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# ON REVERSIBLE $\mathbb{Z}_2$-DOUBLE CYCLIC CODES

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Abstract. A binary linear code is said to be a $\mathbb{Z}_2$-double cyclic code if its coordinates can be partitioned into two subsets such that any simultaneous cyclic shift of the coordinates of the subsets leaves the code invariant. These codes were introduced in [6]. A $\mathbb{Z}_2$-double cyclic code is called reversible if reversing the order of the coordinates of the two subsets leaves the code invariant. In this note, we give necessary and sufficient conditions for a $\mathbb{Z}_2$-double cyclic code to be reversible. We also give a relation between reversible $\mathbb{Z}_2$-double cyclic code and LCD $\mathbb{Z}_2$-double cyclic code for the separable case and we present a few examples to show that such a relation doesn't hold in the non-separable case. Furthermore, we list examples of reversible $\mathbb{Z}_2$-double cyclic codes of length $\leq 10$.

## 1. Introduction

A linear code $C$ over a finite field $\mathbb{F}_q$ is called reversible if for any codeword $c = (c_1,\ c_2, \ldots, c_n) \in C$, the word $c^R = (c_n, \ldots,\ c_2,\ c_1)$ obtained by reversing the order of coordinates of $c$, is also in $C$. Reversible codes have been studied for many years and have applications in certain data storage and retrieval systems. Massey in [21] first introduced reversible codes. Apart from indicating the potential applications of these codes in [21], he also gave necessary and sufficient conditions for cyclic and convolutional codes to be reversible. In [30], the authors studied the minimum distance of a certain class of reversible cyclic codes. Since then, reversible codes have been an object of great interest for researchers. Various construction techniques for reversible codes have been studied in [3, 8, 9, 11, 23, 24], etc. In [17], the authors gave the necessary and sufficient conditions for cyclic codes over $\mathbb{Z}_{p^k}$ to be reversible. In [27], the authors classified reversible cyclic codes over the ring $R = \mathbb{F}_q + u\mathbb{F}_q$ where $u^2 = 0 \pmod q$. In [2], reversible cyclic codes over $\mathbb{Z}_4$ have been studied. Reversible complement cyclic codes over Galois rings have been studied in [19]. Conditions for reversibility of negacyclic codes, some classes of quasi-cyclic codes

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and constacyclic codes over finite field have been studied in [25], [12] and [5], respectively. In [20], the authors studied the dimension and minimum distance of a family of reversible BCH codes over finite fields. Recently in [12], the authors proved a necessary and sufficient condition for the self-orthogonality of reversible QC codes of index 2. In [9], the authors generalized the construction of reversible codes given in [23] for odd and even length. To deal with a block-wise error, they also studied blockwise reversible codes.

Another generalization of binary cyclic codes is a $\mathbb{Z}_2$-double cyclic code. A $\mathbb{Z}_2$-double cyclic code is a binary linear code of length $n = r + s$ such that its coordinates can be partitioned into two sets of $r$ and $s$ coordinates and any simultaneous cyclic shift of coordinates of both subsets leaves the code invariant. These codes were studied in [6]. In [18], the authors studied double quadratic residue codes (QRC) of length $n = p + q$ for prime numbers $p$ and $q$ in $\mathbb{F}_2^p \times \mathbb{F}_2^q$. In [26], we calculated the weight distribution of $\mathbb{Z}_2$-double cyclic codes for a special case. In [4], the authors studied the properties of the $R$-double cyclic codes and $R$-double constacyclic codes where $R = \mathbb{F}_4 + v\mathbb{F}_4$, $v^2 = v$. Double cyclic codes over $\mathbb{Z}_4$ are studied in [15]. In the case of the ring $R = \mathbb{F}_q + u\mathbb{F}_q + u^2\mathbb{F}_q$, the properties of double cyclic codes over $R$ and its dual have been studied in [33]. Triple cyclic codes over various rings have been studied in [22, 29, 34], etc. The family of $\mathbb{Z}_2$-double cyclic codes is closely related to generalized quasi-cyclic codes of index 2. Generalized quasi-cyclic codes and their properties have been studied in [1, 7, 13, 14, 16, 28, 31], etc.

In [6], the authors determined the polynomial representation of $\mathbb{Z}_2$-double cyclic codes and their duals. Besides this, they have also studied the relations between $\mathbb{Z}_2$-double cyclic and other families of cyclic codes. They showed that a $\mathbb{Z}_2$-double cyclic code is a $\mathbb{Z}_2[x]$-submodule of $\frac{\mathbb{Z}_2[x]}{\langle x^r-1\rangle} \times \frac{\mathbb{Z}_2[x]}{\langle x^s-1\rangle}$ and has the form $C = \langle (b(x) \mid 0), (l(x) \mid a(x))\rangle$, where $b(x),\ l(x) \in \frac{\mathbb{Z}_2[x]}{\langle x^r-1\rangle}$ such that $b(x) \mid (x^r - 1)$ and $a(x) \in \frac{\mathbb{Z}_2[x]}{\langle x^s-1\rangle}$ such that $a(x) \mid (x^s - 1)$. Using the $\mathbb{Z}_2[x]$-module structure of $C$, we obtained results stating necessary and sufficient conditions for reversibility of $\mathbb{Z}_2$-double cyclic codes.

This paper is organized as follows. In Section 2, we discuss the basic properties of $\mathbb{Z}_2$-double cyclic codes, as proved in [6]. In Section 3, we prove our main results in Theorems 3.9 and 3.12 that give necessary and sufficient conditions for a $\mathbb{Z}_2$-double cyclic code to be reversible. In Section 4, we discuss the relation between reversible $\mathbb{Z}_2$-double cyclic codes and LCD $\mathbb{Z}_2$-double cyclic codes for the separable and the non-separable case. Section 5 lists examples of reversible $\mathbb{Z}_2$-double cyclic codes of length $\leq 10$.

## 2. Preliminaries

**Definition** ([6], Definition 1)**.** Let $C$ be a linear code over $\mathbb{Z}_2$ of length $n = r + s$, where $r, s$ are non-negative integers. Then $C$ is called a $\mathbb{Z}_2$-double cyclic code if

$$(a_0, a_1, \ldots, a_{r-1} \mid b_0, b_1, \ldots, b_{s-1}) \in C$$

implies

$$(a_{r-1}, a_0, \ldots, a_{r-2} \mid b_{s-1}, b_0, \ldots, b_{s-2}) \in C.$$

Let $c = (a_0,\ a_1, \ldots, a_{r-1} \mid b_0,\ b_1, \ldots,\ b_{s-1})$ be a codeword in $C$ and let $i$ be an integer. We denote the $i$-th shift of $c$ by

$$c^{(i)} = (a_{0-i},\ a_{1-i}, \ldots,\ a_{r-1-i} \mid b_{0-i},\ b_{1-i}, \ldots, b_{s-1-i}),$$

where the subscripts are read modulo $r$ and $s$, respectively.

Let $C \subseteq \mathbb{Z}_2^r \times \mathbb{Z}_2^s$ be a $\mathbb{Z}_2$-double cyclic code. Let $C_r$ be the canonical projection of $C$ on the first $r$ coordinates and $C_s$ on the last $s$ coordinates. The code $C$ is called separable if it is the direct product of $C_r$ and $C_s$.

Let $R_{r,s}$ denote the ring $\frac{\mathbb{Z}_2[x]}{\langle x^r-1\rangle} \times \frac{\mathbb{Z}_2[x]}{\langle x^s-1\rangle}$. There is a bijective map given by

$$\begin{gathered}\phi : \mathbb{Z}_2^r \times \mathbb{Z}_2^s \to R_{r,s}\\ u = (a_0,\ a_1, \ldots,\ a_{r-1} \mid b_0,\ b_1, \ldots, b_{s-1})\\ \mapsto u(x) = (a_0 + a_1x + \cdots + a_{r-1}x^{r-1} \mid b_0 + b_1x + \cdots + b_{s-1}x^{s-1}).\end{gathered}$$

The operation $\star$ defined as

$$\begin{aligned}&\star : \mathbb{Z}_2[x] \times R_{r,s} \to R_{r,s}\\ &\lambda(x) \star (p(x) \mid q(x)) = (\lambda(x)p(x) \mid \lambda(x)q(x)),\end{aligned}$$

gives the ring $R_{r,s}$ a structure of $\mathbb{Z}_2[x]$-module. Note that $x^i \star u(x) = u^{(i)}(x)$ for all $i$.

We have the following result on the structure of $\mathbb{Z}_2$-double cyclic codes.

**Theorem 2.1** ([6], Theorem 1)**.** *The $\mathbb{Z}_2[x]$-module $R_{r,s}$ is a Noetherian $\mathbb{Z}_2[x]$-module, and every submodule $N$ of $R_{r,s}$ can be written as*

$$N = \langle (b(x) \mid 0),\ (l(x) \mid a(x))\rangle,$$

*where $b(x),\ l(x) \in \frac{\mathbb{Z}_2[x]}{\langle x^r-1\rangle}$ with $b(x) \mid (x^r - 1)$ and $a(x) \in \frac{\mathbb{Z}_2[x]}{\langle x^s-1\rangle}$ with $a(x) \mid (x^s - 1)$.*

Thus, we can identify $\mathbb{Z}_2$-double cyclic codes in $\mathbb{Z}_2^r \times \mathbb{Z}_2^s$ as submodules of $R_{r,s}$. Hence, any submodule of $R_{r,s}$ is a $\mathbb{Z}_2$-double cyclic code. If $C = \langle (b(x) \mid 0),\ (l(x) \mid a(x))\rangle$ is a $\mathbb{Z}_2$-double cyclic code, then the canonical projections $C_r$ and $C_s$ are binary cyclic codes generated by $\gcd(b(x), l(x))$ and $a(x)$, respectively. The following proposition gives some conditions on the generator polynomials of a $\mathbb{Z}_2$-double cyclic code.

**Proposition 2.2** ([6], Proposition 1)**.** *Let $C = \langle (b(x) \mid 0),\ (l(x) \mid a(x))\rangle$ be a $\mathbb{Z}_2[x]$-double cyclic code. Then we can assume that*

1. *$C_s = \langle a(x)\rangle$, with $a(x) \mid (x^s - 1)$,*
2. *Define $C' = \{(p(x) \mid q(x)) \in C \mid q(x) = 0\}$. Then $\pi_r(C') = \langle b(x)\rangle$ with $b(x) \mid (x^r - 1)$.*
3. *$\deg(l(x)) < \deg(b(x))$.*

**Proposition 2.3** ([6], Proposition 2)**.** *Let $C = \langle (b(x) \mid 0),\ (l(x) \mid a(x)) \rangle$ be a $\mathbb{Z}_2$-double cyclic code. Assume that the generator polynomials of $C$ satisfy the conditions in Proposition 2.2. Then $b(x) \mid \frac{x^s-1}{a(x)} l(x)$.*

*Remark* 2.4. Note that the condition $b(x) \mid \frac{x^s-1}{a(x)} l(x)$ is equivalent to the condition $\pi_r(C') = \langle b(x) \rangle$.

**Proposition 2.5** ([6], Proposition 3)**.** *Let $C = \langle (b(x) \mid 0),\ (l(x) \mid a(x)) \rangle$ be a $\mathbb{Z}_2$-double cyclic code. Assume that the generator polynomials of $C$ satisfy the conditions in Proposition 2.2. Define the sets*

$$S_1 = \{(b(x) \mid 0),\ x \star (b(x) \mid 0), \ldots, x^{r-\deg(b(x))-1} \star (b(x) \mid 0)\},$$

$$S_2 = \{(l(x) \mid a(x)),\ x \star (l(x) \mid a(x)), \ldots, x^{s-\deg(a(x))-1} \star (l(x) \mid a(x))\}.$$

*Then, $S_1 \cup S_2$ forms a minimal generating set for $C$ as a $\mathbb{Z}_2$-module.*

*Remark* 2.6. Propositions 2.3 and 2.5 hold true even if the condition $\deg(l(x)) < \deg(b(x))$ is not satisfied.

## 3. Reversible $\mathbb{Z}_2$-double cyclic code

**Definition.** The reverse of a $\mathbb{Z}_2$-double cyclic code $C$, denoted by $C^R$, is a linear code over $\mathbb{Z}_2$ defined as

$$\begin{aligned} C^R = \{(a_{r-1}, a_{r-2}, \ldots, a_0 \mid b_{s-1}, b_{s-2}, \ldots, b_0) :& \\ (a_0, a_1, \ldots, a_{r-1} \mid b_0, b_1, \ldots, b_{s-1}) \in C\}.& \end{aligned}$$

**Proposition 3.1.** *$C^R$ is a $\mathbb{Z}_2$-double cyclic code.*

*Proof.* Let $t := \mathrm{lcm}(r, s) - 1 = ar - 1 = bs - 1$ where $a \geq 1$, $b \geq 1$ integers. Then we have to show that for any codeword $(a_{r-1}, a_{r-2}, \ldots, a_0 \mid b_{s-1}, b_{s-2}, \ldots, b_0) \in C^R$, we have $(a_0, a_{r-1}, \ldots, a_1 \mid b_0, b_{s-1}, \ldots, b_1) \in C^R$. Now,

$$\begin{aligned}
&(a_{r-1} + a_{r-2}x + \cdots + a_0 x^{r-1} \mid b_{s-1} + b_{s-2}x + \cdots + b_0 x^{s-1}) \in C^R \\
\Longrightarrow\ &(a_0 + a_1 x + \cdots + a_{r-1} x^{r-1} \mid b_0 + b_1 x + \cdots + b_{s-1} x^{s-1}) \in C \\
\Longrightarrow\ &x^t \star (a_0 + a_1 x + \cdots + a_{r-1} x^{r-1} \mid b_0 + b_1 x + \cdots + b_{s-1} x^{s-1}) \in C \\
\Longrightarrow\ &(x^{ar-1}(a_0 + a_1 x + \cdots + a_{r-1} x^{r-1}) \mid x^{bs-1}(b_0 + b_1 x + \cdots + b_{s-1} x^{s-1})) \in C \\
\Longrightarrow\ &(x^{(a-1)r+r-1}(a_0 + a_1 x + \cdots + a_{r-1} x^{r-1}) \mid x^{(b-1)s+s-1}(b_0 + b_1 x + \cdots + b_{s-1} x^{s-1})) \in C \\
\Longrightarrow\ &(x^{r-1}(a_0 + a_1 x + \cdots + a_{r-1} x^{r-1}) \mid x^{s-1}(b_0 + b_1 x + \cdots + b_{s-1} x^{s-1})) \in C \\
\Longrightarrow\ &(a_1 + a_2 x + \cdots + a_{r-1} x^{r-2} + a_0 x^{r-1} \mid b_1 + b_2 x + \cdots + b_{s-1} x^{s-2} + b_0 x^{s-1}) \in C \\
\Longrightarrow\ &(a_0 + a_{r-1} x + \cdots + a_1 x^{r-1} \mid b_0 + b_{s-1} x + \cdots + b_1 x^{s-1}) \in C^R.
\end{aligned}$$

This proves the result. □

**Definition.** If $C = C^R$, then $C$ is called a reversible $\mathbb{Z}_2$-double cyclic code.

### 3.1. Generators of $C^R$

Before coming to the results on the generators of code $C^R$, we give a result that is required later. For a polynomial $f(x) \in \mathbb{Z}_2[x]$, we define reciprocal polynomial $f^*(x) \in \mathbb{Z}_2[x]$ as $f^*(x) = x^{\deg(f(x))} f(\frac{1}{x})$. For the properties of reciprocal polynomial, refer to [10].

**Proposition 3.2.** *We have* $C_1 = \langle (b_1(x) \mid 0), (l_1(x) \mid a_1(x)) \rangle$ *and* $C_2 = \langle (b_2(x) \mid 0), (l_2(x) \mid a_2(x)) \rangle$ $\mathbb{Z}_2$*-double cyclic codes of length* $n = r + s$*. If the generators satisfy conditions* (1) *and* (2) *of Proposition 2.2, then* $C_1 = C_2$ *if and only if the following conditions are satisfied*

(1) $b_1(x) = b_2(x)$,
(2) $a_1(x) = a_2(x)$,
(3) $b_1(x) \mid (l_1(x) - l_2(x))$ *or* $b_2(x) \mid (l_1(x) - l_2(x))$.

*Proof.* First suppose $C_1 = C_2$. Then

$$\begin{aligned} & (C_1)_s = (C_2)_s \\ \Longrightarrow\ & \langle a_1(x) \rangle = \langle a_2(x) \rangle \\ \Longrightarrow\ & a_1(x) = \lambda(x) a_2(x) + \mu(x)(x^s - 1) \text{ for some } \lambda(x),\ \mu(x) \in \mathbb{Z}_2[x] \\ \Longrightarrow\ & a_2(x) \mid a_1(x) \text{ (as } a_2(x) \mid (x^s - 1)). \end{aligned}$$

Similarly, we get $a_1(x) \mid a_2(x)$. Thus, $a_1(x) = a_2(x)$.

Proceeding similarly, $C_1' = C_2'$ gives $b_1(x) = b_2(x)$. Now we prove (3).

$$\begin{aligned} & (l_1(x) \mid a_1(x)) = \lambda(x) \star (b_2(x) \mid 0) + \mu(x) \star (l_2(x) \mid a_2(x)) \\ \Longrightarrow\ & l_1(x) = \lambda(x) b_2(x) + \mu(x) l_2(x) \pmod{(x^r - 1)} \text{ and} \\ & a_1(x) = a_1(x)\mu(x) \pmod{(x^s - 1)}, \end{aligned}$$

where $\deg(\lambda(x)) \le r - \deg(b_1(x)) - 1$ and $\deg(\mu(x)) \le s - \deg(a_1(x)) - 1$. This gives $\mu(x) = 1 \left(\mathrm{mod}\ \frac{x^s-1}{a_1(x)}\right)$. It follows

$$l_1(x) = \lambda(x) b_1(x) + \Big(1 + t(x) \frac{x^s - 1}{a_1(x)}\Big) l_2(x) \pmod{(x^r - 1)} \quad \text{for some } t(x) \in \mathbb{Z}_2[x].$$

We get $b_1(x) \mid (l_1(x) - l_2(x))$.

Conversely, suppose that (1), (2) and (3) hold. We want to show $C_1 = C_2$. Let $b_1(x) = b_2(x) =: b(x)$ and $a_1(x) = a_2(x) =: a(x)$. Since $\dim_{\mathbb{Z}_2} C_1 = r + s - \deg(b(x)) - \deg(a(x)) = \dim_{\mathbb{Z}_2} C_2$. So, we only need to show that $(l_1(x) \mid a_1(x)) \in C_2$. From (3), we have $l_1(x) = l_2(x) + \alpha(x) b(x)$ for some $\alpha(x) \in \mathbb{Z}_2[x]$. Then

$$\begin{aligned} (l_1(x) \mid a_1(x)) &= (l_2(x) + \alpha(x) b(x) \mid a(x)) \\ &= (l_2(x) \mid a_2(x)) + \alpha(x) \star (b_2(x) \mid 0) \in C_2. \end{aligned}$$

□

**Corollary 3.3.** *Let* $C = \langle (b(x) \mid 0),\ (l(x) \mid a(x)) \rangle$ *be a* $\mathbb{Z}_2$*-double cyclic code. Then the following conditions determine the generators of* $C$ *uniquely.*

(1) $b(x) \mid (x^r - 1)$ *and* $a(x) \mid (x^s - 1)$.

(2) deg $l(x) <$ deg $b(x)$.
(3) $b(x) \mid \frac{x^s-1}{a(x)}l(x)$ *and* $C_s = \langle a(x)\rangle$.

From Proposition 3.1 we have, $C^R$ is a $\mathbb{Z}_2$-double cyclic code, thus

$$C^R = \langle (B(x) \mid 0),\ (L(x) \mid A(x))\rangle,$$

where $B(x) \in \frac{\mathbb{Z}_2[x]}{\langle x^r-1\rangle}$ such that $B(x) \mid (x^r-1)$, $\pi_r((C^R)') = \langle B(x)\rangle$, where $(C^R)' = \{(p(x) \mid q(x)) \in C^R : q(x) = 0\}$, and $A(x) \in \frac{\mathbb{Z}_2[x]}{\langle x^s-1\rangle}$ such that $A(x) \mid (x^s-1)$ and $(C^R)_s = \langle A(x)\rangle$.

In the following results we use the notation $f^R(x) \in \frac{\mathbb{Z}_2[x]}{\langle x^n-1\rangle}$ for the polynomial corresponding to the word $u^R = (a_{n-1},\ a_{n-2}, \ldots, a_0)$, where $f(x) \in \frac{\mathbb{Z}_2[x]}{\langle x^n-1\rangle}$ is the polynomial corresponding to $u = (a_0,\ a_1, \ldots, a_{n-1})$. Thus, $f^R(x) = x^{n-1}f(\frac{1}{x})$. Also for the $\mathbb{Z}_2$-double cyclic code $C$, $(p(x) \mid q(x)) \in C$ if and only if $(p^R(x) \mid q^R(x)) \in C^R$.

**Proposition 3.4.** *Let* $C = \langle (b(x) \mid 0), (l(x) \mid a(x))\rangle$ *be a* $\mathbb{Z}_2$*-double cyclic code with* $C^R = \langle (B(x) \mid 0), (L(x) \mid A(x))\rangle$ *such that their generators satisfy conditions* (1) *and* (2) *of Proposition 2.2. Then, we have* $B(x) = b^*(x)$.

*Proof.* We have $x^{r-\deg(b(x))-1} \star (b(x) \mid 0) \in C$ which gives $(b^*(x) \mid 0) \in (C^R)'$. So, $B(x) \mid b^*(x)$.

Again, we have $(B(x) \mid 0) \in (C^R)'$, which implies $(B^R(x) \mid 0) \in C$. This gives $B^R(x) = \lambda(x)b(x)$ where $\deg(\lambda(x)) \leq r - \deg\ b(x) - 1$. Therefore, $B(x) = x^{r-1}\lambda(x^{-1})b(x^{-1}) = x^{r-1-\deg(b(x))}\lambda(x^{-1})b^*(x)$. This gives $b^*(x) \mid B(x)$. This proves the result. $\square$

**Proposition 3.5.** *Let* $C = \langle (b(x) \mid 0), (l(x) \mid a(x))\rangle$ *be a* $\mathbb{Z}_2$*-double cyclic code with* $C^R = \langle (B(x) \mid 0), (L(x) \mid A(x))\rangle$ *such that their generators satisfy conditions* (1) *and* (2) *of Proposition 2.2. Then, we have* $A(x) = a^*(x)$.

*Proof.* We have $(l^R(x) \mid a^R(x)) \in C^R$. So, $x^{\deg(a(x))+1} \star (l^R(x) \mid a^R(x)) \in C^R$. This gives $(x^{\deg(a(x))+1}l^R(x) \mid a^*(x)) \in C^R$. (The operations are performed mod $(x^r-1)$ and mod $(x^s-1)$ in first and second part, respectively.) Thus, $A(x) \mid a^*(x)$.

Again, $(L^R(x) \mid A^R(x)) \in C$. This gives $A^R(x) \in C_s = \langle a(x)\rangle$. So, $A^R(x) = \mu(x)a(x)$ where $\deg(\mu(x)) \leq s - \deg(a(x)) - 1$. This implies $A(x) = x^{s-\deg(a(x))-1}\mu(x^{-1})a^*(x)$. This proves the result. $\square$

**Proposition 3.6.** *For* $r - \deg(l(x)) \geq s - \deg(a(x))$, *we have*

$$b^*(x) \mid (L(x) - x^{r-s+\deg(a(x))-\deg(l(x))}l^*(x)).$$

*Thus if* $r - s + \deg(a(x)) - \deg(l(x)) + \deg(l^*(x)) < \deg(b(x))$, *then* $L(x) = x^{r-s+\deg(a(x))-\deg(l(x))}l^*(x)$.

*Proof.* We have

$$x^{s-1-\deg(a(x))} \star (l(x) \mid a(x)) \in C$$

$$\begin{aligned}
&\Longrightarrow (x^{r-s+\deg(a(x))}l(x^{-1}) \mid a^*(x)) \in C^R \\
&\Longrightarrow (x^{r-s+\deg(a(x))-\deg(l(x))}l^*(x) \mid a^*(x)) \in C^R.
\end{aligned}$$

Then

$$(x^{r-s+\deg(a(x))-\deg(l(x))}l^*(x) \mid a^*(x)) = \lambda(x)\star(b^*(x) \mid 0)+\mu(x)\star(L(x) \mid a^*(x)),$$

where $\deg(\lambda(x)) \leq r-\deg(b(x))-1$ and $\deg(\mu(x)) \leq s-\deg(a(x))-1$.

This gives $\mu(x)=1$ and

$$x^{r-s+\deg(a(x))-\deg(l(x))}l^*(x) = \lambda(x)b^*(x)+L(x).$$

This implies $b^*(x) \mid (L(x)-x^{r-s+\deg(a(x))-\deg(l(x))}l^*(x))$. □

**Theorem 3.7.** *Let $C=\langle(b(x) \mid 0),(l(x) \mid a(x))\rangle$ be a $\mathbb{Z}_2$-double cyclic code such that its generators satisfy conditions* (1) *and* (2) *of Proposition 2.2. If $r-\deg(l(x)) \geq s-\deg(a(x))$, then $C$ is reversible if and only if the following conditions are satisfied*

(1) $b(x)=b^*(x)$,
(2) $a(x)=a^*(x)$,
(3) $b(x) \mid (x^{r-s+\deg(a(x))-\deg(l(x))}l^*(x)-l(x))$.

*Proof.* Note that by Proposition 3.2, we have $C=C^R$ if and only if $b(x)=b^*(x)$, $a(x)=a^*(x)$ and $b(x) \mid (L(x)-l(x))$. Then

$$\begin{aligned}
&b(x) \mid (L(x)-l(x)) \\
\Longleftrightarrow\ &b(x) \mid (L(x)-x^{r-s+\deg(a(x))-\deg(l(x))}l^*(x)+x^{r-s+\deg(a(x))-\deg(l(x))}l^*(x)-l(x)) \\
\Longleftrightarrow\ &b(x) \mid (x^{r-s+\deg(a(x))-\deg(l(x))}l^*(x)-l(x)) \text{ (by Proposition 3.6).}
\end{aligned}$$ □

*Remark* 3.8. If we assume that the conditions (1), (2) and (3) of Proposition 2.2 are satisfied by a $\mathbb{Z}_2$-double cyclic code and $r-\deg(l(x)) \geq s-\deg(a(x))$, then we get the same result as Theorem 3.7 (as in that case instead of $b(x) \mid (L(x)-l(x))$ we get $L(x)=l(x)$).

In case $r-\deg(l(x)) < s-\deg(a(x))$, let $s-\deg(a(x))-1+\deg(l(x))=tr+\beta$ where $\beta<r$.

We have

$$x^{s-1-\deg(a(x))}\star(l(x) \mid a(x)) \in C.$$

If $\beta \geq \deg(l(x))$, then

$$\begin{aligned}
&(x^{\beta-\deg(l(x))}l(x) \mid x^{s-1-\deg(a(x))}a(x)) \in C \\
\Longrightarrow\ &(x^{r-1-\beta}l^*(x) \mid a^*(x)) \in C^R.
\end{aligned}$$

If $\beta < \deg(l(x))$, then

$$\begin{aligned}
&(x^{r+\beta-\deg(l(x))}l(x) \mid x^{s-1-\deg(a(x))}a(x)) \in C \\
\Longrightarrow\ &(x^{r-1-\beta}l^*(x) \mid a^*(x)) \in C^R.
\end{aligned}$$

If $r-1-\beta+\deg(l^*(x)) \geq r$, $L'(x) := x^{r-1-\beta}l^*(x)+(x^r+1)(x^{\deg(l^*(x))-1-\beta}+a_{\deg(l^*(x))-1}x^{\deg(l^*(x))-1-\beta-1}+\cdots+a_{\beta+2}x+a_{\beta+1})$, where $l^*(x)=x^{\deg(l^*(x))}+a_{\deg(l^*(x))-1}x^{\deg(l^*(x))-1}+\cdots+a_1x+a_0$.

**Proposition 3.9.** *For $r-\deg(l(x)) < s-\deg(a(x))$, we have*

$$b^*(x) \mid (L(x)-x^{r-1-\beta}l^*(x)),$$

*where $\beta$ is given by $s-\deg(a(x))-1+\deg(l(x)) = tr+\beta,\ \beta < r$.*

*Proof.* The proof is the same as the proof of Proposition 3.6. □

**Theorem 3.10.** *Let $C = \langle (b(x) \mid 0), (l(x) \mid a(x)) \rangle$ be a $\mathbb{Z}_2$-double cyclic code such that its generators satisfy conditions (1) and (2) of Proposition 2.2. If $r-\deg(l(x)) < s-\deg(a(x))$, then $C$ is reversible if and only if the following conditions are satisfied*

(1) *$b(x) = b^*(x)$,*
(2) *$a(x) = a^*(x)$,*
(3) *$b(x) \mid (x^{r-1-\beta}l^*(x) - l(x))$,*

*where $\beta$ is given by $s-\deg(a(x))-1+\deg(l(x)) = tr+\beta,\ \beta < r$.*

*Proof.* Note that by Proposition 3.2, we have $C = C^R$ if and only if $b(x) = b^*(x)$, $a(x) = a^*(x)$ and $b(x) \mid (L(x)-l(x))$. Then

$$\begin{aligned} & b(x) \mid (L(x)-l(x)) \\ \iff & b(x) \mid (L(x)-x^{r-1-\beta}l^*(x)+x^{r-1-\beta}l^*(x)-l(x)) \\ \iff & b(x) \mid (x^{r-1-\beta}l^*(x)-l(x)). \end{aligned}$$

□

*Remark* 3.11. If we assume that the conditions (1), (2) and (3) of Proposition 2.2 are satisfied by a $\mathbb{Z}_2$-double cyclic code and $r-\deg(l(x)) < s-\deg(a(x))$, then we get the same result as Theorem 3.10 (as in that case instead of $b(x) \mid L(x)-l(x)$ we get $L(x) = l(x)$).

We have the following result from [6].

**Proposition 3.12** ([6], Proposition 7)**.** *Let $C = \langle (b(x) \mid 0), (l(x) \mid a(x)) \rangle$ be a separable $\mathbb{Z}_2$-double cyclic code. Assume that the generator polynomials of $C$ satisfy the conditions (1), (2) and (3) in Proposition 2.2. Then $l(x) = 0$. Moreover, $C^\perp$ is a separable $\mathbb{Z}_2$-double cyclic code such that $C^\perp = \langle \big(\frac{x^r-1}{b^*(x)} \mid 0\big), \big(0 \mid \frac{x^s-1}{a^*(x)}\big) \rangle$.*

**Proposition 3.13.** *Let $C$ be a separable $\mathbb{Z}_2$-double cyclic code $C$ such that the generator polynomials of $C$ satisfy the conditions of Proposition 2.2. Then $C = \langle (b(x) \mid 0), (0 \mid a(x)) \rangle$. We have $C^R$ is separable and $C^R = \langle (b^*(x) \mid 0), (0 \mid a^*(x)) \rangle$. Moreover, $C$ is reversible if and only if $b(x)$ and $a(x)$ are self-reciprocal, i.e., $b^*(x) = b(x)$ and $a^*(x) = a(x)$.*

## 4. Relation to LCD $\mathbb{Z}_2$-double-cyclic codes

Let $C$ be a $\mathbb{Z}_2$-double cyclic code of length $n = r + s$. Then its dual $C^\perp$ is also a $\mathbb{Z}_2$-double cyclic code. A linear complementary dual $\mathbb{Z}_2$-double cyclic code is defined as follows.

**Definition.** A $\mathbb{Z}_2$-double cyclic code $C$ is said to be a linear complementary dual $\mathbb{Z}_2$-double cyclic code if $C \cap C^\perp = \{0\}$.

For cyclic codes over $\mathbb{F}_q$, we have the following result:

**Proposition 4.1** ([32]). *A $q$-ary cyclic code, whose length $n$ is relatively prime to the characteristic $p$ of $\mathbb{F}_q$, is an LCD code if and only if it is a reversible code.*

Such a result doesn't hold in the case of non-separable $\mathbb{Z}_2$-double cyclic codes, as can be seen from the following examples.

**Example 4.2.** Let $r = 4$, $s = 3$, $b(x) = (x+1)^2$, $a(x) = x^2+x+1$, $l(x) = x+1$. Then $n = r + s = 7$ is relatively prime to 2, and

$$\begin{aligned} C = \{&(0,0,0,0 \mid 0,0,0), (1,1,0,0 \mid 1,1,1), (0,1,0,1 \mid 0,0,0), (1,0,0,1 \mid 1,1,1),\\ &(1,0,1,0 \mid 0,0,0), (0,1,1,0 \mid 1,1,1), (1,1,1,1 \mid 0,0,0), (0,0,1,1 \mid 1,1,1)\},\\ C^\perp = \{&(0,0,0,0 \mid 0,0,0), (0,0,0,0 \mid 0,1,1), (0,0,0,0 \mid 1,0,1), (0,0,0,0 \mid 1,1,0),\\ &(0,1,0,1 \mid 0,0,1), (0,1,0,1 \mid 0,1,0), (0,1,0,1 \mid 1,0,0), (0,1,0,1 \mid 1,1,1),\\ &(1,0,1,0 \mid 0,0,1), (1,0,1,0 \mid 0,1,0), (1,0,1,0 \mid 1,0,0), (1,0,1,0 \mid 1,1,1),\\ &(1,1,1,1 \mid 0,0,0), (1,1,1,1 \mid 0,1,1), (1,1,1,1 \mid 1,0,1), (1,1,1,1 \mid 1,1,0)\}. \end{aligned}$$

Here $C$ is a reversible $\mathbb{Z}_2$-double cyclic code but $C \cap C^\perp \neq \{0\}$.

**Example 4.3.** Let $r = s = 3$. Let $b(x) = x + 1$, $a(x) = x^2 + x + 1$, $l(x) = 1$. Here $r$ and $s$ are both relatively prime to 2. We have

$$\begin{aligned} C = \{&(0,0,0 \mid 0,0,0), (1,0,0 \mid 1,1,1), (0,1,1 \mid 0,0,0), (1,1,1 \mid 1,1,1),\\ &(1,1,0 \mid 0,0,0), (0,1,0 \mid 1,1,1), (1,0,1 \mid 0,0,0), (0,0,1 \mid 1,1,1)\},\\ C^\perp = \{&(0,0,0 \mid 0,0,0), (0,0,0 \mid 0,1,1), (0,0,0 \mid 1,0,1), (0,0,0 \mid 1,1,0),\\ &(1,1,1 \mid 0,0,1), (1,1,1 \mid 0,1,0), (1,1,1 \mid 1,0,0), (1,1,1 \mid 1,1,1)\}. \end{aligned}$$

Thus, $C$ is a reversible $\mathbb{Z}_2$-double cyclic code but $C \cap C^\perp \neq \{0\}$.

**Example 4.4.** Let $r = s = 5$. Let $b(x) = x^4 + x^3 + x^2 + x + 1,\ a(x) = x + 1,\ l(x) = 1$. Here $r$ and $s$ are both relatively prime to 2. $C$ has $\mathbb{Z}_2$-basis

$$\begin{aligned} \{&(1,1,1,1,1 \mid 0,0,0,0,0), (1,0,0,0,0 \mid 1,1,0,0,0), (0,1,0,0,0 \mid 0,1,1,0,0),\\ &(0,0,1,0,0 \mid 0,0,1,1,0), (0,0,0,1,0 \mid 0,0,0,1,1)\}. \end{aligned}$$

It can be checked that $C$ is not reversible but $C \cap C^\perp = \{0\}$, i.e., $C$ is a LCD $\mathbb{Z}_2$-double cyclic code.

**Example 4.5.** Let $r = 6$, $s = 3$, $b(x) = x^2 + x + 1$, $l(x) = x$, $a(x) = 1$. Then $n = r + s = 9$ is relatively prime to 2, and $C$ has $\mathbb{Z}_2$-basis

$$\{(1,1,1,0,0,0 \mid 0,0,0), (0,1,1,1,0,0 \mid 0,0,0),$$
$$(0,0,1,1,1,0 \mid 0,0,0), (0,0,0,1,1,1 \mid 0,0,0), (0,1,0,0,0,0 \mid 1,0,0),$$
$$(0,0,1,0,0,0 \mid 0,1,0), (0,0,0,1,0,0 \mid 0,0,1)\}.$$

It can be checked that $C$ is not reversible but $C \cap C^{\perp} = \{0\}$, i.e., $C$ is a LCD $\mathbb{Z}_2$-double cyclic code.

### 4.1. For $C$ separable

Recall that a $\mathbb{Z}_2$-double cyclic code $C$ is separable if $C$ is the direct product of $C_r$ and $C_s$. We have $\bar{b}(x) = \frac{x^r-1}{b^*(x)}$ and $\bar{a}(x) = \frac{x^s-1}{a^*(x)}$.

Using the ideas of the paper [32], we have the following conditions for a separable $\mathbb{Z}_2$-double cyclic code to have complementary dual.

**Lemma 4.6.** *Let $C = \langle (b(x) \mid 0), (0 \mid a(x)) \rangle$ be a separable $\mathbb{Z}_2$-double cyclic code of length $n = r + s$. Assume that the generator polynomials of $C$ satisfy the conditions of Proposition 2.2. Then $C$ is a LCD $\mathbb{Z}_2$-double cyclic code if and only if* $\gcd(b(x), \bar{b}(x)) = 1$ *and* $\gcd(a(x), \bar{a}(x)) = 1$.

*Proof.* Since $C$ is separable, we have $C = C_r \times C_s$ and $C^{\perp} = C_r^{\perp} \times C_s^{\perp}$. Also, the components of $C$ and $C^{\perp}$ have the form $C_r = \langle b(x) \rangle$, $C_s = \langle a(x) \rangle$, $C_r^{\perp} = \langle \bar{b}(x) \rangle$ and $C_s^{\perp} = \langle \bar{a}(x) \rangle$. Therefore, $C_r \cap C_r^{\perp} = \langle \mathrm{lcm}(b(x), \bar{b}(x)) \rangle$ and $C_s \cap C_s^{\perp} = \langle \mathrm{lcm}(a(x), \bar{a}(x)) \rangle$.

Now, $C \cap C^{\perp} = (C_r \cap C_r^{\perp}) \times (C_s \cap C_s^{\perp})$. Therefore, $C \cap C^{\perp} = \{0\}$ if and only if $C_r \cap C_r^{\perp} = \{0\}$ and $C_s \cap C_s^{\perp} = \{0\}$. We note that $C_r \cap C_r^{\perp} = \{0\}$ if and only if $\mathrm{lcm}(b(x), \bar{b}(x)) = x^r - 1$ and $C_s \cap C_s^{\perp} = \{0\}$ if and only if $\mathrm{lcm}(a(x), \bar{a}(x)) = x^s - 1$. But $x^r - 1$ is divisible by $b(x)$ and $\bar{b}(x)$, and $\deg(\bar{b}(x)) = r - \deg(b(x))$. Therefore, $\mathrm{lcm}(b(x), \bar{b}(x)) = x^r - 1$ if and only if $\gcd(b(x), \bar{b}(x)) = 1$. Similarly, $\mathrm{lcm}(a(x), \bar{a}(x)) = x^s - 1$ if and only if $\gcd(a(x), \bar{a}(x)) = 1$. □

**Theorem 4.7.** *Let $C$ be a separable $\mathbb{Z}_2$-double cyclic code of length $n = r + s$. Assume that the generator polynomials of $C$ satisfy the conditions of Proposition 2.2. Then $C$ is a LCD code if and only if $b(x)$ and $a(x)$ are self-reciprocal (i.e., $b^*(x) = b(x)$ and $a^*(x) = a(x)$) and all the monic irreducible factors of $b(x)$ have the same multiplicity in $b(x)$ and in $x^r - 1$ and all the monic irreducible factors of $a(x)$ have the same multiplicity in $a(x)$ and in $x^s - 1$.*

*Proof.* Let $r = \tilde{r} \cdot 2^e$ and $s = \tilde{s} \cdot 2^f$ where $e \geq 0$, $f \geq 0$ and $\gcd(2, \tilde{r}) = 1$, $\gcd(2, \tilde{s}) = 1$. First, suppose that $C$ is a LCD code. Then by Lemma 4.6, we have $\gcd(b(x), \bar{b}(x)) = 1$ and $\gcd(a(x), \bar{a}(x)) = 1$. Thus, from

$$x^r - 1 = b(x)(\bar{b}(x))^* = b^*(x)\bar{b}(x), \tag{1}$$

it follows that $b(x)$ must divide $b^*(x)$. Thus, $b^*(x) = b(x)$, i.e., $b(x)$ is self-reciprocal. Also, $\gcd(b(x), \bar{b}(x)) = 1$ implies that $\gcd(b^*(x), \bar{b}(x)) = 1$ which further implies that $\gcd(b(x), (\bar{b}(x))^*) = 1$. Because

$$x^r - 1 = b(x)(\bar{b}(x))^* = (x^{\tilde{r}} - 1)^{2^e}, \tag{2}$$

it follows that all the irreducible factors of $b(x)$ must have multiplicity $2^e$. Similarly, $a(x)$ is self-reciprocal and all the monic irreducible factors of $a(x)$ have the same multiplicity in $a(x)$ and in $x^s - 1$.

Conversely, suppose first that one of $b(x)$ or $a(x)$ is not self-reciprocal. Without loss of generality, suppose that $b(x)$ is not self-reciprocal, i.e., $b(x)$ does not divide $b^*(x)$. It follows then from (1) that $\gcd(b(x), \bar{b}(x)) \neq 1$ and hence, by Lemma 4.6, it follows that $C$ is not a LCD code.

Finally, suppose that $b(x)$ and $a(x)$ are self-reciprocal but some monic irreducible factor of $b(x)$ or $a(x)$ has multiplicity less than $2^e$ and $2^f$, respectively. The assumption $b(x)$ and $a(x)$ are self-reciprocal implies $\bar{b}(x)$ and $\bar{a}(x)$ are self-reciprocal. Without loss of generality, assume that some monic irreducible factor of $b(x)$ has multiplicity less than $2^e$. From (2), we have $1 \neq \gcd(b(x), (\bar{b}(x))^*) = \gcd(b(x), \bar{b}(x))$, and hence by Lemma 4.6, $C$ is not a LCD code. □

Thus, we get the following analogy between reversible $\mathbb{Z}_2$-double cyclic code and LCD $\mathbb{Z}_2$-double cyclic code for the separable case.

**Corollary 4.8.** *Let $C$ be a separable $\mathbb{Z}_2$-double cyclic code of length $n = r + s$. Assume that the generator polynomials of $C$ satisfy the conditions in Proposition 2.2. If $r$ and $s$ are odd positive integers, then $C$ is a LCD code if and only if it is a reversible $\mathbb{Z}_2$-double cyclic code.*

## 5. Examples

In the following tables, we list several examples of non-trivial reversible $\mathbb{Z}_2$-double cyclic codes of length $n = r + s \leq 10$ where $r,\ s \geq 2$.

| $[r,s]$ | $b(x)$ | $a(x)$ | $l(x)$ |
|---|---|---|---|
| $[2,2]$ | $1$ | $1,\ x+1,\ x^2+1$ | $0$ |
| | $x+1$ | $1,\ x+1$ | $0,\ 1$ |
| | $x+1$ | $x^2+1$ | $0$ |
| | $x^2+1$ | $1$ | $0,\ 1,\ x,\ x+1$ |
| | $x^2+1$ | $x+1$ | $0,\ x+1$ |
| $[3,2]$ | $1$ | $1,\ x+1,\ x^2+1$ | $0$ |
| | $x+1$ | $1,\ x+1$ | $0,\ 1$ |
| | $x+1$ | $x^2+1$ | $0$ |
| | $x^2+x+1$ | $1,\ x+1,\ x^2+1$ | $0$ |
| | $x^3+1$ | $1,\ x+1$ | $0,\ x^2+x+1$ |
| $[2,3]$ | $1$ | $1,\ x+1,\ x^2+x+1,\ x^3+1$ | $0$ |
| | $x+1$ | $1,\ x^2+x+1$ | $0,\ 1$ |
| | $x+1$ | $x+1,\ x^3+1$ | $0$ |
| | $x^2+1$ | $1,\ x^2+x+1$ | $0,\ x+1$ |
| | $x^2+1$ | $x+1$ | $0$ |
| $[2,4]$ | $1$ | $1,\ x+1,\ x^2+1,\ (x+1)^3,\ (x+1)^4$ | $0$ |
| | $x+1$ | $1,\ x+1,\ x^2+1,\ (x+1)^3$ | $0,\ 1$ |
| | $x+1$ | $(x+1)^4$ | $0$ |
| | $x^2+1$ | $1,\ x^2+1$ | $0,\ 1,\ x,\ x+1$ |
| | $x^2+1$ | $x+1,\ (x+1)^3$ | $0,\ x+1$ |

| | | | |
|---|---|---|---|
| $[4,2]$ | $1$ | $1,\ x+1,\ x^2+1$ | $0$ |
| | $x+1$ | $1,\ x+1$ | $0,\ 1$ |
| | $x+1$ | $x^2+1$ | $0$ |
| | $x^2+1$ | $1$ | $0,\ 1,\ x,\ x+1$ |
| | $x^2+1$ | $x+1$ | $0,\ x+1$ |
| | $x^2+1$ | $x^2+1$ | $0$ |
| | $(x+1)^3$ | $1,\ x+1$ | $0,\ x^2+1$ |
| | $(x+1)^3$ | $x^2+1$ | $0$ |
| | $x^4+1$ | $1$ | $0,\ x^2+1,\ x(x^2+1),\ (x+1)^3$ |
| | $x^4+1$ | $x+1$ | $0,\ (x+1)^3$ |
| $[3,3]$ | $1$ | $1,\ x+1, x^2+x+1, x^3+1$ | $0$ |
| | $x+1$ | $1,\ x^2+x+1$ | $0,\ 1$ |
| | $x+1$ | $x+1,\ x^3+1$ | $0$ |
| | $x^2+x+1$ | $1$ | $0,\ 1$ |
| | $x^2+x+1$ | $x+1$ | $0,\ x+1$ |
| | $x^2+x+1$ | $x^2+x+1,\ x^3+1$ | $0$ |
| | $x^3+1$ | $1$ | $0,\ 1,\ x^2+x,\ x^2+x+1$ |
| | $x^3+1$ | $x+1$ | $0,\ x+1$ |
| | $x^3+1$ | $x^2+x+1$ | $0,\ x^2+x+1$ |
| $[2,5]$ | $1$ | $1,\ x+1,\ x^4+x^3+x^2+x+1$ $x^5+1$ | $0$ |
| | $x+1$ | $1,\ x^4+x^3+x^2+x+1$ | $0,\ 1$ |
| | $x+1$ | $x+1,\ x^5+1$ | $0$ |
| | $x^2+1$ | $1,\ x^4+x^3+x^2+x+1$ | $0,\ x+1$ |
| | $x^2+1$ | $x+1$ | $0$ |
| $[5,2]$ | $1$ | $1,\ x+1,\ x^2+1$ | $0$ |
| | $x+1$ | $1,\ x+1$ | $0,\ 1$ |
| | $x+1$ | $x^2+1$ | $0$ |
| | $x^4+x^3+x^2+x+1$ | $1,\ x+1,\ x^2+1$ | $0$ |
| | $x^5+1$ | $1,\ x+1$ | $0,\ x^4+x^3+x^2+x+1$ |
| $[3,4]$ | $1$ | $1,\ x+1,\ x^2+1,\ (x+1)^3,\ (x+1)^4$ | $0$ |
| | $x+1$ | $1,\ x+1, x^2+1,\ (x+1)^3,$ | $0,\ 1$ |
| | $x+1$ | $(x+1)^4$ | $0$ |
| | $x^2+x+1$ | $1,\ x+1, x^2+1,\ (x+1)^3,\ (x+1)^4$ | $0$ |
| | $x^3+1$ | $1,\ x+1, x^2+1,\ (x+1)^3$ | $0,\ x^2+x+1$ |
| $[4,3]$ | $1$ | $1,\ x+1, x^2+x+1,\ x^3+1$ | $0$ |
| | $x+1$ | $1,\ x^2+x+1$ | $0,\ 1$ |
| | $x+1$ | $x+1,\ x^3+1$ | $0$ |
| | $x^2+1$ | $1,\ x^2+x+1$ | $0,\ x+1$ |
| | $x^2+1$ | $x+1,\ x^3+1$ | $0$ |
| | $(x+1)^3$ | $1,\ x^2+x+1$ | $0,\ x^2+1$ |
| | $(x+1)^3$ | $x+1,\ x^3+1$ | $0$ |
| | $(x+1)^4$ | $1,\ x^2+x+1$ | $0,\ (x+1)^3$ |
| | $(x+1)^4$ | $x+1$ | $0$ |
| $[2,6]$ | $1$ | $1,\ x+1,\ x^2+1,\ x^2+x+1,\ x^3+1,\ (x^2+x+1)^2,\ (x+1)(x^2+x+1)^2,\ (x+1)^2(x^2+x+1),\ x^6+1$ | $0$ |
| | $x+1$ | $1,\ x+1,\ x^2+x+1$ $(x^2+x+1)^2,\ x^3+1,\ (x+1)(x^2+x+1)^2$ | $0,\ 1$ |
| | $x+1$ | $x^2+1,\ (x+1)^2(x^2+x+1),\ x^6+1$ | $0$ |
| | $x^2+1$ | $1,\ x^2+x+1,\ (x^2+x+1)^2$ | $0,\ 1,\ x,\ x+1$ |
| | $x^2+1$ | $x+1,\ x^3+1,\ (x+1)(x^2+x+1)^2$ | $0,\ x+1$ |
| | $x^2+1$ | $x^2+1,\ (x+1)^2(x^2+x+1)$ | $0$ |
| $[3,5]$ | $1$ | $1,\ x+1,\ x^4+x^3+x^2+x+1,\ x^5+1$ | $0$ |
| | $x+1$ | $1,\ x^4+x^3+x^2+x+1$ | $0,\ 1$ |
| | $x+1$ | $x+1,\ x^5+1$ | $0$ |

| | | | |
|---|---|---|---|
| | $x^2+x+1$ | $1,\ x+1,\ x^4+x^3+x^2+x+1,\ x^5+1$ | $0$ |
| | $x^3+1$ | $1,\ x^4+x^3+x^2+x+1$ | $0,\ x^2+x+1$ |
| | $x^3+1$ | $x+1$ | $0$ |
| $[4,4]$ | $1$ | $1,\ x+1,\ x^2+1,\ (x+1)^3,\ (x+1)^4$ | $0$ |
| | $x+1$ | $1,\ x+1,\ x^2+1,\ (x+1)^3$ | $0,\ 1$ |
| | $x+1$ | $(x+1)^4$ | $0$ |
| | $x^2+1$ | $1,\ x^2+1$ | $0,\ 1,\ x,\ x+1$ |
| | $x^2+1$ | $x+1,\ (x+1)^3$ | $0,\ x+1$ |
| | $x^2+1$ | $(x+1)^4$ | $0$ |
| | $(x+1)^3$ | $1$ | $0,\ 1,\ x^2,\ x^2+1$ |
| | $(x+1)^3$ | $x+1$ | $0,\ x+1,\ x^2+1,\ x^2+x$ |
| | $(x+1)^3$ | $x^2+1,\ (x+1)^3$ | $0,\ x^2+1$ |
| | $(x+1)^3$ | $(x+1)^4$ | $0$ |
| | $(x+1)^4$ | $1$ | $0,\ 1,\ x^2,\ x^3+x,\ x^3+x+1,\ x^3+x^2+x,\ x^3+x^2+x+1\ x^2+1$ |
| | $(x+1)^4$ | $x+1$ | $0,\ x+1,\ x^3+x^2,\ x^3+x^2+x+1$ |
| | $(x+1)^4$ | $x^2+1$ | $0,\ x^2+1,\ x^3+x,\ x^3+x^2+x+1$ |
| | $(x+1)^4$ | $(x+1)^3$ | $0,\ (x+1)^3$ |
| $[5,3]$ | $1$ | $1,\ x+1,\ x^2+x+1,\ x^3+1$ | $0$ |
| | $x+1$ | $1,\ x^2+x+1$ | $0,\ 1$ |
| | $x+1$ | $x+1,\ x^3+1$ | $0$ |
| | $x^4+x^3+x^2+x+1$ | $1,\ x+1,\ x^2+x+1,\ x^3+1$ | $0$ |
| | $x^5+1$ | $1,\ x^2+x+1$ | $0,\ x^4+x^3+x^2+x+1$ |
| | $x^5+1$ | $x+1$ | $0$ |
| $[6,2]$ | $1$ | $1,\ x+1, x^2+1$ | $0$ |
| | $x+1$ | $1,\ x+1$ | $0,\ 1$ |
| | $x+1$ | $x^2+1$ | $0$ |
| | $x^2+1$ | $1$ | $0,\ 1,\ x,\ x+1$ |
| | $x^2+1$ | $x+1$ | $0,\ x+1$ |
| | $x^2+1$ | $x^2+1$ | $0$ |
| | $x^2+x+1$ | $1,\ x+1,\ x^2+1$ | $0$ |
| | $(x^2+x+1)^2$ | $1,\ x+1,\ x^2+1$ | $0$ |
| | $x^3+1$ | $1,\ x+1$ | $0,\ x^2+x+1$ |
| | $x^3+1$ | $x^2+1$ | $0$ |
| | $(x+1)^2(x^2+x+1)$ | $1$ | $0,\ x^2+x+1,\ x^3+x^2+x,\ x^3+1$ |
| | $(x+1)^2(x^2+x+1)$ | $x+1$ | $0,\ x^3+1$ |
| | $(x+1)^2(x^2+x+1)$ | $x^2+1$ | $0$ |
| | $(x+1)(x^2+x+1)^2$ | $1,\ x+1$ | $0,\ (x^2+x+1)^2$ |
| | $(x+1)(x^2+x+1)^2$ | $x^2+1$ | $0$ |
| | $x^6+1$ | $1$ | $0,\ (x^2+x+1)^2,\ x(x^2+x+1)^2,\ (x+1)(x^2+x+1)^2$ |
| | $x^6+1$ | $x+1$ | $0,\ (x+1)(x^2+x+1)^2$ |
| $[2,7]$ | $1$ | $1,\ x+1,\ x^7+1,\ x^6+x^5+x^4+x^3+x^2+x+1$ | $0$ |
| | $x+1$ | $1,\ x^6+x^5+x^4+x^3+x^2+x+1$ | $0,\ 1$ |
| | $x+1$ | $x+1,\ x^7+1$ | $0$ |
| | $x^2+1$ | $1,\ x^6+x^5+x^4+x^3+x^2+x+1$ | $0,\ x+1$ |
| | $x^2+1$ | $x+1$ | $0$ |
| $[3,6]$ | $1$ | $1,\ x+1,\ x^2+1,\ x^2+x+1,\ (x^2+x+1)^2,\ (x+1)^2(x^2+x+1),\ x^3+1,\ (x+1)(x^2+x+1)^2,\ x^6+1$ | $0$ |
| | $x+1$ | $1,\ x+1,\ x^2+x+1,\ x^3+1\ (x^2+x+1)^2, (x+1)(x^2+x+1)^2$ | $0,\ 1$ |
| | $x+1$ | $x^2+1,\ (x+1)^2(x^2+x+1),\ x^6+1$ | $0$ |

| | | | |
|---|---|---|---|
| | $x^2+x+1$ | $1,\ x^3+1$ | $0,\ 1$ |
| | $x^2+x+1$ | $x+1,\ (x+1)^2(x^2+x+1)$ | $0,\ x+1$ |
| | $x^2+x+1$ | $x^2+1,\ x^2+x+1$ | $0,\ x$ |
| | $x^2+x+1$ | $(x^2+x+1)^2,\ (x+1)(x^2+x+1)^2$ $x^6+1$ | $0$ |
| | $x^3+1$ | $1,\ x^3+1$ | $0,\ 1,\ x^2+x,\ x^2+x+1$ |
| | $x^3+1$ | $x+1$ | $0,\ x+1,\ x^2,\ x^2+x+1$ |
| | $x^3+1$ | $x^2+1$ | $0,\ x^2+1$ |
| | $x^3+1$ | $x^2+x+1$ | $0,\ x,\ x^2+1,\ x^2+x+1$ |
| | $x^3+1$ | $(x^2+x+1)^2$ | $0,\ x^2+x+1$ |
| | $x^3+1$ | $(x+1)^2(x^2+x+1)$ | $0,\ x+1$ |
| | $x^3+1$ | $(x+1)(x^2+x+1)^2$ | $0,\ x^2+x+1$ |
| $[4,5]$ | $1$ | $1,\ x+1,\ x^5+1$ $x^4+x^3+x^2+x+1$ | $0$ |
| | $x+1$ | $1,\ x^4+x^3+x^2+x+1$ | $0,\ 1$ |
| | $x+1$ | $x+1,\ x^5+1$ | $0$ |
| | $x^2+1$ | $1,\ x^4+x^3+x^2+x+1$ | $0,\ x+1$ |
| | $x^2+1$ | $x+1,\ x^5+1$ | $0$ |
| | $(x+1)^3$ | $1,\ x^4+x^3+x^2+x+1$ | $0,\ x^2+1$ |
| | $(x+1)^3$ | $x+1,\ x^5+1$ | $0$ |
| | $(x+1)^4$ | $1,\ x^4+x^3+x^2+x+1$ | $0,\ (x+1)^3$ |
| | $(x+1)^4$ | $x+1$ | $0$ |
| $[5,4]$ | $1$ | $1,\ x+1,\ x^2+1$ $(x+1)^3,\ (x+1)^4$ | $0$ |
| | $x+1$ | $1,\ x+1,\ x^2+1,\ (x+1)^3$ | $0,\ 1$ |
| | $x+1$ | $(x+1)^4$ | $0$ |
| | $x^4+x^3+x^2+x+1$ | $1,\ x+1,\ x^2+1$ $(x+1)^3,\ (x+1)^4$ | $0$ |
| | $x^5+1$ | $1,\ x+1,\ x^2+1,\ (x+1)^3$ | $0,\ x^4+x^3+x^2+x+1$ |
| $[6,3]$ | $1$ | $1,\ x+1,\ x^2+x+1,\ x^3+1$ | $0$ |
| | $x+1$ | $1,\ x^2+x+1$ | $0,\ 1$ |
| | $x+1$ | $x+1,\ x^3+1$ | $0$ |
| | $x^2+1$ | $1,\ x^2+x+1$ | $0,\ x+1$ |
| | $x^2+1$ | $x+1,\ x^3+1$ | $0$ |
| | $x^2+x+1$ | $1$ | $0,\ 1$ |
| | $x^2+x+1$ | $x+1$ | $0,\ x+1$ |
| | $x^2+x+1$ | $x^2+x+1,\ x^3+1$ | $0$ |
| | $(x^2+x+1)^2$ | $1$ | $0,\ x^3+1$ |
| | $(x^2+x+1)^2$ | $x+1$ | $0,\ x(x^2+x+1),$ |
| | $(x^2+x+1)^2$ | $x^2+x+1,\ x^3+1$ | $0$ |
| | $x^3+1$ | $1$ | $0,\ 1,\ x^2+x,\ x^2+x+1$ |
| | $x^3+1$ | $x+1$ | $0,\ x+1$ |
| | $x^3+1$ | $x^2+x+1$ | $0,\ x^2+x+1$ |
| | $x^3+1$ | $x^3+1$ | $0$ |
| | $(x+1)^2(x^2+x+1)$ | $1$ | $0,\ x^2+x,\ (x+1)^3,$ $x^3+1$ |
| | $(x+1)^2(x^2+x+1)$ | $x+1$ | $0,\ x^3+x$ |
| | $(x+1)^2(x^2+x+1)$ | $x^2+x+1$ | $0,\ x^3+1$ |
| | $(x+1)^2(x^2+x+1)$ | $x^3+1$ | $0$ |
| | $(x+1)(x^2+x+1)^2$ | $1$ | $0,\ x^3+1,\ x^2(x^2+x+1),$ $(x^2+x+1)^2$ |
| | $(x+1)(x^2+x+1)^2$ | $x+1$ | $0,\ (x+1)^2(x^2+x+1)$ |
| | $(x+1)(x^2+x+1)^2$ | $x^2+x+1$ | $0,\ (x^2+x+1)^2$ |
| | $(x+1)(x^2+x+1)^2$ | $x^3+1$ | $0$ |
| | $x^6+1$ | $1$ | $0,\ x^3+1,$ $x(x+1)^2(x^2+x+1),$ $(x+1)(x^2+x+1)^2$ |
| | $x^6+1$ | $x+1$ | $0,\ (x+1)^2(x^2+x+1)$ |
| | $x^6+1$ | $x^2+x+1$ | $0,\ (x+1)(x^2+x+1)^2$ |
| $[7,2]$ | $1$ | $1,\ x+1,\ x^2+1$ | $0$ |
| | $x+1$ | $1,\ x+1$ | $0,\ 1$ |
| | $x+1$ | $x^2+1$ | $0$ |
| | $(x^3+x+1)(x^3+x^2+1)$ | $1,\ x+1,\ x^2+1$ | $0$ |
| | $x^7+1$ | $1,\ x+1$ | $0,\ (x^3+x+1)(x^3+x^2+1)$ |
| $[2,8]$ | $1$ | $(x+1)^i,\ 0\le i\le 8$ | $0$ |

| | | | |
|---|---|---|---|
| | $x+1$ | $(x+1)^i,\ 0 \le i \le 7$ | $0,\ 1$ |
| | $x+1$ | $(x+1)^8$ | $0$ |
| | $x^2+1$ | $1,\ x^2+1,\ (x+1)^4,\ (x+1)^6$ | $0,\ 1,\ x,\ x+1$ |
| | $x^2+1$ | $x+1,\ (x+1)^3,\ (x+1)^5, (x+1)^7$ | $0,\ x+1$ |
| $[3,7]$ | $1$ | $1,\ x+1,\ x^7+1$ $(x^3+x+1)(x^3+x^2+1)$ | $0$ |
| | $x+1$ | $1,\ (x^3+x+1)(x^3+x^2+1)$ | $0,\ 1$ |
| | $x+1$ | $x+1,\ x^7+1$ | $0$ |
| | $x^2+x+1$ | $1,\ x+1,\ x^7+1$ $(x^3+x+1)(x^3+x^2+1)$ | $0$ |
| | $x^3+1$ | $1,\ (x^3+x+1)(x^3+x^2+1)$ | $0,\ x^2+x+1$ |
| | $x^3+1$ | $x+1$ | $0$ |
| $[4,6]$ | $1$ | $1,\ x+1,\ x^2+x+1,\ x^2+1$ $(x^2+x+1)^2,\ (x+1)(x^2+x+1)^2$ $(x+1)^2(x^2+x+1),\ x^3+1,\ x^6+1$ | $0$ |
| | $x+1$ | $1,\ x+1,\ x^2+x+1$ $(x^2+x+1)^2,\ x^3+1,$ $(x+1)(x^2+x+1)^2$ | $0,\ 1$ |
| | $x+1$ | $x^2+1,\ (x+1)^2(x^2+x+1)$ $x^6+1$ | $0$ |
| | $x^2+1$ | $1,\ x^2+x+1,\ (x^2+x+1)^2$ | $0,\ 1,\ x,\ x+1$ |
| | $x^2+1$ | $x+1,\ x^3+1,\ (x+1)(x^2+x+1)^2$ | $0,\ x+1$ |
| | $x^2+1$ | $x^2+1,\ (x+1)^2(x^2+x+1),\ x^6+1$ | $0$ |
| | $(x+1)^3$ | $1,\ x+1,\ (x+1)^2(x^2+x+1),$ $(x^2+x+1)^2,\ x^3+1,$ $x^2+x+1,\ (x+1)(x^2+x+1)^2$ | $0,\ x^2+1$ |
| | $(x+1)^3$ | $x^2+1,\ x^6+1$ | $0$ |
| | $(x+1)^4$ | $1,\ x^2+x+1,\ (x^2+x+1)^2$ | $0,\ x^2+1,\ x(x^2+1), (x+1)^3$ |
| | $(x+1)^4$ | $x+1,\ x^3+1,$ $(x+1)(x^2+x+1)^2$ | $0,\ (x+1)^3$ |
| | $(x+1)^4$ | $x^2+1,\ (x+1)^2(x^2+x+1)$ | $0$ |
| $[5,5]$ | $1$ | $1,\ x+1,\ x^5+1$ $x^4+x^3+x^2+x+1$ | $0$ |
| | $x+1$ | $1,\ x^4+x^3+x^2+x+1$ | $0,\ 1$ |
| | $x+1$ | $x+1,\ x^5+1$ | $0$ |
| | $x^4+x^3+x^2+x+1$ | $1$ | $0,\ 1,\ x^3+x^2,\ x^3+x^2+1$ |
| | $x^4+x^3+x^2+x+1$ | $x+1$ | $0,\ x+1,\ x^3,\ x^3+x+1$ |
| | $x^4+x^3+x^2+x+1$ | $x^4+x^3+x^2+x+1,\ x^5+1$ | $0$ |
| | $x^5+1$ | $1$ | $0,\ 1,\ x^3+x^2,\ x^3+x^2+1,$ $x^4+x,\ x^4+x+1,$ $x^4+x^3+x^2+x$ $x^4+x^3+x^2+x+1$ |
| | $x^5+1$ | $x+1$ | $0,\ x+1,\ x^2(x+1)^2,$ $x^4+x^2+x+1$ |
| | $x^5+1$ | $x^4+x^3+x^2+x+1$ | $0,\ x^4+x^3+x^2+x+1$ |
| $[6,4]$ | $1$ | $(x+1)^i,\ 0 \le i \le 4$ | $0$ |
| | $x+1$ | $(x+1)^i,\ 0 \le i \le 3$ | $0,\ 1$ |
| | $x+1$ | $(x+1)^4$ | $0$ |
| | $x^2+1$ | $1,\ x^2+1$ | $0,\ 1,\ x,\ x+1$ |
| | $x^2+1$ | $x+1,\ (x+1)^3$ | $0,\ x+1$ |
| | $x^2+1$ | $(x+1)^4$ | $0$ |
| | $x^2+x+1,\ (x^2+x+1)^2$ | $(x+1)^i,\ 0 \le i \le 4$ | $0$ |
| | $x^3+1$ | $(x+1)^i,\ 0 \le i \le 3$ | $0,\ x^2+x+1$ |
| | $x^3+1$ | $(x+1)^4$ | $0$ |
| | $(x+1)^2(x^2+x+1)$ | $1,\ x^2+1$ | $0,\ x^2+x+1,\ x(x^2+x+1),$ $x^3+1$ |
| | $(x+1)^2(x^2+x+1)$ | $x+1,\ (x+1)^3$ | $0,\ x^3+1$ |
| | $(x+1)^2(x^2+x+1)$ | $(x+1)^4$ | $0$ |

| | | | |
|---|---|---|---|
| | $(x+1)(x^2+x+1)^2$ | $(x+1)^i,\ 0 \leq i \leq 3$ | $0,\ (x^2+x+1)^2$ |
| | $(x+1)(x^2+x+1)^2$ | $(x+1)^4$ | $0$ |
| | $x^6+1$ | $1,\ x^2+1$ | $0,\ (x^2+x+1)^2,\ x(x^2+x+1)^2,$ $(x+1)(x^2+x+1)^2$ |
| | $x^6+1$ | $x+1,\ (x+1)^3$ | $0,\ (x+1)(x^2+x+1)^2$ |
| $[7,3]$ | $1$ | $1,\ x+1,\ x^2+x+1,\ x^3+1$ | $0$ |
| | $x+1$ | $1,\ x^2+x+1$ | $0,\ 1$ |
| | $x+1$ | $x+1,\ x^3+1$ | $0$ |
| | $(x^3+x+1)(x^3+x^2+1)$ | $1,\ x+1,\ x^2+x+1,\ x^3+1$ | $0$ |
| | $x^7+1$ | $1,\ x^2+x+1$ | $0$ $x^6+x^5+x^4+x^3+x^2+x+1$ |
| | $x^7+1$ | $x+1$ | $0$ |
| $[8,2]$ | $1$ | $1,\ x+1,\ x^2+1$ | $0$ |
| | $x+1$ | $1,\ x+1$ | $0,\ 1$ |
| | $x+1$ | $x^2+1$ | $0$ |
| | $x^2+1$ | $1$ | $0,\ 1,\ x,\ x+1$ |
| | $x^2+1$ | $x+1$ | $0,\ x+1$ |
| | $x^2+1$ | $x^2+1$ | $0$ |
| | $(x+1)^3$ | $1,\ x+1$ | $0,\ x^2+1$ |
| | $(x+1)^3$ | $x^2+1$ | $0$ |
| | $(x+1)^4$ | $1$ | $0,\ x^2+1,\ x(x^2+1),\ (x+1)^3$ |
| | $(x+1)^4$ | $x+1$ | $0,\ (x+1)^3$ |
| | $(x+1)^4$ | $x^2+1$ | $0$ |
| | $(x+1)^5$ | $1,\ x+1$ | $0,\ (x+1)^4$ |
| | $(x+1)^5$ | $x^2+1$ | $0$ |
| | $(x+1)^6$ | $1$ | $0,\ (x+1)^4,\ x(x+1)^4,\ (x+1)^5$ |
| | $(x+1)^6$ | $x+1$ | $0,\ (x+1)^5$ |
| | $(x+1)^6$ | $x^2+1$ | $0$ |
| | $(x+1)^7$ | $1,\ x+1$ | $0,\ (x+1)^6$ |
| | $(x+1)^7$ | $x^2+1$ | $0$ |
| | $(x+1)^8$ | $1$ | $0,\ (x+1)^6,\ x(x+1)^6,\ (x+1)^7$ |
| | $(x+1)^8$ | $x+1$ | $0,\ (x+1)^7$ |

## 6. Conclusion

In this note, we have given necessary and sufficient conditions for a $\mathbb{Z}_2$-double cyclic code to be reversible. We have also shown a relation between LCD $\mathbb{Z}_2$-double cyclic codes and reversible $\mathbb{Z}_2$-double cyclic codes, in case the code is separable. For non-separable codes, we have given a few examples demonstrating that the relation between LCD and reversible codes, which hold for cyclic codes, doesn't hold for $\mathbb{Z}_2$-double cyclic codes. We also listed several examples of reversible $\mathbb{Z}_2$-double cyclic codes.

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