose that ω is a state on D_r such that $\omega(E_X) = 1$ for all $X \in J$, $X \neq \emptyset$, and τ is an α-invariant state of A. Let $\tau' = \tau \circ \mathcal{E}$. Then*

$$\tau'(x\lambda_h) = \tau'(\lambda_h x)$$

for all $h \in P$ and $x \in A \rtimes_{\alpha,r} P$.

Proof. For each $x = (I_{\mathcal{H}} \otimes E_P)a_{(\alpha)}(I_{\mathcal{H}} \otimes E_X)(I_{\mathcal{H}} \otimes U_g)(I_{\mathcal{H}} \otimes E_P) \in B$, we have

$$\begin{aligned}
&\tau'(x(I_{\mathcal{H}} \otimes E_P)(I_{\mathcal{H}} \otimes U_h)(I_{\mathcal{H}} \otimes E_P)) \\
&= \tau'((I_{\mathcal{H}} \otimes E_P)a_{(\alpha)}(I_{\mathcal{H}} \otimes E_X)(I_{\mathcal{H}} \otimes U_g)(I_{\mathcal{H}} \otimes E_P)(I_{\mathcal{H}} \otimes U_h)(I_{\mathcal{H}} \otimes E_P)) \\
&= \tau'((I_{\mathcal{H}} \otimes E_P)a_{(\alpha)}(I_{\mathcal{H}} \otimes E_X)(I_{\mathcal{H}} \otimes E_{g\cdot P})(I_{\mathcal{H}} \otimes U_g)(I_{\mathcal{H}} \otimes U_h)(I_{\mathcal{H}} \otimes E_P)) \\
&= \tau'((I_{\mathcal{H}} \otimes E_P)a_{(\alpha)}(I_{\mathcal{H}} \otimes E_{X\cap g\cdot P})(I_{\mathcal{H}} \otimes U_{gh})(I_{\mathcal{H}} \otimes E_P)) \\
&= \begin{cases} \tau(a), & \text{if } g = h^{-1} \text{ and } P \cap X \cap g \cdot P \neq \emptyset; \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

for all $h \in P$, and

$$\begin{aligned}
&\tau'((I_{\mathcal{H}} \otimes E_P)(I_{\mathcal{H}} \otimes U_h)(I_{\mathcal{H}} \otimes E_P)x) \\
&= \tau'((I_{\mathcal{H}} \otimes E_P)(I_{\mathcal{H}} \otimes U_h)(I_{\mathcal{H}} \otimes E_P)a_{(\alpha)}(I_{\mathcal{H}} \otimes E_X)(I_{\mathcal{H}} \otimes U_g)(I_{\mathcal{H}} \otimes E_P))
\end{aligned}$$

$$
\begin{aligned}
&= \tau'((I_{\mathcal{H}} \otimes E_P)\alpha_h(a)_{(\alpha)}(I_{\mathcal{H}} \otimes E_{h\cdot P})(I_{\mathcal{H}} \otimes E_{h\cdot X})(I_{\mathcal{H}} \otimes U_{hg})(I_{\mathcal{H}} \otimes E_P)) \\
&= \tau'((I_{\mathcal{H}} \otimes E_P)\alpha_h(a)_{(\alpha)}(I_{\mathcal{H}} \otimes E_{h\cdot(P\cap X)})(I_{\mathcal{H}} \otimes U_{hg})(I_{\mathcal{H}} \otimes E_P)) \\
&= \begin{cases} \tau(\alpha_h(a)) = \tau(a), & \text{if } g = h^{-1} \text{ and } P \cap h\cdot(P\cap X) \neq \emptyset; \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

for all $h \in P$. Since

$$P \cap h\cdot(P\cap X) = h\cdot(h^{-1}\cdot P \cap P \cap X),$$

it is easy to see that $\tau'(x\lambda_h) = \tau'(\lambda_h x)$ for all $h \in P$ and $x \in A \rtimes_{\alpha,r} P$. □

From now on, assume that (G,P) is a quasi-lattice ordered group such that the inequality $g \leq h$ implies $sgt \leq sht$ for all $g,h,s,t \in G$. Let $Q = \{x \in G : x \leq e\}$. Then $PQ = QP = \{pq : p \in P, q \in Q\} = \{x \in G : x \text{ has upper bounds in } P\}$.

Remark 3.5. If $s_1,\dots,s_n$ are arbitrary elements of PQ, then there is an element $u \in P$ such that $s_i \leq u$ for all $1 \leq i \leq n$. To see this, write $s_i = g_i^{-1}h_i$, where $g_i, h_i \in P$, set $u = h_1\cdots h_n$, then $s_i \leq h_i \leq u$.

Using a similar argument of [12, Proposition 2.2], we have the following results.

Lemma 3.6. *Let B be a unital C^*-algebra. If $W : P \to B$ is an isometric homomorphism, then there exists a unique extension $W : PQ \to B$ such that $W_{g^{-1}h} = W_g^*W_h$ for all $g \in P$ and $h \in PQ$. Moreover, if $g_1,\dots,g_m \in PQ$, then the matrix $(W_{g_i^{-1}g_j})_{ij}$ is positive in $M_m(B)$.*

Hence, the regular isometric representation λ of P has a unique extension $\lambda : PQ \to B(\ell^2(G^+))$ such that $\lambda_s = \lambda_g^*\lambda_h$, where $s = g^{-1}h \in PQ$.

The quasi-lattice ordered group (G,P) is said to have *approximation property for positive definite functions* if there exists a net $\{\varphi_i\}_{i\in I}$ of positive definite functions with finite support on PQ (see [14]) such that $\varphi_i(x) \to 1$ for all $x \in PQ$.

Theorem 3.7. *Suppose that ω is a state on D_r such that $\omega(E_X) = 1$ for all $X \in J$, $X \neq \emptyset$, and A has an α-invariant state τ. If $A \rtimes_{\alpha,r} P$ is nuclear, then (G,P) has an approximation property for positive definite functions.*

Proof. Suppose that the nets $\varphi_n : A \rtimes_{\alpha,r} P \to M_{k_n}(\mathbb{C})$ and $\psi_n : M_{k_n}(\mathbb{C}) \to A \rtimes_{\alpha,r} P$ of completely positive maps satisfy the conditions of nuclearity. By the argument of [11, Theorem 4.3], we can assume that the range of ψ_n is in B_0. Let $\Phi_n = \psi_n \circ \varphi_n$. By Lemma 3.6, we define

$$\varphi_n(g) = \tau'(\Phi_n(\lambda_g)\lambda_g^*)$$

for all $g \in PQ$. If $\{g_1,\dots,g_m\}$ is an arbitrary finite set in PQ, use Remark 3.5 to choose $u \in P$ such that $s_i = ug_i \in P$ for all $i = 1,2,\dots,m$. For all

$c_1, \dots, c_m \in \mathbb{C}$, it follows from the positivity of τ' and Lemma 3.6 that

$$\begin{aligned}\sum_{i,j=1}^{m} c_i\bar{c_j}\varphi_n(g_j^{-1}g_i) &= \sum_{i,j=1}^{m} c_i\bar{c_j}\tau'((\Phi_n(\lambda_{g_j^{-1}g_i}))(\lambda_{g_j^{-1}g_i})^*)\\ &= \sum_{i,j=1}^{m} c_i\bar{c_j}\tau'((\Phi_n(\lambda_{s_j^{-1}s_i}))(\lambda_{s_j^{-1}s_i})^*)\\ &= \sum_{i,j=1}^{m} \tau'(\bar{c_j}\lambda_{s_j}\Phi_n(\lambda_{s_j}^*\lambda_{s_i})c_i\lambda_{s_i}^*) \geq 0.\end{aligned}$$

Hence, φ_n is positive definite on PQ. Moreover, as $n \to +\infty$,

$$\begin{aligned}|\varphi_n(g) - 1| &= |\tau'(\Phi_n(\lambda_g)\lambda_g^*) - 1| = |\tau'(\Phi_n(\lambda_g)\lambda_g^*) - \tau'(\lambda_g\lambda_g^*)|\\ &= |\tau'((\Phi_n(\lambda_g) - \lambda_g)\lambda_g^*)| \leq \|\Phi_n(\lambda_g) - \lambda_g\| \to 0\end{aligned}$$

for all $g \in PQ$. Since $\{\Phi_n\}_{n\geq 1}$ is finite dimensional, φ_n is finite supported. This shows that (G, P) has approximation property for positive definite functions. □

Remark 3.8. If the conditions in the above theorem is satisfied, then it follows from [14, Propositions 2] that (G, P) is amenable in the sense of [14]. Moreover, [7, Corollary 3.8] shows that (G, P) is amenable in the sense of [7].

4. Examples

In this section, we only consider lattice ordered groups. We will construct the conditional expectation from $A \rtimes_{\alpha,r} P$ to A in a different way.

Definition. A lattice ordered group is a pair $(G, \leq)$ consisting of a discrete group G and a partially ordered $\leq$ on G such that if e is the unit of G and $G^+ = \{s \in G \mid e \leq s\}$, then

1. Every pair x, y of elements of G has a least common upper bound $\sigma(x, y)$ in G^+.
2. The inequality $g \leq h$ implies $sgt \leq sht$ for all $g, h, s, t \in G$.

If the order is a total order, we call (G, G^+) an *ordered group*. It follows from [3] that if (G, G^+) is a lattice ordered group, then G is quasi-lattice ordered and

$$G = G^+(G^+)^{-1} = (G^+)^{-1}G^+.$$

The class of all lattice ordered groups is large. It contains all torsion-free abelian groups, all torsion-free nilpotent groups, free groups, Thompson's group, surface groups, pure braid groups, the group of all order automorphisms of a totally ordered space, the group of orientation-preserving homeomorphisms of the line and so on (see [4]).

Lemma 4.1. *Let (G, G^+) be a lattice ordered group. Then G^+ is right reversible (left reversible), i.e., for every p_1, $p_2 \in G^+$, we have $G^+p_1 \cap G^+p_2 \neq \emptyset$.*

Proof. We only prove that G^+ is right reversible. For any $p_1,\ p_2 \in G^+$, we have

$$\sigma(p_1,p_2) = \sigma(p_1,p_2)p_1^{-1}p_1 = \sigma(p_1,p_2)p_2^{-1}p_2 \in G^+p_1 \cap G^+p_2.$$

This shows that G^+ is right reversible. □

Let (A,G,α) be a C^*-dynamical system, where (G,G^+) is a lattice ordered group. Then it follows from the properties of lattice ordered groups that $A \rtimes_{\alpha,r} G^+$ is the closed linear span of $\{a\lambda_s\lambda_t^*, a \in A, s,t \in G^+\}$.

Let us begin with the following fact concerning the reduced semigroup crossed product. We denote the vector state $x \to \langle x\delta_p, \delta_p\rangle$ by ρ_p.

Lemma 4.2. $\bar{\rho} = \lim_p \rho_p$ *is a tracial state on* $C_r^*(G^+)$.

Proof. For sufficiently large p, we have

$$\rho_p(\lambda_{p_1}\lambda_{q_1}^*) = \begin{cases} 1, & p_1 = q_1 \\ 0, & \text{otherwise} \end{cases}$$

for all $p_1, q_1 \in G^+$. Since the linear span of $\{\lambda_s\lambda_t^*, s,t \in G^+\}$ is dense in $C_r^*(G^+)$, a routine $\varepsilon/3$-argument shows the convergence for general x in $C_r^*(G^+)$. Moreover,

$$\bar{\rho}(\lambda_{p_1}\lambda_{q_1}^*\lambda_{p_2}\lambda_{q_2}^*) = \bar{\rho}(\lambda_{p_1q_1^{-1}\sigma(q_1,p_2)}\lambda^*_{q_2p_2^{-1}\sigma(q_1,p_2)}) = \begin{cases} 1, & p_1q_1^{-1} = q_2p_2^{-1} \\ 0, & \text{otherwise} \end{cases}$$

and

$$\bar{\rho}(\lambda_{p_2}\lambda_{q_2}^*\lambda_{p_1}\lambda_{q_1}^*) = \bar{\rho}(\lambda_{p_2q_2^{-1}\sigma(q_2,p_1)}\lambda^*_{q_1p_1^{-1}\sigma(q_2,p_1)}) = \begin{cases} 1, & p_2q_2^{-1} = q_1p_1^{-1} \\ 0, & \text{otherwise} \end{cases}$$

for all $p_1, p_2, q_1, q_2 \in G^+$. Hence, $\bar{\rho}$ is a tracial state on $C_r^*(G^+)$. □

Remark 4.3. In general, $\bar{\rho}$ is not faithful. For example, let $G = \mathbb{Z}$ and $x = \lambda_1\lambda_2^* - \lambda_2\lambda_3^*$, then

$$\begin{aligned}\bar{\rho}(xx^*) &= \bar{\rho}((\lambda_1\lambda_2^* - \lambda_2\lambda_3^*)(\lambda_1\lambda_2^* - \lambda_2\lambda_3^*)^*) \\ &= \bar{\rho}((\lambda_1\lambda_2^* - \lambda_2\lambda_3^*)(\lambda_2\lambda_1^* - \lambda_3\lambda_2^*)) \\ &= \bar{\rho}(\lambda_1\lambda_1^* - \lambda_2\lambda_2^* - \lambda_2\lambda_2^* + \lambda_2\lambda_2^*) = 0.\end{aligned}$$

Using the similar argument of [2, Proposition 4.1.7], we have the Fell's absorbtion principle of the semigroup C^*-dynamical system.

Lemma 4.4 (Fell's absorbtion principle). *Let* $(u, \mathrm{id}_A, \mathcal{H})$ *be a covariant representation of* (A,G,α). *Then the covariant representation*

$$(u \otimes \lambda, \mathrm{id}_A \otimes 1, \mathcal{H} \otimes \ell^2(G^+))$$

is unitarily equivalent to a regular representation. In fact, we have a $*$*-isomorphism*

$$C^*((u \otimes \lambda)(G^+), A \otimes 1) \cong A \rtimes_{\alpha,r} G^+.$$

Theorem 4.5. *The map* $\mathcal{E}(\sum_{s,t\in G^+} a_{s,t}\lambda_s\lambda_t^*) = \sum_{s\in G^+} a_{s,s}$ *extends to a conditional expectation from* $A \rtimes_{\alpha,r} G^+$ *to* A.

Proof. Let $(u, \mathrm{id}_A, \mathcal{H})$ be a covariant representation of (A, G, α). By Lemma 4.4, the reduced semigroup crossed product $A \rtimes_{\alpha,r} G^+$ can be viewed as the C^*-algebra generated by $A\otimes 1$ and $u\otimes\lambda(G^+)$, which is a subalgebra of $B(\mathcal{H})\otimes C_r^*(G^+)$. In fact, the map $\mathcal{E}$ is the restriction of $\mathrm{id}_{B(\mathcal{H})}\otimes\bar{\rho}$ on $A \rtimes_{\alpha,r} G^+$. $\square$

Remark 4.6. It is easy to see that $\bar{\rho}$ is a state ω on D_r such that $\omega(E_X) = 1$ for all $X \in J$, $X \neq \emptyset$. Hence, the conditional expectation $\mathcal{E}$ is the same as the one defined in Section 3.

As a special case of Theorem 3.2, we have the following result.

Theorem 4.7. *Consider the following statements.*

(1) $A \rtimes_\alpha G^+$ *is nuclear.*
(2) $A \rtimes_{\alpha,r} G^+$ *is nuclear.*
(3) A *is nuclear.*

Then $(1) \Rightarrow (2) \Rightarrow (3)$. *If* G *is amenable, then they are equivalent.*

Example 4.8. Let α be an automorphism of a nuclear C^*-algebra A. We also use α to denote the induced action of $\mathbb{Z}$ given by $n \mapsto \alpha^n$. Since $\mathbb{Z}$ is amenable, it follows from Theorem 4.7 that $A \rtimes_\alpha \mathbb{Z}^+$ is nuclear.

Since $G = G^+(G^+)^{-1}$, the following result is a special case of Theorem 3.7.

Theorem 4.9. *Suppose that* A *has an* α*-invariant state* τ. *If* $A \rtimes_{\alpha,r} G^+$ *is nuclear, then* G *is amenable.*

An application of Theorem 4.9 is the following corollary, which can be regarded as a generalization of [2, Theorem 2.6.8]. Let $C^*(G)$ ($C_r^*(G)$) be the full (reduced) group C^*-algebra of G.

Corollary 4.10. *Assume that* (G, G^+) *is countable. Then the following statements are equivalent.*

(1) $C^*(G^+)$ *is nuclear.*
(2) $C_r^*(G^+)$ *is nuclear.*
(3) $C^*(G)$ *is nuclear.*
(4) $C_r^*(G)$ *is nuclear.*
(5) G *is amenable.*
(6) G^+ *is right amenable.*
(7) G^+ *is left amenable.*
(8) $C^*(G^+) = C_r^*(G^+)$.

Proof. We always have $(1) \Rightarrow (2)$ and $(3) \Leftrightarrow (4) \Leftrightarrow (5)$.

$(5) \Rightarrow (6)$ It follows from the right version of [15, Proposition 1.28] and Lemma 4.1.

$(6) \Rightarrow (1)$ It follows from [9, Proposition 4.15].

(2) $\Rightarrow$ (5) Since $A = \mathbb{C}$ always has an invariant tracial state, then Theorem 4.9 shows the amenability of G.

(5) $\Leftrightarrow$ (7) It follows from [15, Propositions 1.27 and 1.28].

(7) $\Leftrightarrow$ (8) Since G^+ is left reversible, then there exists a non-zero character on $C^*(G^+)$ (see [9, Lemma 4.6]). The conclusion follows from the statements of [9, Section 4]. $\square$

Remark 4.11. If one of the conditions in Corollary 4.10 holds, it follows from [10, Theorem 6.1] that for any action α of G on A, the $*$-homomorphism $\lambda_{(A,G^+,\alpha)} : A \rtimes_\alpha G^+ \to A \rtimes_{\alpha,r} G^+$ is an isomorphism.

Another application of Theorem 4.9 is the following corollary.

Corollary 4.12. *If A is a nuclear C^*-algebra with an α-invariant state τ, then the following statements are equivalent.*

(1) *$A \rtimes_\alpha G^+$ is nuclear.*
(2) *$A \rtimes_{\alpha,r} G^+$ is nuclear.*
(3) *G is amenable.*

We conclude this article with the following examples.

Example 4.13. Let $(\mathbb{F}_2, \mathbb{F}_2^+)$ be the free group on two generators with the total order (see [4]). The conjugation action will be denoted by γ.

(1) Since $\mathbb{F}_2$ is not amenable, the full group C^*-algebra $C^*(\mathbb{F}_2)$ is not nuclear. It follows from Theorem 3.2 that $C^*(\mathbb{F}_2) \rtimes_\gamma \mathbb{F}_2^+$ is not nuclear.

(2) We define a map $\tau : C_b(\mathbb{F}_2) \to \mathbb{C}$ by $\tau(f) = f(e)$ for every $f \in C_b(\mathbb{F}_2)$. Then τ is a γ-invariant tracial state on $C_b(\mathbb{F}_2)$. It follows from Theorem 4.9 that $C_b(\mathbb{F}_2) \rtimes_{\gamma,r} \mathbb{F}_2^+$ is not nuclear.

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