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UNIQUENESS RESULTS ON MEROMORPHIC FUNCTIONS AND THEIR DIFFERENCE OPERATORS SHARING TARGETS WITH WEIGHT

Thu Thuy Hoang, Hong Nhat Nguyen, and Duc Thoan Pham

Abstract. Let f be a nonconstant meromorphic function of hyper-order strictly less than 1, and let $c \in \mathbb{C} \setminus \{0\}$ such that $f(z+c) \not\equiv f(z)$. We prove that if f and its exact difference $\Delta_c f(z) = f(z+c) - f(z)$ share partially $0, \infty$ CM and share 1 IM, then $\Delta_c f = f$, where all 1-points with multiplicities more than 2 do not need to be counted. Some similar uniqueness results for such meromorphic functions partially sharing targets with weight and their shifts are also given. Our results generalize and improve the recent important results.

1. Introduction

In the past decade, uniqueness questions of meromorphic functions and their shifts or their difference operators sharing values have been well treated by many authors with many important results (see [1, 6, 10, 12, 14, 15, 18, 19]). Among them, there are results found by J. Heittokangas, R. Korhonen, I. Laine and J. Rieppo [9], by S. Chen and W. Lin [3] or by S. Chen and A. Xu [4]. To state their results and some related ones in this direction, first of all we recall the following notations.

Let f be a nonconstant meromorphic function on complex plane $\mathbb{C}$. We use the standard notations of the Nevanlinna theory as given in [8,11,17]. We recall that $T(r, f)$ denotes the characteristic function of f. In particular, we denote $S(f)$ as the family of all meromorphic functions α such that $T(r, \alpha) = o(T(r, f))$ as $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure. Such function α is said to be a small function with respect to f.

As usual, the order $\rho(f)$ and the hyper-order $\rho_2(f)$ of f are defined, respectively, by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}, \ \rho_2(f) = \limsup_{r\to\infty} \frac{\log^+ \log^+ T(r,f)}{\log r}.$$

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For a positive integer k (maybe $k = +\infty$) and a function $a \in S(f) \cup \{\infty\}$, we denote by $E_{k)}(a, f)$ the set of zeros of $f - a$ with multiplicity $l \leq k$, where a zero with multiplicity l is counted exactly l times in the set. The counting function corresponding to $E_{k)}(a, f)$ is denoted by $N_{k)}\big(r, \frac{1}{f-a}\big)$. Similarly, we also denote by $N_{(k}\big(r, \frac{1}{f-a}\big)$ the counting function of those zeros of $f - a$ whose multiplicities are not less than k in counting the zeros of $f - a$. We mean that $\overline{E}_{k)}(a, f)$ is the set of zeros of $f - a$ with multiplicity $l \leq k$, where a zero with multiplicity l is counted only once in the set. The reduced counting functions are denoted by $\overline{N}_{k)}\big(r, \frac{1}{f-a}\big)$ and $\overline{N}_{(k}\big(r, \frac{1}{f-a}\big)$. If $k = +\infty$, we omit character $_{k)}$ in the symbol.

Let f, g be two nonconstant meromorphic functions. If $\overline{E}(a, f) = \overline{E}(a, g)$ we say that f and g share a IM and if $E(a, f) = E(a, f)$ we say that f and g share a CM.

In 2011, Z. X. Chen and H. X. Yi, [5] proved a uniqueness theorem for a transcendental meromorphic function $f(z)$ and its first order exact difference $\Delta_c f(z) = f(z+c) - f(z)$ with three shared values CM when the order of growth $\rho(f)$ is not an integer or infinite. They conjectured that the condition "order of growth $\rho(f)$ is not an integer or infinite" can be omitted. Later, J. Zhang and L. Liao [20] partially answer this conjecture when f is a transcendental entire function of finite order.

Theorem A ([20])**.** *Let f be a transcendental entire function of finite order, and a, b be two distinct constants. If $\Delta f(z) = f(z+1) - f(z)$ $(\not\equiv 0)$ and f share a, b CM, then $\Delta f = f$. Furthermore, f must be of the following form $f(z) = 2^z h(z)$, where $h(z)$ is a periodic entire function with period* 1.

In 2016, F. Lü and W. Lü [13] improved Theorem A from "entire function" to "meromorphic function" and obtained a uniqueness result on meromorphic function f of finite order sharing three values CM with its exact difference $\Delta_c f$. Later, S. Chen [2] proved that this result still holds when the meromorphic functions of hyper-order less than 1 share one value CM and share partially two values CM. Here, the concept "partially shared value CM" of two meromorphic functions f and g sharing value a means that $E(a, f) \subseteq E(a, g)$.

Very recently, S. Chen and A. Xu [4] have been successful to improve Chen's result. They obtained the uniqueness theorem for f and its exact difference $\Delta_c f$ with two shared values CM and one shared value IM as follows.

Theorem B ([4])**.** *Let f be a nonconstant meromorphic function of hyper-order $\rho_2(f) < 1$ and $c \in \mathbb{C}\setminus\{0\}$ such that $f(z+c) \not\equiv f(z)$. If $\Delta_c f(z) = f(z+c) - f(z)$ and $f(z)$ share $0, \infty$ CM and 1 IM, then $\Delta_c f = f$.*

Shared values results related to a meromorphic function of finite order $f(z)$ and its shift $f(z+c)$ were studied by J. Heittokangas et al. [9]. Later, these results were improved for the case of meromorphic function $f(z)$ of hyper-order less then 1 by S. Chen and W. Lin [3].

Theorem C ([3]). *Let f be a meromorphic function of hyper-order $\rho_2(f) < 1$, let $c \in \mathbb{C}$, and let $a_1, a_2, a_3 \in S(f) \cup \{\infty\}$ be three distinct periodic functions with period c. If $f(z)$ and $f(z+c)$ share a_1, a_2 CM and a_3 IM, then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.*

In this article, our aim is to extend and improve the above mentioned theorems from considering the partially shared values and omitting the zeros whose multiplicities are greater than a certain value. Before giving our results, we have the following definition.

Definition. We say that two meromorphic functions f and g share value $a \in S(f) \cap S(g) \cup \{\infty\}$ IM with weight k if $\overline{E}_{k)}(a, f) = \overline{E}_{k)}(a, g)$. If $\overline{E}_{k)}(a, f) \subseteq \overline{E}_{k)}(a, g)$, then f and g is said to share partially a IM with weight k.

First of all, we obtain the conclusion for Theorem B when all 1-points with multiplicities more than 2 do not need to be counted. This is a great improvement of that theorem. Note that the proof of Theorem B is quite long and complicated. Here, we use a simple method which is very different from those of that theorem.

Theorem 1.1. *Let f be a nonconstant meromorphic function of hyper-order $\rho_2(f) < 1$ and let $c \in \mathbb{C} \setminus \{0\}$ such that $f(z+c) \not\equiv f(z)$. Assume that $\Delta_c f(z) = f(z+c) - f(z)$ and $f(z)$ share partially $0, \infty$ CM and share 1 IM with weight k, i.e.,*

$$E(0, f) \subseteq E(0, \Delta_c f), \;\; E(\infty, f) \supseteq E(\infty, \Delta_c f)$$

and

$$\overline{E}_{k)}(1, f) = \overline{E}_{k)}(1, \Delta_c f).$$

If $k \geq 2$, then $\Delta_c f(z) = f(z)$ for all $z \in \mathbb{C}$.

When f is an entire function, we obtain directly the first conclusion of Theorem A from Theorem 1.1 when $\Delta_c f$ and f share partially 0 CM and sharing 1 IM, in which we do not need to count all 1-points with multiplicities more than 2. This result improves strongly Theorem A. However, in this situation we are able to prove a better result than that. Here, we only need to consider the 1-points with multiplicity 1.

Theorem 1.2. *Let f be a nonconstant entire function of hyper-order $\rho_2(f) < 1$ and let $c \in \mathbb{C} \setminus \{0\}$ such that $f(z+c) \not\equiv f(z)$. If $\Delta_c f(z) = f(z+c) - f(z)$ and $f(z)$ share partially 0 CM and share 1 IM with weight 1, then $\Delta_c f(z) = f(z)$ for all $z \in \mathbb{C}$.*

Naturally, we are interested in finding what happens in details when the shared value IM in Theorem C has weight k.

Theorem 1.3. *Let f be a nonconstant meromorphic function of hyper-order $\rho_2(f) < 1$ and let $c \in \mathbb{C} \setminus \{0\}$. Assume that $f(z)$ and its shift $f_c(z) = f(z+c)$ share partially $0, \infty$ CM and share partially 1 IM with weight k, i.e.,*

$$E(0, f) \subseteq E(0, f_c), \;\; E(\infty, f) \supseteq E(\infty, f_c)$$

and

$$\overline{E}_{k)}(1, f) \subseteq \overline{E}_{k)}(1, f_c).$$

If $k \geq 3$, then $f_c(z) = f(z)$ or $f_c(z) = -f(z)$ for all $z \in \mathbb{C}$.

The case of $f_c(z) = -f(z)$ can occur, as is shown by the following example.

Example 1.4. Let $f(z) = \frac{\sin^3 z + 3\sin z}{3\sin^2 z + 1}$ and $c = \pi$. Obviously, $f(z)$ and its shift share $0, \infty$ CM with its shift $f(z+\pi) = -f(z)$. It is easy to see that $(\sin z - 1)^3 = 0$ is equivalent to $f(z) = 1$ and $(\sin z + 1)^3 = 0$ is equivalent to $f_\pi(z) = 1$. Thus, $\overline{E}_{5)}(1, f(z)) = \overline{E}_{5)}(1, f_\pi(z) = \emptyset$. The assumptions of Theorem 1.3 are fully satisfied and here, we see that $f_\pi(z) = -f(z)$ for all $z \in \mathbb{C}$.

When $k = \infty$, we obtain the following corollary which can be considered as a refinement of Theorem C.

Corollary 1.5. *Let f be a nonconstant meromorphic function of of hyper-order $\rho_2(f) < 1$, let $c \in \mathbb{C} \setminus \{0\}$ and let $a_1, a_2, a_3 \in S(f) \cup \{\infty\}$ be three distinct periodic functions with period c. If $f(z)$ and its shift $f_c(z) = f(z+c)$ share partially a_1, a_2 CM and share partially a_3 IM, i.e.,*

$$E(a_1, f) \subseteq E(a_1, f_c),\ \ E(a_2, f) \supseteq E(a_2, f_c)$$

and

$$\overline{E}(a_3, f) \subseteq \overline{E}(a_3, f_c),$$

then $f_c(z) = f(z)$ for all $z \in \mathbb{C}$.

When f is an entire function, we obtain the following.

Theorem 1.6. *Let f be a nonconstant entire function of of hyper-order $\rho_2(f) <$ 1 and let $c \in \mathbb{C} \setminus \{0\}$. If f and f_c share partially 0 CM and share partially 1 IM with weight 1, then $f_c(z) = f(z)$ or $f_c(z) = -f(z)$ for all $z \in \mathbb{C}$.*

Similar to the example above, we show that the case of $f_c(z) = -f(z)$ could be occurred as follows.

Example 1.7. Consider the entire function $f(z) = \sin z$ and $c = \pi$. Obviously, this function and its shift $f_\pi(z) = -f(z)$ share 0 CM and share 1 IM with weight 1 since $\overline{E}_{1)}(1, f) = \overline{E}_{1)}(1, f_\pi) = \emptyset$.

We would like to emphasize that in the situation of Theorem 1.6, the case of the entire function $f_c(z) = -f(z)$ could not happen only when $\overline{E}_{k)}(1, f) \subseteq \overline{E}_{k)}(1, f_c)$ with $k \geq 2$ by [14, Corollary 1.2]. Moreover, for any k $(k \geq 3)$, we can find a meromorphic function f such that $f_\pi = -f$ satisfying fully the assumptions of Theorem 1.3 by the same way as in Example 1.4.

2. Some lemmas

Lemma 2.1 ([7]). *Let f be a nonconstant meromorphic function and $c \in \mathbb{C}$. If f is of finite order, then*

$$m\big(r, \frac{f(z+c)}{f(z)}\big) = O\left(\frac{\log r}{r} T(r,f)\right)$$

for all r outside of a subset E zero logarithmic density. If the hyper-order $\rho_2(f)$ of f is less than one, then for each $\epsilon > 0$, we have

$$m\big(r, \frac{f(z+c)}{f(z)}\big) = o\left(\frac{T(r,f)}{r^{1-\rho_2(f)-\epsilon}}\right)$$

for all r outside of a subset finite logarithmic measure.

Lemma 2.2 ([7]). *Let $f(z)$ be a nonconstant meromorphic function of hyper-order $\rho_2(f) < 1$, and $c \in \mathbb{C} \setminus \{0\}$. Then*

$$m\big(r, \frac{\Delta_c f}{f}\big) = m\big(r, \frac{f(z+c) - f(z)}{f(z)}\big) = S_1(r,f),$$

where $S_1(r,f) = o(T(r,f)$ for all r outside of a set of finite logarithmic measure.

Lemma 2.3 ([7]). *Let $T : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing continuous function, and let $s \in (0, +\infty)$ such that hyper-order of T is strictly less than one, i.e.,*

$$\rho_2 = \limsup_{r\to\infty} \frac{\log^+ \log^+ T(r)}{\log r} < 1,$$

then

$$T(r+s) = T(r) + o\left(\frac{T(r)}{r^{1-\rho_2-\epsilon}}\right),$$

where $\epsilon > 0$ and $r \to \infty$ outside a subset of finite logarithmic measure.

Lemma 2.4 ([7, Theorem 2.1]). *Let $c \in \mathbb{C}$, and let f be a meromorphic function of hyper-order < 1 such that $\Delta_c f \not\equiv 0$. Let $q \geq 2$ and $a_1(z), \ldots, a_q(z)$ be distinct meromorphic periodic small functions of f with period c. Then*

$$m(r,f) + \sum_{k=1}^{q} m\big(r, \frac{1}{f - a_k}\big) \leq 2T(r,f) - N_{pair}(r,f) + S_1(r,f),$$

where

$$N_{pair}(r,f) = 2N(r,f) - N(r, \Delta_c f) + N\big(r, \frac{1}{\Delta_c f}\big).$$

Lemma 2.5 ([16]). *Let f be a nonconstant meromorphic function on $\mathbb{C}$. Let $a_1, a_2, \ldots, a_q$ $(q \geq 3)$ be q distinct small meromorphic functions of f on $\mathbb{C}$. Then the following holds*

$$(q-2)T(r,f) \leq \sum_{i=1}^{q} \overline{N}\big(r, \frac{1}{f - a_i}\big) + S(r,f),$$

where $S(r,f) = o(T(r,f))$ for all $r \in [1, \infty)$ outside a finite Borel measure set.

Lemma 2.6 ([17, Lemma 5.1]). *If f is a nonconstant periodic meromorphic, then $\rho(f) \geq 1$ and $\mu(f) \geq 1$, where $\mu(f) = \liminf_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}$ is the lower order of f.*

Lemma 2.7 ([17, Theorem 1.45]). *Suppose that $h(z)$ is a nonconstant entire function and $f(z) = e^{h(z)}$, then $\rho_2(f) = \rho(h)$.*

3. Proof of Theorem 1.1

Assume on the contrary that $\Delta_c f \not\equiv f$. By the assumption $E(0, f) \subseteq E(0, \Delta_c f)$, we have

$$N\big(r, \frac{1}{f}\big) \leq N\big(r, \frac{1}{\Delta_c f}\big).$$

By the First main theorem and Lemma 2.2, we obtain

$$\begin{aligned} T(r, f) &= T\big(r, \frac{1}{f}\big) + O(1) \\ &= m\big(r, \frac{1}{f}\big) + N\big(r, \frac{1}{f}\big) + O(1) \\ &\leq m\big(r, \frac{\Delta_c f}{f}\big) + m\big(r, \frac{1}{\Delta_c f}\big) + N\big(r, \frac{1}{\Delta_c f}\big) + O(1) \\ &= T(r, \Delta_c f) + S_1(r, f). \end{aligned}$$

On the other hand, by Lemma 2.3, we have

$$T(r, \Delta_c f) \leq T(r, f_c) + T(r, f) + O(1) \leq 2T(r, f) + S_1(r, f),$$

where $f_c(z) := f(z + c)$. These imply that

$$T(r, f) = T(r, \Delta_c f) + S_1(r, f),$$

and hence $S(r) := S_1(r, f) = S_1(r, \Delta_c f)$ which implies $\rho_2(\Delta_c f) = \rho_2(f) < 1$. We set

$$h = \frac{\Delta_c f}{f}. \tag{3.1}$$

Then by Lemma 2.2 again, we have

$$m\big(r, h\big) = S(r).$$

The assumptions $E(0, f) \subseteq E(0, \Delta_c f)$ and $E(\infty, f) \supseteq E(\infty, \Delta_c f)$ imply that h is an entire function and hence h is a small function with respect to f.

Since the assumption $E_{k)}(1, f) = E_{k)}(1, \Delta_c f)$, it is easy to see that

$$\overline{N}_{k)}\big(r, \frac{1}{f - 1}\big) = \overline{N}_{k)}\big(r, \frac{1}{\Delta_c f - 1}\big) \leq N\big(r, \frac{1}{h - 1}\big) = S(r). \tag{3.2}$$

From (3.1), we get

$$f_c = (h + 1)f.$$

It is easy to see that $S_1(r, f_c) = S_1(r, f) = S(r)$. By Lemma 2.3 again, we have

$$\begin{aligned}\overline{N}_{k)}\big(r, \frac{1}{f_c-1}\big) &\le \overline{N}_{k)}\big(r+|c|, \frac{1}{f-1}\big)\\ &= \overline{N}_{k)}\big(r, \frac{1}{f-1}\big) + o\big(\overline{N}_{k)}\big(r, \frac{1}{f-1}\big)\big)\\ &\le \overline{N}_{k)}\big(r, \frac{1}{f-1}\big) + S_1(r, f).\end{aligned}$$

By applying this inequality to $f = (f_c)_{-c}$, we obtain

$$\begin{aligned}\overline{N}_{k)}\big(r, \frac{1}{f-1}\big) &= \overline{N}_{k)}\big(r, \frac{1}{(f_c)_{-c}-1}\big)\\ &\le \overline{N}_{k)}\big(r, \frac{1}{f_c-1}\big) + S_1(r, f) = S_1(r, f).\end{aligned}$$

Combining these inequalities and (3.2), we get

$$\overline{N}_{k)}\big(r, \frac{1}{f-1}\big) = \overline{N}_{k)}\big(r, \frac{1}{f_c-1}\big) + S_1(r, f) \le S(r). \tag{3.3}$$

Then, we have

$$\begin{aligned}\overline{N}_{k)}\big(r, \frac{1}{f-\frac{1}{h+1}}\big) &= \overline{N}_{k)}\big(r, \frac{1}{\frac{1}{h+1}(f_c-1)}\big)\\ &= \overline{N}_{k)}\big(r, h+1\big) + \overline{N}_{k)}\big(r, \frac{1}{f_c-1}\big)\\ &\le S(r).\end{aligned} \tag{3.4}$$

On the other hand, also from (3.2), we get

$$\begin{aligned}\overline{N}_{k)}\big(r, \frac{1}{f-\frac{1}{h}}\big) &= \overline{N}_{k)}\big(r, \frac{1}{\frac{1}{h}(\Delta_c f-1)}\big)\\ &= \overline{N}_{k)}\big(r, h\big) + \overline{N}_{k)}\big(r, \frac{1}{\Delta_c f-1}\big)\\ &\le S(r).\end{aligned} \tag{3.5}$$

Then, since (3.3) and (3.5), we get

$$\begin{aligned}\overline{N}_{k)}\big(r, \frac{1}{f_c-(h+1)}\big) &= \overline{N}_{k)}\big(r, \frac{1}{(h+1)(f-1)}\big)\\ &= \overline{N}_{k)}\big(r, \frac{1}{h+1}\big) + \overline{N}_{k)}\big(r, \frac{1}{f-1}\big)\\ &\le S(r),\end{aligned}$$

and

$$\begin{aligned}\overline{N}_{k)}\big(r, \frac{1}{f_c - \frac{h+1}{h}}\big) &= \overline{N}_{k)}\big(r, \frac{1}{(h+1)(f - \frac{1}{h})}\big)\\ &= \overline{N}_{k)}\big(r, \frac{1}{h+1}\big) + \overline{N}_{k)}\big(r, \frac{1}{f - \frac{1}{h}}\big)\\ &\leq S(r).\end{aligned}$$

Replacing f in (3.3) by $f - h_{-c}$ and $f - \frac{1}{h_{-c}}$ and using two inequalities above, we have

$$\overline{N}_{k)}\big(r, \frac{1}{f - h_{-c} - 1}\big) = \overline{N}_{k)}\big(r, \frac{1}{f_c - h - 1}\big) + S(r) \leq S(r), \tag{3.6}$$

and

$$\overline{N}_{k)}\big(r, \frac{1}{f - \frac{1}{h_{-c}} - 1}\big) = \overline{N}_{k)}\big(r, \frac{1}{f_c - \frac{1}{h} - 1}\big) + S(r) \leq S(r). \tag{3.7}$$

Obviously, $a_1 = 1, a_2 = \frac{1}{h+1}, a_3 = \frac{1}{h}$ are distinct and $a_4 = h_{-c}+1, a_5 = \frac{1}{h_{-c}}+1$ are distinct too.

If $a_4 = a_2$, it is easy to see that $-\frac{1}{h} = \frac{1}{h_{-c}} + 1 = a_5$. This implies that $a_5 \neq a_3$ since $h \neq \pm 1$.

If $a_4 = a_3$, we also have $\frac{1}{hh_{-c}} = \frac{1}{h_{-c}} + 1 = a_5$. This yields that $a_5 \neq a_2$ since $h \neq 0$.

Therefore, there exist four distinct elements in the set $\{a_1, \ldots, a_5\}$, for instance $a_{j_1}, a_{j_2}, a_{j_3}, a_{j_4}$. So, by applying Lemma 2.5 to these functions and together this with (3.2)-(3.7), we get

$$\begin{aligned}2T(r,f) &\leq \sum_{i=1}^{4} \overline{N}\big(r, \frac{1}{f - a_{j_i}}\big) + S(r)\\ &= \sum_{i=1}^{4} \left(\overline{N}_{k)}\big(r, \frac{1}{f - a_{j_i}}\big) + \overline{N}_{(k+1}\big(r, \frac{1}{f - a_{j_i}}\big)\right) + S(r)\\ &\leq \frac{1}{k+1} \sum_{i=1}^{4} N\big(r, \frac{1}{f - a_{j_i}}\big) + S(r)\\ &\leq \frac{4}{k+1} T\big(r, f\big) + S(r).\end{aligned}$$

Letting $r \to +\infty$, we obtain $1 \leq \frac{2}{k+1}$, i.e., $k \leq 1$. This is a contradiction. Therefore, we must have $\Delta_c f = f$. The proof of Theorem 1.1 is completed.

4. Proof of Theorem 1.2

Similar to the proof of Theorem 1.1, we assume that $\Delta_c f \neq f$. Then,

$$h := \frac{\Delta_c f}{f}$$

must be a small function with respect to f and $f_c = (h+1)f$ and

$$\overline{N}_{1)}\big(r, \frac{1}{f-1}\big) = S(r), \quad \overline{N}_{1)}\big(r, \frac{1}{f-\frac{1}{h+1}}\big) = S(r)$$

and

$$\overline{N}_{1)}\big(r, \frac{1}{f-\frac{1}{h}}\big) = S(r).$$

Since the assumption that f is entire, applying Lemma 2.5 to $a_1 = 1, a_2 = \frac{1}{h+1}, a_3 = \frac{1}{h}, a_4 = \infty$, we obtain

$$\begin{aligned} 2T(r,f) &\le \sum_{i=1}^{3} \overline{N}\big(r, \frac{1}{f-a_i}\big) + \overline{N}(r,f) + S(r) \\ &\le \sum_{i=1}^{3} \overline{N}_{1)}\big(r, \frac{1}{f-a_i}\big) + \sum_{i=1}^{3} \overline{N}_{(2}\big(r, \frac{1}{f-a_i}\big) + S(r) \\ &\le \frac{3}{2}T\big(r,f\big) + S(r). \end{aligned}$$

This is a contradiction. Therefore, $\Delta_c f = f$. The proof of Theorem 1.2 is completed.

5. Proof of Theorem 1.3

Assume that $f_c \not\equiv f$. Put

$$h = \frac{f_c}{f},$$

then $h \not\equiv 1$. By Lemma 2.1, we have $m(r,h) = S(r)$. Since the assumption $E(0,f) \subseteq E(0,f_c)$ and $E(\infty,f) \supseteq E(\infty,f_c)$, it is easy to see that h is an entire function. Hence, h is small with respect to f. By the same arguments as in the proof of Theorem 1.1, we obtain

$$\overline{N}_{k)}\big(r, \frac{1}{f_c-1}\big) = \overline{N}_{k)}\big(r, \frac{1}{f-1}\big) + S(r) \le S(r),$$

$$\overline{N}_{k)}\big(r, \frac{1}{f-\frac{1}{h}}\big) = \overline{N}_{k)}\big(r, \frac{1}{\frac{f_c}{h}-\frac{1}{h}}\big) \le \overline{N}_{k)}\big(r, \frac{1}{f_c-1}\big) \le S(r),$$

$$\overline{N}_{k)}\big(r, \frac{1}{f_c-h}\big) = \overline{N}_{k)}\big(r, \frac{1}{h(f-1)}\big) \le \overline{N}_{k)}\big(r, \frac{1}{f-1}\big) \le S(r).$$

Similar to (3.3), we have

$$\overline{N}_{k)}\big(r, \frac{1}{f-h_{-c}}\big) \le S(r).$$

Case 1. $h_{-c} \neq \frac{1}{h}$. Applying Lemma 2.5 to $1, \frac{1}{h}, h_{-c}$, we obtain

$$\begin{aligned} T(r,f) &\leq \overline{N}\big(r, \frac{1}{f-1}\big) + \overline{N}\big(r, \frac{1}{f-\frac{1}{h}}\big) + \overline{N}\big(r, \frac{1}{f-h_{-c}}\big) + S(r) \\ &\leq \frac{3}{k+1} T\big(r,f\big) + S(r). \end{aligned}$$

This implies that $1 \leq \frac{3}{k+1}$, i.e., $k \leq 2$ which is a contradiction. This case can not happen.

Case 2. $h_{-c} = \frac{1}{h}$. This implies that

$$h \cdot h_c = 1 \tag{5.1}$$

and hence the entire h can not have any zeros. By Hadamard's factorization theorem, h must be in the form $h = e^{\tau}$, where τ is an entire function. Since h is a small function with respect to f, $\rho_2(h) \leq \rho_2(f) < 1$. Then by Lemma 2.7, we get

$$\rho(\tau) = \rho_2(h) < 1. \tag{5.2}$$

On the other hand, since (5.1) we have $e^{\tau} \cdot e^{\tau_c} = 1$ which implies that

$$\tau'_c = -\tau'. \tag{5.3}$$

From this, we get $\tau'_{2c} = \tau'$, and hence τ' is a periodic entire function. By Lemma 2.6, if τ' is a nonconstant function, then $\rho(\tau) \geq 1$ which is a contradiction to (5.2). Therefore, τ' must be a constant and hence $\tau' \equiv 0$ since (5.3). This implies that τ and hence h must be a constant. By (5.1), we obtain $h^2 = 1$ which yields $h = \pm 1$.

From two cases above, we obtain $f_c(z) = f(z)$ or $f_c(z) = -f(z)$ for all $z \in \mathbb{C}$. The proof of Theorem 1.3 is completed.

6. Proof of Corollary 1.5

In the each of the following cases, we always have $T(r,g) = T(r,f) + S_1 r, f$.

If $a_1, a_2, a_3 \in S(f)$, then we set $g = \frac{f-a_1}{f-a_2} \cdot \frac{a_3-a_2}{a_3-a_1}$. This implies that $h = \frac{g_c}{g} = \frac{f_c-a_1}{f_c-a_2} \cdot \frac{f-a_2}{f-a_1}$.

If $a_1 = \infty$, then we set $g = \frac{a_3-a_2}{f-a_2}$. This implies that $h = \frac{g_c}{g} = \frac{f-a_2}{f_c-a_2}$.

If $a_2 = \infty$, then we set $g = \frac{f-a_1}{a_3-a_1}$. This implies that $h = \frac{g_c}{g} = \frac{f_c-a_1}{f-a_1}$.

If $a_3 = \infty$, then we set $g = \frac{f-a_1}{f-a_2}$. This implies that $h = \frac{g_c}{g} = \frac{f_c-a_1}{f_c-a_2} \cdot \frac{f-a_2}{f-a_1}$.

It is easy to see from the assumptions that h is a entire function. This implies that $E(0,g) \subseteq E(0,g_c)$ and $E(\infty,g) \supseteq E(\infty,g_c)$. By simple calculation, we also get

$$\overline{E}(1,g) \subseteq \overline{E}(1,g_c). \tag{6.1}$$

Applying Theorem 1.3, we obtain $g_c = g$ or $g_c = -g$. If the second case happens, then $\Delta_c g = -2g$. It follows from (6.1) that 1 is a Picard value. By

Lemma 2.4, we get

$$m(r,g)+m\big(r,\frac{1}{g}\big)+m\big(r,\frac{1}{g-1}\big)\le 2T\big(r,g\big)-N\big(r,g\big)-N\big(r,\frac{1}{g}\big)+S(r),$$

which implies that

$$T(r,g)\le N\big(r,\frac{1}{g-1}\big)+S(r)=S(r).$$

This is impossible. Then, the first case must happen and hence $f_c(z)=f(z)$ for all $z\in\mathbb{C}$. The proof of Corollary 1.5 is completed.

7. Proof of Theorem 1.6

Assume that $f_c\not\equiv\pm f$ and take h as in the proof of Theorem 1.3. Based on the proof of this theorem, we see that $h_{-c}=\frac{1}{h}$ can not happen. Hence, by applying Lemma 2.5 to $1,\frac{1}{h},h_{-c},\infty$, we obtain

$$\begin{aligned}2T(r,f)&\le\overline{N}\big(r,\frac{1}{f-1}\big)+\overline{N}\big(r,\frac{1}{f-\frac{1}{h}}\big)+\overline{N}\big(r,\frac{1}{f-h_{-c}}\big)+\overline{N}\big(r,f\big)+S(r)\\&\le\frac{3}{2}T\big(r,f\big)+S(r),\end{aligned}$$

which is impossible. So we have $f_c(z)=f(z)$ or $f_c(z)=-f(z)$ for all $z\in\mathbb{C}$. The proof of Theorem 1.6 is completed.

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Thu Thuy Hoang
Department of Mathematics
Hanoi University of Civil Engineering
55 Giai Phong str., Hai Ba Trung
Hanoi, Vietnam
Email address: thuyht@huce.edu.vn

Hong Nhat Nguyen
Faculty of Mathematical economics
National Economics University
207 Giai Phong str., Hai Ba Trung
Hanoi, Vietnam
Email address: nhatnh@neu.edu.vn

Duc Thoan Pham
Department of Mathematics
Hanoi University of Civil Engineering
55 Giai Phong str., Hai Ba Trung
Hanoi, Vietnam
Email address: thoanpd@huce.edu.vn