ON MEROMORPHIC SOLUTIONS OF NONLINEAR
PARTIAL DIFFERENTIAL-DIFFERENCE EQUATIONS OF
FIRST ORDER IN SEVERAL COMPLEX VARIABLES

QIBIN CHENG, YEZHOU LI, AND ZHIXUE LIU

Abstract. This paper is concerned with the value distribution for meromorphic solutions \( f \) of a class of nonlinear partial differential-difference equation of first order with small coefficients. We show that such solutions \( f \) are uniquely determined by the poles of \( f \) and the zeros of \( f - c, f - d \) (counting multiplicities) for two distinct small functions \( c, d \).

1. Introduction and the main theorem

For a point \( z_0 \in \mathbb{C}^m \), \( f \) is holomorphic at \( z_0 \) if it can be written as \( f(z) = \sum_{i=0}^{\infty} Q_i(z - z_0) \) on a neighborhood \( U \subset \mathbb{C}^m \) of \( z_0 \), where the term \( Q_i(z - z_0) \) is either identically zero or a homogeneous polynomial of degree \( i \). We say \( f \) is a nonzero “meromorphic” function on \( \mathbb{C}^m \) if for all \( z_0 \in \mathbb{C}^m \), one can choose non-zero holomorphic functions \( f_1 \) and \( f_2 \) on a neighborhood \( U \) of \( z_0 \) such that \( f = f_1 f_2 \) on \( U \) and \( \dim \{ z \in \mathbb{C}^m \mid f_1(z) = f_2(z) = 0 \} \leq m - 2 \).

We assume that the readers are familiar with standard notations of Nevanlinna value distribution theory (see, e.g., [7,12,21,22,29]), such as the proximity function \( m(r, f) \), the (integrated) counting function \( N(r, f) \), and the Nevanlinna characteristic function \( T(r, f) \). The order and hyper-order of \( f \) are defined respectively by

\[
\rho(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}, \quad \rho_2(f) = \limsup_{r \to \infty} \frac{\log \log T(r, f)}{\log r}.
\]

A meromorphic function \( \alpha(z) \) is called a small function with respect to \( f \) if \( T(r, \alpha) = o(T(r, f)) \) holds for all \( r \) possibly outside of a set \( E \) with finite logarithmic measure, i.e., \( \text{lm}(E) = \int_E \frac{dt}{t} < \infty \). Denote by \( S(f) \) the family of all small functions with respect to \( f \), and write \( \tilde{S}(f) = S(f) \cup \{ \infty \} \). For

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\( \alpha \in \hat{S}(f) \cap \hat{S}(g), \) we say that two meromorphic functions \( f \) and \( g \) share \( \alpha \) \( \text{CM} \) (IM), provided that \( f - \alpha \) and \( g - \alpha \) have the same zeros counting multiplicities (ignoring multiplicities).

In 1926, Nevanlinna [20] proved the celebrated five-value theorem that for two non-constant meromorphic functions \( f \) and \( g \) on the complex plane \( \mathbb{C} \), if they have the same preimages (ignoring multiplicities) for five distinct values in \( \mathbb{P}^1(\mathbb{C}) \), then \( f = g \). Afterwards, Nevanlinna five-value theorem was also considered into the case of small functions [18,31]. Various uniqueness theories of meromorphic or entire functions in different research directions had been established by many scholars (see, e.g., [6, 12, 18, 19, 29]). For instance, in 1983, Mues and Steinmetz [19] and Gundersen [6] proved that a nonconstant meromorphic function \( f \) and its first derivative \( f' \) are identical if they share two finite values CM.

The study of uniqueness problems of meromorphic functions plays a significant role in value distribution theory of complex analysis. The investigation of uniqueness problems has also been extended to the case when \( f \) is a meromorphic solution of differential equations (see, e.g., [3, 8, 10, 11, 14, 15]). A typical example is that Brosch [3] pointed out that a meromorphic solution to the complex differential equation of Malmquist-Yosida type \((w')^n = \sum_{k=0}^{2n} a_k w^k\) is uniquely determined by three distinct values counting multiplicities, where the coefficients \( a_k \) \((k = 1, 2, \ldots, 2n)\) are small with respect to the solution.

In recent years, many scholars have shown great interest in the uniqueness problems in the case of higher dimension (see, e.g., [9, 13, 16, 24, 25]). In 2011, Hu and Li [9] focused their attentions on a special class of nonlinear partial differential equation of first order in \( m \) \((\geq 1)\) independent complex variables, that is,

\[
(1) \quad \sum_{0 < |i| < n} a_i (\partial u)^i = \sum_{j=0}^n b_j u^j,
\]

where \( i = (i_1, i_2, \ldots, i_m) \in \mathbb{Z}^m \) denotes an index of \( m \) dimension with \( |i| = i_1 + i_2 + \cdots + i_m \), \( a_i = a_i(z) \) and \( b_j = b_j(z) \) \((j = 0, 1, \ldots, n)\) are meromorphic functions in \( z = (z_1, z_2, \ldots, z_m) \in \mathbb{C}^m \),

\[
(\partial u)^i = (u_{z_1})^{i_1} (u_{z_2})^{i_2} \cdots (u_{z_m})^{i_m}, \quad u_{z_k} = \frac{\partial u}{\partial z_k} \quad (k = 1, 2, \ldots, m).
\]

Hu and Li [9] showed that a meromorphic solution \( f \) to partial differential equation (1) could be uniquely determined by the poles of \( f \) and the zeros of \( f - c_l \), where \( c_l \) \((l = 1, 2)\) are two distinct finite complex numbers.

In the 1970s and 1980s, Bank and Kaufman [1], Shimomura [23], Yanagihara [27, 28] and other scholars had acquired some results on the existence of meromorphic solutions to several classes of difference equations. Compared with a few decades ago, complex difference equations, the discrete counterparts of differential equations, has gained much more attention today. Xu and Cao [24] studied entire and meromorphic solutions of a Fermat-type partial
differential-difference equation
\begin{align}
(2) \quad \left( \frac{\partial u(z_1, z_2)}{\partial z_1} \right)^p + u^q(z_1 + \zeta_1, z_2 + \zeta_2) &= 1
\end{align}

in \( \mathbb{C}^2 \), and Xu and Wang [25] investigated the existence of transcendental entire solutions of finite order for a kind of Fermat-type partial differential-difference equation
\begin{align}
(3) \quad \left( \frac{\partial u(z_1, z_2)}{\partial z_1} + \frac{\partial u(z_1, z_2)}{\partial z_2} \right)^p + u^q(z_1 + \zeta_1, z_2 + \zeta_2) &= 1
\end{align}
in \( \mathbb{C}^2 \).

Based on the above results, we will make further study on meromorphic solutions of a class of nonlinear partial differential-difference equation of first order in \( \mathbb{C}^m \). The equation is of the form
\begin{align}
(4) \quad \sum_{0 < |i| < n} a_i \prod_{k=1}^m \left[ u_{zk}(z + \xi_k) \right]^{i_k} = \sum_{j=0}^n b_j u^j(z + \eta_j),
\end{align}
where \( i = (i_1, i_2, \ldots, i_m) \in \mathbb{Z}^m \) denotes an index of \( m \) dimension with \( |i| = i_1 + i_2 + \cdots + i_m \), \( a_i = a_i(z) \) and \( b_j = b_j(z) \) \( (j = 0, 1, \ldots, n) \) are meromorphic functions in \( z = (z_1, z_2, \ldots, z_m) \in \mathbb{C}^m \), \( u_{zk}(z + \xi_k) = \frac{\partial}{\partial z_k}(z + \xi_k) \) \( (k = 1, 2, \ldots, m) \), and \( \xi_k, \eta_j \) \( (k = 1, 2, \ldots, m; j = 0, 1, \ldots, n) \) are \( m \)-dimensional vectors in \( \mathbb{C}^m \).

As we can see, (1), (2) and (3) can be regarded as some specific cases of (4). The left and the right of (4) are shifts of the corresponding sides of (1), respectively. This paper is concerned with the value distribution for meromorphic solutions \( f \) of (4) with small functions coefficients, which shows that \( f \) could be uniquely determined by its poles and the zeros of \( f - c \), \( f - d \) with \( c, d \in S(f) \) (Notice that \( c \), \( d \) are not limited to finite complex constants). We prove the following result:

**Theorem 1.1.** Suppose that \( f \) is a nonconstant meromorphic solution of (4) in \( \mathbb{C}^m \) with \( b_n \neq 0 \) and \( a_i, b_j \in S(f) \) \( (0 < |i| < n, 0 \leq j \leq n) \) such that
\begin{align}
(5) \quad \lim_{r \to \infty} \sup_{r \notin E} \frac{\log T(r, f)}{r} = 0,
\end{align}
holds for all \( r \notin E \), where \( E \) is a set with zero upper density measure, i.e.,
\begin{align}
\text{dens} E = \lim_{r \to \infty} \frac{1}{r} \int_{E \cap [1, r]} dt = 0.
\end{align}

Let \( c(z), d(z) \) be distinct small functions with respect to \( f \) satisfying \( H(z, c) = \sum_{j=0}^n b_j(z) c^j(z + \eta_j) \neq 0 \) and \( H(z, d) = \sum_{j=0}^n b_j(z) d^j(z + \eta_j) \neq 0 \). If \( f \) and a meromorphic function \( g \) in \( \mathbb{C}^m \) share \( c(z), d(z) \) and \( \infty \) CM, then \( f = g \).

Noting that the set of meromorphic functions satisfying (5) consists of all meromorphic functions with some being of hyper-order \( \rho_2(f) < 1 \) and the others being of hyper-order \( \rho_2(f) = 1 \). In fact, let \( f(z) \) be a meromorphic
function such that \( T(r, f) = \exp\{r \cdot (\log r)^{-\lambda}\} \) (\( \lambda > 0 \)), we know easily that 
\[
\limsup_{r \to \infty} \frac{\log T(r, f)}{r} = 0 \quad \text{and} \quad p_2(f) = 1.
\]

The following counterexamples indicate the strictness of the conditions in Theorem 1.1 in some aspects.

**Example 1.2.** The condition \(|i| < n\) in (4) cannot be improved. Let \( m = 3, n = 1 \), we consider the following partial differential-difference equation:
\[
\frac{\partial u}{\partial z_1}(z_1 + \pi i, z_2 + \pi i, z_3) + \frac{\partial u}{\partial z_2}(z_1 + 2\pi i, z_2 + \pi i, z_3 + \pi i) = 2u \left( z_1 + \frac{\pi i}{2}, z_2 + \frac{3\pi i}{2}, z_3 \right),
\]
where \( i \) is the imaginary unit. Then \( f(z_1, z_2, z_3) = e^{z_1+z_2+z_3} \) is an entire solution of the above equation. We can easily check that \( f(z_1, z_2, z_3) \) and \( g(z_1, z_2, z_3) = e^{-(z_1+z_2+z_3)} \) share 1, \(-1, \infty\) CM with \( H(z, 1) \neq 0, H(z, -1) \neq 0 \). However, \( f \neq g \).

**Example 1.3.** The conditions \( H(z, c) \neq 0 \) and \( H(z, d) \neq 0 \) in Theorem 1.1 cannot be dropped. For instance, \( f(z_1, z_2, z_3) = \tan(\alpha z_1 + \beta z_2 + \gamma z_3) \) is a meromorphic solution of the partial differential-difference equation
\[
\frac{\partial u}{\partial z_1} \left( z_1 + \frac{\pi}{4\alpha}, z_2 + \frac{\pi}{4\beta}, z_3 + \frac{\pi}{2\gamma} \right) + \frac{\partial u}{\partial z_2} \left( z_1 + \frac{\pi}{6\alpha}, z_2 + \frac{\pi}{2\beta}, z_3 + \frac{\pi}{3\gamma} \right) + \frac{\partial u}{\partial z_3} \left( z_1 + \frac{\pi}{8\alpha}, z_2 + \frac{3\pi}{4\beta}, z_3 + \frac{\pi}{8\gamma} \right)
= (\alpha + \beta + \gamma) \left[ 1 + u^2 \left( z_1 + \frac{2\pi}{3\alpha}, z_2 + \frac{\pi}{6\beta}, z_3 + \frac{\pi}{6\beta} \right) \right].
\]
Let \( i \) be the imaginary unit, we know that \( i, -i \) are two exceptional values of \( f \), which implies \( f(z_1, z_2, z_3) \) and \( g(z_1, z_2, z_3) = -\tan(\alpha z_1 + \beta z_2 + \gamma z_3) \) share \( i, -i, \infty \) CM. However, \( f \neq g \). The reason is that \( H(z, i) \equiv 0, H(z, -i) \equiv 0 \).

**Example 1.4.** The number of shared small functions cannot be reduced. Considering the following equation:
\[
\frac{\partial u}{\partial z_1} \left( z_1 + \frac{\pi i}{2}, z_2 - \frac{\pi i}{2}, z_3 \right) + \frac{\partial u}{\partial z_2} (z_1, z_2, z_3 + 2\pi i)
= -u^2 (z_1 + \pi i, z_2 - \pi i, z_3 + 4\pi i),
\]
where \( i \) is the imaginary unit. We have that \( f(z_1, z_2, z_3) = \frac{2}{z_1 + z_2 + e^{z_3}} \) is a meromorphic solution to the equation. Further, we can verify that \( f(z_1, z_2, z_3) \) and \( g(z_1, z_2, z_3) = z_1 + z_2 + e^{z_3} \) share \( \sqrt{2} \) and \( -\sqrt{2} \) CM with \( H(z, \sqrt{2}) \neq 0 \) and \( H(z, -\sqrt{2}) \neq 0 \). Obviously, \( f \neq g \).
2. Preliminary lemmas

Here, we give some auxiliary lemmas which are of great importance to the proof of our main result.

**Lemma 2.1** (The First Main Theorem [17]). Let \( f \) be a meromorphic function on \( \mathbb{C}^m \), and \( a \in \hat{S}(f) \). Then, we have

\[
T\left(r, \frac{1}{f-a}\right) = T(r, f) + o(T(r, f))
\]

for all \( r \notin E \) with \( \ln(E) < \infty \).

**Lemma 2.2** (The Second Main Theorem [4, 5, 26]). Let \( f \) be a meromorphic function on \( \mathbb{C}^m \), and \( a_1, a_2, \ldots, a_q \in \hat{S}(f) \) are distinct (\( q \geq 3 \)). Then, we have

\[
(q-2)T(r, f) \leq \sum_{j=1}^{q} N\left(r, \frac{1}{f-a_j}\right) + o(T(r, f))
\]

for all \( r \notin E \) with \( \ln(E) < \infty \).

**Lemma 2.3** (The Logarithmic Derivative Lemma [12, 30]). Suppose that \( f \) is a nonconstant meromorphic function on \( \mathbb{C}^m \). Then

\[
m\left(r, \frac{f_{z_k}}{f}\right) = o(T(r, f))
\]

for all \( r \notin E \) with \( \ln(E) < \infty \) and for any \( k \in \{1, 2, \ldots, m\} \), where \( f_{z_k} = \frac{\partial f}{\partial z_k} \).

**Lemma 2.4** (The Logarithmic Difference Lemma [4, 32]). Let \( f \) be a nonconstant meromorphic function on \( \mathbb{C}^m \), and let \( c \in \mathbb{C}^m \setminus \{0\} \). If

\[
\limsup_{r \to \infty} \frac{\log T(r, f)}{r} = 0,
\]

then

\[
m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = o(T(r, f))
\]

for all \( r \notin E \) with \( \overline{\text{dens}}E = 0 \).

**Lemma 2.5** ([4]). Let \( f \) be a nonconstant meromorphic function on \( \mathbb{C}^m \). If

\[
\limsup_{r \to \infty} \frac{\log T(r, f)}{r} = 0,
\]

then

\[
T(r, f(z+c)) = T(r, f) + o(T(r, f)),
\]

\[
N(r, f(z+c)) = N(r, f) + o(T(r, f))
\]

for any \( c \in \mathbb{C}^m \setminus \{0\} \) and all \( r \notin E \) with \( \overline{\text{dens}}E = 0 \).
Remark 2.6. Both Lemma 2.4 and Lemma 2.5 imply

\[ m(r, f(z + c)) = m(r, f) + o(T(r, f)) \]

for any \( c \in \mathbb{C}^m \setminus \{0\} \) and all \( r \notin E \) with \( \overline{\text{dens}} E = 0 \), provided that \( f \) is a nonconstant meromorphic function on \( \mathbb{C}^m \) satisfying \( \limsup_{r \to \infty} \frac{\log T(r, f)}{r} = 0 \).

Lemma 2.7 ([2,12]). Let \( f_j \) (\( j = 1, 2, \ldots, n \)) \( (n \geq 2) \) be meromorphic functions and \( g_j \) (\( j = 1, 2, \ldots, n \)) be entire functions in \( \mathbb{C}^m \) satisfying

(i) \( \sum_{j=1}^{n} f_j e^{g_j} = 0 \);
(ii) \( g_j - g_k \) are not constants for \( 1 \leq j < k \leq n \);
(iii) For \( 1 \leq j \leq n, 1 \leq h < k \leq n, T(r, f_j) = o(T(r, e^{g_h - g_k})) \) for all \( r \notin E \) with \( \text{lm}(E) < \infty \).

Then, \( f_j = 0 \) (\( j = 1, 2, \ldots, n \)).

A slight modification to the proof of [9, Lemma 2.2] yields the following lemma, which lays a foundation for the proof of Theorem 1.1.

Lemma 2.8. Let \( f \) be a nonconstant meromorphic solution of (4) with \( b_i \neq 0 \) on \( \mathbb{C}^m \) such that

\begin{equation}
\sum_{0 < |l| < n} T(r, a_l) + \sum_{j=0}^{n} T(r, b_j) = o(T(r, f))
\end{equation}

and

\[ \limsup_{r \to \infty} \frac{\log T(r, f)}{r} = 0 \]

hold for all \( r \notin E = E_1 \cup E_2 \) with \( \text{lm}(E_1) < \infty \) and \( \overline{\text{dens}} E_2 = 0 \). Then we have

\begin{equation}
m(r, f) = o(T(r, f))
\end{equation}

holds for all \( r \notin E = E_1 \cup E_2 \) with \( \text{lm}(E_1) < \infty \) and \( \overline{\text{dens}} E_2 = 0 \). Furthermore, if \( d \in S(f) \) satisfies

\[ H(z, d) = \sum_{j=0}^{n} b_j(z)d^j(z + \eta_j) \neq 0, \]

then

\begin{equation}
m \left( r, \frac{1}{f - d} \right) = o(T(r, f))
\end{equation}

holds for all \( r \notin E = E_1 \cup E_2 \) with \( \text{lm}(E_1) < \infty \) and \( \overline{\text{dens}} E_2 = 0 \).

Proof. From the assumption that \( f \) is a nonconstant meromorphic solution of (4), we get

\[ \sum_{0 < |l| < n} a_l[f_{z_1}(z + \xi_1)]^{l_1}[f_{z_2}(z + \xi_2)]^{l_2} \cdots [f_{z_m}(z + \xi_m)]^{l_m} = \sum_{j=0}^{n} b_j f^j(z + \eta_j). \]
Thus,

\[ f(z + \eta_n) = \frac{1}{b_n} \left( \sum_{0 < |\xi| < n} a_1 \left[ f_{z_1}(z + \xi_1) \right]^{i_1} \cdots \left[ f_{z_m}(z + \xi_m) \right]^{i_m} \frac{f^{n-1}(z + \eta_n)}{f^{n-1}(z + \eta_n)} \right. \]

\[ \left. - \sum_{j=0}^{n-1} b_j \frac{f^j(z + \eta_j)}{f^{n-1}(z + \eta_n)} \right). \]

(9)

As we know, \( m(r, f(z + \eta_n)) = 0 \) when \( |f(z + \eta_n)| < 1 \). So it suffices to consider the term \( m(r, f(z + \eta_n)) \) in the case of \( |f(z + \eta_n)| \geq 1 \). For \( 0 \leq k \leq n - 1 \), \( \frac{1}{|f(z + \eta_n)|} \leq 1 \). Hence, by (9) we have the following estimation:

\[ |f(z + \eta_n)| \leq \frac{1}{b_n} \left( \sum_{0 < |\xi| < n} |a_1| \left| f_{z_1}(z + \xi_1) \right|^{i_1} \cdots \left| f_{z_m}(z + \xi_m) \right|^{i_m} \frac{f^{n-1}(z + \eta_n)}{f^{n-1}(z + \eta_n)} \right. \]

\[ \left. + \sum_{j=0}^{n-1} |b_j| \left| \frac{f(z + \eta_j)}{f(z + \eta_n)} \right|^j \right) \]

when \( |f(z + \eta_n)| \geq 1 \), which implies that

\[ m(r, f) = m(r, f(z + \eta_n)) + o(T(r, f)) \]

\[ \leq \sum_{0 < |\xi| < n} \left\{ m(r, a_1) + i_1 m \left( r, \frac{f_{z_1}(z + \xi_1)}{f(z + \eta_n)} \right) \right. \]

\[ \left. + \cdots + i_m m \left( r, \frac{f_{z_m}(z + \xi_m)}{f(z + \eta_n)} \right) \right\} + \sum_{j=0}^{n-1} m(r, b_j) \]

\[ + \sum_{j=0}^{n-1} j m \left( r, \frac{f(z + \eta_j)}{f(z + \eta_n)} \right) + m \left( r, \frac{1}{b_n} \right) + o(T(r, f)) \]

holds for all \( r \notin E = E_1 \cup E_2 \) with \( \text{lm}(E_1) < \infty \) and \( \text{denns} E_2 = 0 \). By Lemmas 2.3, 2.4 and Remark 2.6, we further get from the assumption (6) that \( m(r, f) = o(T(r, f)) \), i.e., (7) holds.

Next, we prove (8). Let \( g = f - d \), where \( d \in S(f) \). Since \( m \left( r, \frac{1}{g} \right) = 0 \) for \( |g(z)| > 1 \), we only need to consider the case that \( |g(z)| \leq 1 \). The solution \( f = g + d \) of (4) ensures that

\[ \sum_{0 < |\xi| < n} a_1 \prod_{k=1}^{m} \left[ g_{z_k}(z + \xi_k) + d_{z_k}(z + \xi_k) \right]^{i_k} = \sum_{j=0}^{n} b_j [g(z + \eta_j) + d(z + \eta_j)]^j. \]

(10)

For the term \( \prod_{k=1}^{m} [g_{z_k}(z + \xi_k) + d_{z_k}(z + \xi_k)]^{i_k} \), we rewrite it as follows:

\[ \prod_{k=1}^{m} [g_{z_k}(z + \xi_k) + d_{z_k}(z + \xi_k)]^{i_k} = [g_{z_1}(z + \xi_1)]^{i_1} \cdots [g_{z_m}(z + \xi_m)]^{i_m} + Q_i \]
where $Q_i$ is a polynomial in $g_{z_1}(z + \xi_1), g_{z_2}(z + \xi_2), \ldots, g_{z_m}(z + \xi_m)$ with degree less than $|i| = i_1 + i_2 + \cdots + i_m$ and with coefficients being small functions with respect to $f$ (thus are also small with respect to $g$). Hence, the left-hand side of (10) can be reformulated as follows:

$$
\sum_{0 < |i| < n} \sum_{k=1}^{m} a_i \prod_{k=1}^{m} [g_{z_k}(z + \xi_k)]^{i_k} + Q_i,
$$

where $Q_* = \sum_{0 < |i| < n} a_i Q_i$ is a polynomial in $g_{z_1}(z + \xi_1), g_{z_2}(z + \xi_2), \ldots, g_{z_m}(z + \xi_m)$ with degree less than $n - 1$ and with small coefficients.

For the right-hand side of (10), we have

$$
\sum_{j=0}^{n} b_j(z)[g(z + \eta_j) + d(z + \eta_j)]^j = \sum_{j=0}^{n} b_j(z) \sum_{s=0}^{j} \binom{j}{s} g^s(z + \eta_j) d^{j-s}(z + \eta_j)
$$

$$
= \sum_{j=1}^{n} b_j(z) \sum_{s=1}^{j} \binom{j}{s} g^s(z + \eta_j) d^{j-s}(z + \eta_j) + H(z, d),
$$

where $H(z, d) = \sum_{j=0}^{n} b_j(z) d^j(z + \eta_j)$. It follows from (10)-(12) that

$$
\sum_{0 < |i| < n} \sum_{k=1}^{m} a_i \prod_{k=1}^{m} [g_{z_k}(z + \xi_k)]^{i_k} + Q_*
$$

$$
= \sum_{j=1}^{n} b_j(z) \sum_{s=1}^{j} \binom{j}{s} g^s(z + \eta_j) d^{j-s}(z + \eta_j) + H(z, d).
$$

Divide both sides of the above equation by $g(z)H(z, d)$, we obtain

$$
\frac{1}{g(z)} = \frac{1}{H(z, d)} \sum_{0 < |i| < n} \sum_{k=1}^{m} a_i [g_{z_1}(z + \xi_1)]^{i_1} [g_{z_2}(z + \xi_2)]^{i_2} \cdots [g_{z_m}(z + \xi_m)]^{i_m} \frac{g^s(z + \eta_j)}{g(z)} d^{j-s}(z + \eta_j).
$$

$$
(13) \quad + \frac{Q_*}{H(z, d) g(z)} - \frac{1}{H(z, d)} \sum_{j=1}^{n} b_j(z) \sum_{s=1}^{j} \binom{j}{s} \frac{g^s(z + \eta_j)}{g(z)} d^{j-s}(z + \eta_j).
$$
When $|g(z)| \leq 1$, we get the following inequality

\[
\frac{1}{|g(z)|} \leq \frac{1}{H(z,d)} \sum_{0<|i|<n} |a_i| |\frac{g_{z_1}(z + \xi_1)}{g(z)}|^{i_1} |\frac{g_{z_2}(z + \xi_2)}{g(z)}|^{i_2} \cdots |\frac{g_{z_m}(z + \xi_m)}{g(z)}|^{i_m} + \frac{1}{H(z,d)} \left| \frac{Q_1}{g} \right| + \frac{1}{H(z,d)} \sum_{j=1}^{n} \sum_{i=1}^{j} \left| \frac{g(z + \eta_i)}{g(z)} \right|^{j-i} |d(z + \eta_j)|^{j-i},
\]

which implies

\[
m\left( r, \frac{1}{g} \right) \leq O \left( \sum_{0<|i|<n} m(r, a_i) + \sum_{k=1}^{m} m\left( r, \frac{g_{z_k}(z + \xi_k)}{g(z)} \right) + m\left( r, \frac{1}{H(z,d)} \right) \right) + O \left( \sum_{j=1}^{n} \left\{ m\left( r, \frac{g(z + \eta_j)}{g(z)} \right) + m(r, b_j) + m(r, d(z + \eta_j)) \right\} \right).
\]

By Lemmas 2.3, 2.4, we know

\[
m\left( r, \frac{g_{z_k}(z + \xi_k)}{g(z)} \right) \leq m\left( r, \frac{g_{z_k}(z + \xi_k)}{g(z + \xi_k)} \right) + m\left( r, \frac{g(z + \xi_k)}{g(z)} \right) = o(T(r,g)),
\]

and

\[
m\left( r, \frac{g(z + \eta_j)}{g(z)} \right) = o(T(r,g))
\]

for $k = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Together with (6), it can be concluded from (14) that

\[
m\left( r, \frac{1}{f-d} \right) = m\left( r, \frac{1}{g} \right) = o(T(r,f))
\]

holds for all $r \notin E = E_1 \cup E_2$ with $lm(E_1) < \infty$ and $\text{dens} E_2 = 0$. \hfill \Box

3. Proof of Theorem 1.1

Since $f$ and $g$ share $c$, $d$, $\infty$ CM, we get

\[
\frac{f-c}{g-c} = e^\alpha, \quad \frac{f-d}{g-d} = e^\beta,
\]

where $\alpha$ and $\beta$ are two entire functions on $\mathbb{C}^n$. By Lemma 2.2,

\[
T(r,g) \leq N(r,g) + N\left( r, \frac{1}{g-c} \right) + N\left( r, \frac{1}{g-d} \right) + o(T(r,g))
\]

\[
= N(r,f) + N\left( r, \frac{1}{f-c} \right) + N\left( r, \frac{1}{f-d} \right) + o(T(r,g))
\]

\[
= 3T(r,f) + o(T(r,g))
\]

holds for all $r \notin E$ with $lm(E) < \infty$. Furthermore, we deduce that

\[
T(r,g) \leq (3 + o(1)) T(r,f).
\]
In view of (15), we have
\begin{align}
T(r, e^\alpha) &
\leq T(r, f) + T(r, g) + o(T(r, f)) 
\leq (4 + o(1))T(r, f), \\
T(r, e^\beta) &
\leq T(r, f) + T(r, g) + o(T(r, f)) 
\leq (4 + o(1))T(r, f)
\end{align}
hold for all \( r \notin \mathbb{E} \) with \( \text{lm}(\mathbb{E}) < \infty \).

By (15), if \( z_0 \in \mathbb{C}^m \) is a zero of \( f - c \), \( z_0 \) must be a zero of \( g - c \). It is easy to check that \( z_0 \) is also a zero of \( \frac{-d}{f - c} - 1 \) or \( c - \bar{d} \). So by Lemma 2.1 and Lemma 2.8, we obtain
\begin{align}
T(r, f) &= \mathcal{N} \left( r, \frac{1}{f - c} \right) + m \left( r, \frac{1}{f - c} \right) + o(T(r, f)) \\
&\leq \mathcal{N} \left( r, \frac{1}{\frac{d}{f - c} - 1} \right) + m \left( r, \frac{1}{e - d} \right) + o(T(r, f)) \\
&\leq T \left( r, \frac{g - d}{f - d} \right) + o(T(r, f)) = m \left( r, \frac{g - d}{f - d} \right) + o(T(r, f)) \\
&\leq m(r, g) + o(T(r, f))
\end{align}
holds for all \( r \notin E = E_1 \cup E_2 \) with \( \text{lm}(E_1) < \infty \) and \( \text{den}(E_2) = 0 \).

By simple calculation and analysis, it can be concluded from (15) that \( f = g \)
if and only if \( e^\alpha = 1 \) or \( e^\beta = 1 \) or \( e^\alpha = e^\beta \). We now suppose to the contrary
that \( f \neq g \), which means
\[ e^\alpha \neq 1, \ e^\beta \neq 1, \ e^\alpha \neq e^\beta, \ e^{\beta - \alpha} \neq 1. \]

From (15), we get the following expression for \( f \) and \( g \):
\[ f = c + (d - c) \frac{e^\beta - 1}{e^\beta - \alpha - 1}, \ g = d + (d - c) \frac{1 - e^{-\alpha}}{e^{\beta - \alpha} - 1}. \]

For \( k \in \{1, 2, \ldots, m\} \) and \( j \in \{0, 1, \ldots, n\} \), we have
\[ f_{z_k} = c_{z_k} + \frac{(d_{z_k} - c_{z_k})(e^\beta - 1)}{(e^{\beta - \alpha} - 1)} + (d - c)(\alpha_{z_k} e^{2\beta - \alpha} - \beta_{z_k} e^\beta + \beta_{z_k} e^{\beta - \alpha} - \alpha_{z_k} e^{\beta - \alpha}) \]
and
\[ f(z + \eta_j) = c(z + \eta_j) + (d(z + \eta_j) - c(z + \eta_j)) \frac{(e^{\beta(z + \eta_j)} - 1)}{e^{\beta(z + \eta_j)} - \alpha(z + \eta_j) - 1}. \]

For the sake of simplicity, we will use short notations
\[ \mathcal{F}^{1k}(z) = f(z + \xi_k), \ \mathcal{F}^{1k}(z) = f_{z_k}(z + \xi_k), \ \mathcal{F}^{2j}(z) = f(z + \eta_j). \]
for meromorphic function $f$ and $k \in \{1, 2, \ldots, m\}$, $j \in \{0, 1, \ldots, n\}$. Since $f$ is a solution to equation (4), we get the equality

$$
\sum_{0<|i|<n} a_i \prod_{k=1}^{m} \left\{ \frac{d^{1k} - \bar{\tau}_i^{1k}}{(e^{|d|^{1k} - \bar{\tau}_i^{1k}} - 1)} \right\}^2 \left[ \frac{d^{1k}}{\bar{\tau}_i^{1k}} e^{d^{1k} - \bar{\tau}_i^{1k}} - \frac{\bar{\tau}_i^{1k}}{\bar{\tau}_i^{1k}} e^{d^{1k}} \right] 
+ \left( \frac{d^{1k}}{\bar{\tau}_i^{1k}} - \frac{\bar{\tau}_i^{1k}}{\bar{\tau}_i^{1k}} \right) e^{d^{1k} - \bar{\tau}_i^{1k}} + e^{d^{1k} - 1} \left( \frac{d^{1k}}{\bar{\tau}_i^{1k}} - \frac{\bar{\tau}_i^{1k}}{\bar{\tau}_i^{1k}} \right) \right)^j e^{d^{1k} - \bar{\tau}_i^{1k}} = \sum_{j=0}^{n} b_j \left[ e^{2j} - \bar{\tau}_i^{2j} \right] (\bar{\tau}_i^{2j} - 1)^j.
$$

Multiplying $\prod_{k=1}^{m} (e^{|d|^{1k} - \bar{\tau}_i^{1k}} - 1) \prod_{j=0}^{n} (e^{2j} - \bar{\tau}_i^{2j} - 1)^j$ to both sides of the above equality, we obtain

$$
\prod_{j=0}^{n} (e^{2j} - \bar{\tau}_i^{2j} - 1)^j \sum_{0<|i|<n} a_i \prod_{k=1}^{m} \left\{ \frac{d^{1k} - \bar{\tau}_i^{1k}}{(e^{|d|^{1k} - \bar{\tau}_i^{1k}} - 1)} \right\}^2 \left[ \frac{d^{1k}}{\bar{\tau}_i^{1k}} e^{d^{1k} - \bar{\tau}_i^{1k}} - \frac{\bar{\tau}_i^{1k}}{\bar{\tau}_i^{1k}} e^{d^{1k}} \right] + \left( \frac{d^{1k}}{\bar{\tau}_i^{1k}} - \frac{\bar{\tau}_i^{1k}}{\bar{\tau}_i^{1k}} \right) e^{d^{1k} - \bar{\tau}_i^{1k}} + e^{d^{1k} - 1} \left( \frac{d^{1k}}{\bar{\tau}_i^{1k}} - \frac{\bar{\tau}_i^{1k}}{\bar{\tau}_i^{1k}} \right) \right)^j e^{d^{1k} - \bar{\tau}_i^{1k}} = \prod_{k=1}^{m} (e^{|d|^{1k} - \bar{\tau}_i^{1k}} - 1) \sum_{j=0}^{n} b_j \left[ e^{2j} - \bar{\tau}_i^{2j} \right] (\bar{\tau}_i^{2j} - 1)^j \prod_{s \neq j} (e^{2s} - \bar{\tau}_i^{2s} - 1)^s.
$$

For any vector $\tau \in \mathbb{C}^n \setminus \{0\}$, by (16), (17) and Lemma 2.5 we know

$$
T \left( r, e^{a(z + r)} \right) = T \left( r, e^{a} \right) + o(T \left( r, e^{a} \right)) = T \left( r, e^{a} \right) + o(T \left( r, f \right)),
$$

and

$$
T \left( r, e^{\beta(z + r)} \right) = T \left( r, e^{\beta} \right) + o(T \left( r, e^{\beta} \right)) = T \left( r, e^{\beta} \right) + o(T \left( r, f \right)),
$$

hold for all $r \notin E = E_1 \cup E_2$ with $lm(E_1) < \infty$ and $\text{dens}E_2 = 0$, which indicate that $e^{a(z)}$ (resp. $e^{\beta(z)}$) grows as fast as its shift $e^{a(z + r)}$ (resp. $e^{\beta(z + r)}$) does. Be aware that

$$
e^{a(z + r)} = e^{a(z)}e^{o(z)} = o(z),
$$

$e^{a(z + r)}$ can be represented as the product of $e^{a(z)}$ and a small function with respect to $f$ on account of (16), (17) and Lemma 2.4. Similarly, $e^{\beta(z + r)}$ can be represented as the product of $e^{\beta(z)}$ and a small function with respect to $f$. 
Based on the discussion above, (19) can be rewritten as the form

\[
\sum_{p=0}^{K} \sum_{q=0}^{K} \varphi_{p,q} e^{p\beta - q\alpha} = \sum_{p=0}^{K} \sum_{q=0}^{K} \psi_{p,q} e^{p\beta - q\alpha},
\]

where \( K = 2nm + \frac{n(n+1)}{2} \), \( \varphi_{p,q} \) are either 0 or polynomials in \( r_k \) and \( s_k \) with coefficients being products of a nonzero integer, \( a_k, r_k, s_k, d_k \) with \( e^{r_k}, e^{s_k}, e^{d_k}, e^{2r_k}, e^{2s_k}, e^{2d_k}, e^{r_k+s_k}, e^{r_k+d_k}, e^{s_k+d_k}, e^{r_k+s_k+d_k}, e^{r_k+s_k+d_k} \), where \( k \in \{1,2,\ldots,m\} \), \( j \in \{0,1,\ldots,n\} \).

And \( \psi_{p,q} \) are either 0 or polynomials in \( b_j \) with coefficients being products of a nonzero integer, \( e^{2j}, e^{2j}, e^{2j}, e^{2j}, e^{2j}, e^{2j}, e^{2j}, e^{2j} \), \( e^{2j}, e^{2j}, e^{2j}, e^{2j}, e^{2j}, e^{2j}, e^{2j}, e^{2j} \), where \( j \in \{0,1,\ldots,n\} \), \( k \in \{1,2,\ldots,m\} \). In conclusion, \( \varphi_{p,q} \) and \( \psi_{p,q} \) are small with respect to meromorphic function \( f \) for any \( p \in \{0,1,\ldots,2nm + \frac{n(n+1)}{2} \} \) and \( q \in \{0,1,\ldots,2nm + \frac{n(n+1)}{2} \} \). From the observation of \( e^{\beta - \alpha} \) on each side of (20), we deduce \( \varphi_{n,0} = 0 \) and

\[
\psi_{n,0} = (-1)^{2nm + \frac{n(n+1)}{2}} b_n \left( \frac{\alpha}{\alpha - \beta} \right) e^{n(\beta - \alpha)} \neq 0.
\]

Furthermore, we rewrite (20) as

\[
\sum_{p=0}^{K} \sum_{q=0}^{K} \phi_{p,q} e^{p\beta - q\alpha} = 0,
\]

where \( K = 2nm + \frac{n(n+1)}{2} \), \( \phi_{p,q} = \varphi_{p,q} - \psi_{p,q} \) and \( \phi_{n,0} = \varphi_{n,0} - \psi_{n,0} \neq 0 \). By Lemma 2.3, it is concluded from (16) and (17) that

\[
m(r, \alpha z_k) = m \left( r, \left( \frac{e^{\alpha z_k}}{e^{\alpha}} \right) \right) = o(T(r, e^{\alpha})), \quad o(T(r, f)),
\]

\[
m(r, \beta z_k) = m \left( r, \left( \frac{e^{\beta z_k}}{e^{\beta}} \right) \right) = o(T(r, e^{\beta})), \quad o(T(r, f))
\]

hold for all \( r \notin E = E_1 \cup E_2 \) with \( \operatorname{lm}(E_1) < \infty \) and \( \operatorname{den}(E_2) = 0 \) and for any \( k \in \{1,2,\ldots,m\} \). Since \( a_k, c, d, c_{z_k}, d_{z_k}, e^{\beta - \alpha}, e^{\alpha - \beta}, e^{\beta - \alpha}, e^{\alpha - \beta} \), and \( b_j \) (\( 1 \leq k \leq m, 0 \leq j \leq n \)) are all small functions with respect to \( f \), it follows that

\[
T(r, \varphi_{p,q}) = o(T(r, f)), \quad T(r, \psi_{p,q}) = o(T(r, f))
\]

hold for all \( r \notin E = E_1 \cup E_2 \) with \( \operatorname{lm}(E_1) < \infty \) and \( \operatorname{den}(E_2) = 0 \).

Next, we are going to make an estimation of the proximity function of \( e^{u\alpha + v\beta} \) for any pair of integers \( u, v \) with \( (u,v) \neq (0,0) \). Our discussion will be divided into four cases.

**Case 1:** \( u \geq 0 \) and \( v \geq 0 \). By Lemma 2.8, it is easy to verify that

\[
m(r, e^{u\alpha + v\beta}) = m \left( r, \left( \frac{f - c}{g - c} \right)^u \left( \frac{f - d}{g - d} \right)^v \right)
\]
\[ \leq m \left( r, \frac{1}{(g-c)^u(g-d)^v} \right) + o(T(r,f)) \]
\[ \leq m \left( r, \frac{(f-c)^u}{(g-c)^u} \left( \frac{f-d}{g-d} \right)^v \right) + m \left( r, \frac{1}{(f-c)^u(f-d)^v} \right) + o(T(r,f)) \]
\[ = m \left( r, e^{\alpha u+\beta v} \right) + o(T(r,f)), \]

which implies

\[ m \left( r, e^{\alpha u+\beta v} \right) = m \left( r, \frac{1}{(g-c)^u(g-d)^v} \right) + o(T(r,f)) \]

holds for all \( r \notin E = E_1 \cup E_2 \) with \( \mu(E_1) < \infty \) and \( \text{dens} E_2 = 0 \). Notice that

\[ m \left( r, \frac{1}{(g-c)^u(g-d)^v} \right) = T \left( r, \frac{1}{(g-c)^u(g-d)^v} \right) - N \left( r, \frac{1}{(g-c)^u(g-d)^v} \right) \]
\[ = (u + v)T(r,g) - N \left( r, \frac{1}{(g-c)^u(g-d)^v} \right) + o(T(r,f)) \]

(24)

holds for all \( r \notin E = E_1 \cup E_2 \) with \( \mu(E_1) < \infty \) and \( \text{dens} E_2 = 0 \). Owing to the assumption that \( f \) and \( g \) share \( c, d \) CM, by Lemma 2.8 we further gets

\[ N \left( r, \frac{1}{(g-c)^u(g-d)^v} \right) = uN \left( r, \frac{1}{g-c} \right) + vN \left( r, \frac{1}{g-d} \right) \]
\[ = uN \left( r, \frac{1}{f-c} \right) + vN \left( r, \frac{1}{f-d} \right) \]
\[ = (u + v)T(r,f) + o(T(r,f)) \]

(25)

holds for all \( r \notin E = E_1 \cup E_2 \) with \( \mu(E_1) < \infty \) and \( \text{dens} E_2 = 0 \). Combine (23)-(25) with the assumption that meromorphic functions \( f \) and \( g \) share \( \infty \) CM, we know

\[ m \left( r, e^{\alpha u+\beta v} \right) = (u + v)(T(r,g) - T(r,f)) + o(T(r,f)) \]
\[ = (u + v)(m(r,g) - m(r,f)) + o(T(r,f)) \]
\[ = (u + v)m(r,g) + o(T(r,f)) \geq m(r,g) + o(T(r,f)), \]

that is, when \( (u,v) \neq (0,0) \), \( m \left( r, e^{\alpha u+\beta v} \right) \geq m(r,g) + o(T(r,f)) \) holds for all \( r \notin E = E_1 \cup E_2 \) with \( \mu(E_1) < \infty \) and \( \text{dens} E_2 = 0 \).

**Case 2:** \( u \geq 0, v \leq 0 \) and \( (u,v) \neq (0,0) \). Making use of a similar reasoning as in Case 1, we have

\[ m \left( r, e^{\alpha u+\beta v} \right) = m \left( r, \frac{f-c}{g-c} \left( \frac{f-d}{g-d} \right)^v \right) \]
\[ = m \left( r, \frac{1}{(g-c)^u(g-d)^v} \right) + o(T(r,f)) \]
$$= T \left( \frac{1}{(g-c)^u(g-d)^v} \right) - N \left( \frac{1}{(g-c)^u(g-d)^v} \right) + o(T(r,f))$$

$$= \max\{u, -v\} T(r,g) - N \left( \frac{1}{(g-c)^u(g-d)^v} \right) + o(T(r,f)).$$

For the second term in the right-hand side of the equality above, we obtain the following estimation

$$N \left( \frac{1}{(g-c)^u(g-d)^v} \right)$$

$$\leq uN \left( \frac{1}{g-c} \right) + \max\{0, -u - v\} N(r, g - d)$$

$$= uN \left( \frac{1}{g-c} \right) + \max\{0, -u - v\} N(r, f) + o(T(r,f))$$

$$= (uT(r,f) + \max\{0, -u - v\} T(r,f)) + o(T(r,f))$$

$$= \max\{u, -v\} T(r,f) + o(T(r,f)).$$

Hence,

$$m \left( r, e^{u_0 + v_0} \right) = \max\{u, -v\} T(r,g) - \max\{u, -v\} T(r,f) + o(T(r,f))$$

$$= \max\{u, -v\} m(r,g) + o(T(r,f)),$$

which yields that when \( u \geq 0, \) \( v \leq 0 \) and \( (u, v) \neq (0, 0) \), \( m \left( r, e^{u_0 + v_0} \right) \geq m(r,g) + o(T(r,f)) \) holds for all \( r \notin E = E_1 \cup E_2 \) with \( \text{lm}(E_1) < \infty \) and \( \text{dens}_{E_2} = 0. \)

**Case 3:** \( u \leq 0, \) \( v \geq 0 \) and \( (u, v) \neq (0, 0) \). In virtue of Lemma 2.1, we have

$$m \left( r, e^{u_0 + v_0} \right) = T \left( r, e^{u_0 + v_0} \right) - N \left( r, e^{u_0 + v_0} \right) = T \left( r, e^{-u_0 - v_0} \right) + O(1)$$

$$= m \left( r, e^{-u_0 - v_0} \right) + o(T(r,f)),$$

that is,

(26)

$$m \left( r, e^{u_0 + v_0} \right) = m \left( r, e^{-u_0 - v_0} \right) + o(T(r,f))$$

holds for all \( r \notin E = E_1 \cup E_2 \) with \( \text{lm}(E_1) < \infty \) and \( \text{dens}_{E_2} = 0 \). Thus, by a similar discussion as in Case 2, we deduce from (26) that \( m \left( r, e^{u_0 + v_0} \right) \geq m(r,g) + o(T(r,f)) \) holds for all \( r \notin E = E_1 \cup E_2 \) with \( \text{lm}(E_1) < \infty \) and \( \text{dens}_{E_2} = 0 \) when \( u \leq 0, \) \( v \geq 0 \) and \( (u, v) \neq (0, 0) \).

**Case 4:** \( u \leq 0, \) \( v \leq 0 \) and \( (u, v) \neq (0, 0) \). In this case, \( -u \geq 0 \) and \( -v \geq 0 \). As discussed in Case 1 for \( -u, -v \), we see from (26) that \( m \left( r, e^{u_0 + v_0} \right) \geq m(r,g) + o(T(r,f)) \) holds for all \( r \notin E = E_1 \cup E_2 \) with \( \text{lm}(E_1) < \infty \) and \( \text{dens}_{E_2} = 0 \) when \( u \leq 0, \) \( v \leq 0 \) and \( (u, v) \neq (0, 0) \).

Therefore, for any pair of integers \( u, v \) with \( (u, v) \neq (0, 0) \),

$$m \left( r, e^{u_0 + v_0} \right) \geq m(r,g) + o(T(r,f))$$
holds for all \( r \notin E = E_1 \cup E_2 \) with \( \text{lm}(E_1) < \infty \) and \( \overline{\text{dens}}E_2 = 0 \). Combine this with (18), for \((u,v) \neq (0,0)\), we have

\[
T(r,f) \leq m(r,g) + o(T(r,f)) \leq m(r,e^{u\alpha+v\beta}) + o(T(r,f))
\]

(27)

holds for all \( r \notin E = E_1 \cup E_2 \) with \( \text{lm}(E_1) < \infty \) and \( \overline{\text{dens}}E_2 = 0 \). In view of (22) and (27), we get

\[
T(r,\phi_{p,q}) = o(T(r,e^{u\alpha+v\beta})), \ (u,v) \neq (0,0)
\]

holds for all \( r \notin E = E_1 \cup E_2 \) with \( \text{lm}(E_1) < \infty \) and \( \overline{\text{dens}}E_2 = 0 \). Then by Lemma 2.7, we deduce that all coefficients of the exponential function in (21) vanish identically, that is, \( \phi_{p,q} = 0 \) for all \( p \in \{0,1,\ldots,2nm+n(n+1)/2\} \) and \( q \in \{0,1,\ldots,2nm+n(n+1)/2\} \). It yields a contradiction to \( \phi_{n,0} \neq 0 \). Consequently, \( f = g \) holds on \( \mathbb{C}^m \).

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References


Qibin Cheng  
School of Science  
Beijing University of Posts and Telecommunications  
Beijing 100876, P. R. China  
Email address: 18707672503@163.com

Yezhou Li  
School of Science  
Beijing University of Posts and Telecommunications  
Beijing 100876, P. R. China  
Email address: yezhouli2019@outlook.com

Zhixue Liu  
School of Science  
Beijing University of Posts and Telecommunications  
Beijing 100876, P. R. China  
Email address: zxliumath@bupt.edu.cn