Bull. Korean Math. Soc. **60** (2023), No. 2, pp. 389–404
https://doi.org/10.4134/BKMS.b220149
pISSN: 1015-8634 / eISSN: 2234-3016

HANKEL DETERMINANTS FOR STARLIKE FUNCTIONS WITH RESPECT TO SYMMETRICAL POINTS

Nak Eun Cho, Young Jae Sim, and Derek K. Thomas

Abstract. We prove sharp bounds for Hankel determinants for starlike functions f with respect to symmetrical points, i.e., f given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ for $z \in \mathbb{D}$ satisfying

$$\mathrm{Re}\, \frac{zf'(z)}{f(z) - f(-z)} > 0, \quad z \in \mathbb{D}.$$

We also give sharp upper and lower bounds when the coefficients of f are real.

1. Introduction

Let $\mathcal{H}$ be the class of all analytic functions in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ with functions $f \in \mathcal{A}$ having Taylor series

$$f(z) = z + \sum_{n=1}^{\infty} a_n z^n. \tag{1}$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$, consisting of univalent functions, and $\mathcal{S}^*$ denote the class of starlike functions. Then it is well-known that a function $f \in \mathcal{A}$ belongs to $\mathcal{S}^*$ if and only if

$$\mathrm{Re}\ \frac{zf'(z)}{f(z)} > 0, \quad z \in \mathbb{D}.$$

A function $f \in \mathcal{A}$ belongs to $\mathcal{K}$, the class of close-to-convex functions if and only if there exists $g \in \mathcal{S}^*$ such that $\mathrm{Re}[e^{i\tau} (zf'(z)/g(z))] > 0$ for $z \in \mathbb{D}$, and $\tau \in (-\pi/2, \pi/2)$. The class $\mathcal{K}$ was first formally introduced by Kaplan in 1952 [8], who showed that $\mathcal{K} \subset \mathcal{S}$, so that $\mathcal{S}^* \subset \mathcal{K} \subset \mathcal{S}$.

In 1959, Sakaguchi [17] introduced the class $\mathcal{S}_S^*$ of starlike functions with respect to symmetrical points $\mathcal{S}_S^*$ satisfy the condition

$$\mathrm{Re}\, \frac{zf'(z)}{f(z) - f(-z)} > 0, \quad z \in \mathbb{D}, \tag{2}$$

Received February 23, 2022; Revised June 13, 2022; Accepted August 1, 2022.
2020 *Mathematics Subject Classification.* Primary 30C45.
Key words and phrases. Starlike functions, close-to-convex functions, Hankel determinant, coefficient problems.

noting that $\mathcal{S}^*_S$ forms a subclass of $\mathcal{K}$.

For $q, n \in \mathbb{N}$, the Hankel determinant $H_{q,n}(f)$ of functions $f \in \mathcal{A}$ given by (1) is defined as

$$H_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}.$$

General results for Hankel determinants with applications can be found in [2], [14], [15] and [18]. For subclasses of $\mathcal{A}$, finding bounds of $|H_{q,n}(f)|$ for $q, n \in \mathbb{N}$, is an interesting and significant area of study. Hayman [7] examined the second Hankel determinant $H_{2,2}(f) = a_2a_4 - a_3^2$ for really mean univalent functions, and recently many other authors have also examined the second Hankel determinant for a variety of subclasses of $\mathcal{A}$, (see e.g., [3, 4] for further references), often obtaining sharp bounds for $|H_{2,2}(f)|$. The problem of finding sharp bounds for the third Hankel determinant

$$|H_{3,1}(f)| = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = |a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2)| \tag{3}$$

is technically much more difficult.

However, sharp bounds for $|H_{3,1}(f)|$ have been found e.g. for convex functions [9], and recently for starlike functions, [10].

In this paper, we give the sharp bound for $|H_{3,1}(f)|$ when $f \in \mathcal{S}^*_S$, and sharp upper and lower bounds for $H_{3,1}(f)$ when the coefficients of f are real, noting that the sharp inequality $|H_{2,2}(f)| \le 1$ was obtained in [12]. We also find the sharp bound for $|H_{2,3}(f)| = |a_3a_5 - a_4^2|$, and when the coefficients of f are real, give sharp upper and lower bounds for $H_{2,3}(f)$, and find sharp upper and lower bounds for $H_{2,2}(f)$, showing that the bound $|H_{2,2}(f)| \le 1$ can be improved.

Since functions in $\mathcal{S}^*_S$ can be represented using the Carathéodory class $\mathcal{P}$ [1], *i.e.*, the class of functions $p \in \mathcal{H}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}, \tag{4}$$

having a positive real part in $\mathbb{D}$, the coefficients of functions in $\mathcal{S}^*_S$ can be expressed in terms of the coefficients of functions in $\mathcal{P}$. We base our analysis on the following lemmas [11] and [16], and the lemma of Sugawa *et al.* [5] below.

Lemma 1.1 ([11]). *If $p \in \mathcal{P}$ and is given by (4) with $c_1 \ge 0$, then*

$$c_1 = 2\zeta_1,$$

$$c_2 = 2\zeta_1^2 + 2(1 - \zeta_1^2)\zeta_2,$$

$$c_3 = 2\zeta_1^3 + 4(1 - \zeta_1^2)\zeta_1\zeta_2 - 2(1 - \zeta_1^2)\zeta_1\zeta_2^2 + 2(1 - \zeta_1^2)(1 - |\zeta_2|^2)\zeta_3$$

and

$$\begin{aligned} c_4 = {} & 2\zeta_1^4 + 2(1-\zeta_1^2)\zeta_2\left(\zeta_1^2\zeta_2^2 - 3\zeta_1^2\zeta_2 + 3\zeta_1^2 + \zeta_2\right) \\ & + 2(1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_3\left(2\zeta_1 - 2\zeta_1\zeta_2 - \overline{\zeta_2}\zeta_3\right) \\ & + 2(1-\zeta_1^2)(1-|\zeta_2|^2)(1-|\zeta_3|^2)\zeta_4 \end{aligned}$$

for some $\zeta_1 \in [0,1]$ *and* $\zeta_2, \zeta_3, \zeta_4 \in \overline{\mathbb{D}}$.

Lemma 1.2 ([16])**.** *Let* $p \in \mathcal{P}$ *and be given by* (4)*. Then*

$$|\mu c_n c_m - c_{m+n}| \le \begin{cases} 2, & 0 \le \mu \le 1, \\ 2|2\mu - 1|, & \textit{elsewhere}, \end{cases}$$

for all n, $m \in \mathbb{N}$. *If* $0 < \mu < 1$, *then equality holds for the function* $p(z) = (1 + z^{n+m})/(1 - z^{m+n})$. *In all other cases, equality holds for the function* $p(z) = (1+z)/(1-z)$.

The next lemma is a special case of more general results due to Choi, Kim and Sugawa [5] (see also [13]). Define

$$Y(A,B,C) := \max_{z \in \overline{\mathbb{D}}} \left(|A + Bz + Cz^2| + 1 - |z|^2\right), \quad A, B, C \in \mathbb{R}.$$

Lemma 1.3 ([5])**.** *If* $AC \ge 0$, *then*

$$Y(A,B,C) = \begin{cases} |A| + |B| + |C|, & |B| \ge 2(1-|C|), \\ 1 + |A| + \dfrac{B^2}{4(1-|C|)}, & |B| < 2(1-|C|). \end{cases}$$

If $AC < 0$, *then*

$$Y(A,B,C)$$
$$= \begin{cases} 1 - |A| + \dfrac{B^2}{4(1-|C|)}, & \Big(-4AC(C^{-2}-1) \le B^2\Big) \wedge \Big(|B| < 2(1-|C|)\Big), \\ 1 + |A| + \dfrac{B^2}{4(1+|C|)}, & B^2 < \min\left\{4(1+|C|)^2, -4AC(C^{-2}-1)\right\}, \\ R(A,B,C), & \textit{otherwise}, \end{cases}$$

where

$$R(A,B,C) = \begin{cases} |A| + |B| - |C|, & |C|(|B| + 4|A|) \le |AB|, \\ -|A| + |B| + |C|, & |AB| \le |C|(|B| - 4|A|), \\ (|C| + |A|)\sqrt{1 - \dfrac{B^2}{4AC}}, & \textit{otherwise}. \end{cases} \tag{5}$$

2. Main results

Theorem 2.1. *If* $f \in \mathcal{S}_S^*$ *and is given by* (1)*, then*

$$|H_{3,1}(f)| \le \alpha, \tag{6}$$

where

$$\alpha := \frac{1}{144}\left(3879 - 2218\sqrt{3} - 1356\sqrt{15+4\sqrt{3}} + 783\sqrt{45+12\sqrt{3}}\right) \approx 0.26547.$$

The inequality is sharp.

Proof. Let $f \in \mathcal{S}_S^*$ and be given by (1). Then by (2),

$$\frac{2zf'(z)}{f(z)-f(-z)} = p(z), \quad z \in \mathbb{D}, \tag{7}$$

for some function $p \in \mathcal{P}$ given by (4). Since the class $\mathcal{P}$ and the functional $H_{3,1}(f)$ are rotationally invariant, we may assume that $c_1 \in [0,2]$ ([1], see also [6, Vol. I, p. 80, Theorem 3]), *i.e.*, in view of Lemma 1.1 that $\zeta_1 \in [0,1]$. Substituting (1) and (4) into (7) and equating coefficients we obtain

$$\begin{aligned} &a_2 = \frac{c_1}{2}, \quad a_3 = \frac{c_2}{2}, \quad a_4 = \frac{1}{4}\Big(\frac{c_1c_2}{2} + c_3\Big), \\ &a_5 = \frac{1}{4}\Big(\frac{c_2^2}{2} + c_4\Big). \end{aligned} \tag{8}$$

Hence from (3),

$$H_{3,1}(f) = \frac{1}{64}\Big(c_1^2c_2^2 - 4c_2^3 + 4c_1c_2c_3 - 4c_3^2 - 4(c_1^2 - 2c_2)c_4\Big).$$

From Lemma 1.1 a straightforward algebraic computation gives

$$\begin{aligned} H_{3,1}(f) = \frac{1}{4}(1-\zeta_1^2)^2\Big[&\zeta_1^2\zeta_2^2(1-\zeta_2)^2 + 2(1-|\zeta_2|^2)\zeta_1\zeta_2(1-\zeta_2)\zeta_3 \\ &- (1-|\zeta_2|^4)\zeta_3^2 + 2(1-|\zeta_2|^2)(1-|\zeta_3|^2)\zeta_2\zeta_4\Big] \end{aligned} \tag{9}$$

for some $\zeta_1 \in [0,1]$ and $\zeta_2,\ \zeta_3,\ \zeta_4 \in \overline{\mathbb{D}}$. Since $|\zeta_4| \le 1$, using the triangle inequality in (9) we obtain

$$|H_{3,1}(f)| \le \frac{1}{4}(1-\zeta_1^2)^2[|A_1^2 + 2A_1A_2\zeta_3 - A_3\zeta_3^2| + 2(1-|\zeta_2|^2)(1-|\zeta_3|^2)|\zeta_2|], \tag{10}$$

where

$$A_1 = \zeta_1\zeta_2(1-\zeta_2), \quad A_2 = 1-|\zeta_2|^2, \quad A_3 = 1-|\zeta_2|^4.$$

When $\zeta_1 = 1$, (9) gives $H_{3,1}(f) = 0$, and when $\zeta_2 = 0$, we have

$$|H_{3,1}(f)| \le \frac{1}{4}(1-\zeta_1^2)^2|\zeta_3|^2 \le \frac{1}{4} < \alpha.$$

When $|\zeta_2| = 1$, (9) gives

$$|H_{3,1}(f)| \le \frac{1}{4}\zeta_1^2(1-\zeta_1^2)^2|(1-\zeta_2)^2| \le \zeta_1^2(1-\zeta_1^2)^2 \le \frac{4}{27} < \alpha.$$

Now we assume that $0 \le \zeta_1 < 1$ and $\zeta_2 \in \mathbb{D}^* := \mathbb{D} \setminus \{0\}$. A suitable rotation for $\zeta_3 \in \overline{\mathbb{D}}$ ($\zeta_3 \mapsto \zeta_3 e^{i\theta}$ with $\theta = \arg A_1$) in (10) gives

$$|H_{3,1}(f)| \le \frac{1}{4}(1-\zeta_1^2)^2 \times 2|\zeta_2|(1-|\zeta_2|^2)\Psi(B_1,B_2,B_3),$$

where

$$\Psi(B_1, B_2, B_3) = |B_1 + B_2\zeta_3 + B_3\zeta_3^2| + 1 - |\zeta_3|^2$$

and B_1, B_2, $B_3 \in \mathbb{R}$ are defined by

$$B_1 = \frac{\zeta_1^2|\zeta_2||1-\zeta_2|^2}{2(1-|\zeta_2|^2)}, \quad B_2 = \zeta_1|1-\zeta_2|, \quad B_3 = \frac{-(1+|\zeta_2|^2)}{2|\zeta_2|}.$$

Then the following inequalities hold for all $\zeta_1 \in (0,1]$ and $\zeta_2 \in \mathbb{D}^*$,

(a) $B_1B_3 < 0$,
(b) $B_2^2 + 4B_1B_3(B_3^{-2} - 1) \geq 0$,
(c) $|B_2| \geq 2(1 - |B_3|)$.

Thus by Lemma 1.3 we have

$$\max_{\zeta_3 \in \overline{\mathbb{D}}} \Psi(B_1, B_2, B_3) = R(B_1, B_2, B_3),$$

where R is given in (5). Moreover, since

$$|B_1B_2| \leq |B_3|(|B_2| + 4|B_1|)$$

holds for $\zeta_1 \in (0,1]$ and $\zeta_2 \in \mathbb{D}^*$, Lemma 1.3 gives

$$\Psi(B_1, B_2, B_3) \tag{11}$$
$$\leq \begin{cases} -|B_1| + |B_2| + |B_3|, & \text{when } |B_1B_2| \leq |B_3|(|B_2| - 4|B_3|), \\ (|B_3| + |B_1|)\sqrt{1 - \dfrac{B_2^2}{4B_1B_3}}, & \text{otherwise.} \end{cases}$$

A. We assume that

$$|B_1B_2| \leq |B_3|(|B_2| - 4|B_3|).$$

Then

$$\begin{aligned} |H_{3,1}|(f)| &\leq \frac{1}{2}(1-\zeta_1^2)^2|\zeta_2|(1-|\zeta_2|^2)\left(-|B_1| + |B_2| + |B_3|\right) \\ &= \frac{1}{4}(1-\zeta_1^2)^2 F_1(\zeta_2), \end{aligned}$$

where

$$F_1(\zeta_2) = -\zeta_1^2|\zeta_2|^2|1-\zeta_2|^2 + 2\zeta_1|\zeta_2|(1-|\zeta_2|^2)|1-\zeta_2| + 1 - |\zeta_2|^4.$$

Setting $\zeta_2 = re^{i\theta}$ and $t = \cos\theta \in [-1,1]$ we obtain

$$F_1(\zeta_2) = -\zeta_1^2 r^2(1 + r^2 - 2rt) + 2\zeta_1 r(1-r^2)\sqrt{1 + r^2 - 2rt} + 1 - r^4 =: F_2(t),$$

and F_2 has its unique critical point at $t = t_0$, where

$$t_0 = \frac{1}{2r}\left[1 + r^2 - \frac{(1-r^2)^2}{\zeta_1^2 r^2}\right].$$

We consider the following three cases:

(i) $F_2(t) \leq F_2(-1)$ for $t \in [-1,1]$,
(ii) $F_2(t) \leq F_2(1)$ for $t \in [-1,1]$,

(iii) $F_2(t)$, $t \in [-1,1]$, has its maximum at a point in $(-1,1)$.

When condition (i) is satisfied we have

$$|H_{3,1}(f)| \le \frac{1}{4}(1-\zeta_1^2)^2 F_2(-1) = L(\zeta_1, r), \tag{12}$$

where

$$L(x,y) = \frac{1}{4}(1-x^2)^2[-x^2y^2(1+y)^2 + 2xy(1-y^2)(1+y) + 1 - y^4].$$

Differentiating L with respect to x and y gives

$$\frac{\partial L}{\partial x}(x,y) = \frac{1}{2}(1+y)(1-x^2)L_1(x,y),$$

and

$$\frac{\partial L}{\partial y}(x,y) = -\frac{1}{2}(1-x^2)^2 L_2(x,y),$$

where

$$L_1(x,y) = -2x + y^3(1+x)^2(-1+3x) + y(1+2x-5x^2) + 3y^2x(-1+x^2),$$

and

$$L_2(x,y) = -x + y(-2+x)x + 3y^2x(1+x) + 2y^3(1+x)^2.$$

The system $L_1(x,y) = L_2(x,y) = 0$ has the unique solution (x_0, y_0) in $(0,1) \times (0,1)$, where

$$\begin{aligned} x_0 = \frac{1}{4704}\Big(&-5880 + 3528\sqrt{3} + 3741\sqrt{15+4\sqrt{3}} + 77(15+4\sqrt{3})^{3/2} \\ &- 114\sqrt{3}(15+4\sqrt{3})^{3/2} - 558\sqrt{3(15+4\sqrt{3})}\Big), \end{aligned}$$

and

$$y_0 = \frac{1}{14}\left(-4 + 3\sqrt{3} + 7\sqrt{\frac{15}{49} + \frac{4\sqrt{3}}{49}}\right).$$

Substituting gives

$$L(x_0,y_0) = \frac{1}{144}\left(3879 - 2218\sqrt{3} - 1356\sqrt{15+4\sqrt{3}} + 783\sqrt{45+12\sqrt{3}}\right) = \alpha$$

in $(0,1) \times (0,1)$.

It is easy to see that $L(x,y) \le 1/4$ holds on the boundary of $[0,1] \times [0,1]$, since

$$L(0,y) = \frac{1}{4}(1-y^4) \le \frac{1}{4}, \quad L(1,y) \equiv 0,$$

$$L(x,0) = \frac{1}{4}(1-x^2)^2 \le \frac{1}{4}, \quad L(x,1) = -x^2(1-x^2)^2 \le 0$$

for $x \in [0,1]$ and $y \in [0,1]$. Thus we have shown that if condition (i) is satisfied, then

$$\max_{(x,y)\in[0,1]\times[0,1]} L(x,y) = L(x_0,y_0) = \alpha,$$

and so by (12) we have $|H_{3,1}(f)| \le L(\zeta_1, r) \le \alpha$.

We next assume that condition (ii) is satisfied, then

$$|H_{3,1}(f)| \le \frac{1}{4}(1-\zeta_1^2)^2 F_2(1) = M(\zeta_1, r), \tag{13}$$

where

$$M(x,y) = \frac{1}{4}(1-x^2)^2\left[-x^2y^2(1-y)^2 + 2xy(1-y^2)(1-y) + 1 - y^4\right].$$

A similar analysis to that in case (i) gives

$$\max_{(x,y)\in[0,1]\times[0,1]} M(x,y) = M(x_1,y_1) \approx 0.25274\ldots, \tag{14}$$

where the approximate values of x_1 and y_1 are given by

$$x_1 \approx 0.0835\ldots \quad \text{and} \quad y_1 \approx 0.2490\ldots.$$

Thus by (13) and (14), we obtain $|H_{3,1}(f)| \le M(\zeta_1, r) < \alpha$.

When condition (iii) is satisfied, we have $-1 < t_0 < 1$, which implies $r > 1/(1+\zeta_1)$. Therefore

$$\begin{aligned} |H_{3,1}(f)| &\le \frac{1}{4}(1-\zeta_1^2)^2 F_2(t_0) = \frac{1}{2}(1-\zeta_1^2)^2(1-r^2) \\ &\le \frac{1}{2}\zeta_1(1-\zeta_1)^2(2+\zeta_1) \le \frac{3}{8}(-3+2\sqrt{3}) < \alpha. \end{aligned}$$

B. Next we consider the condition

$$|B_1B_2| \ge |B_3|(|B_2| - 4|B_3|), \tag{15}$$

which is equivalent to

$$\zeta_1^2|\zeta_2|^2|1-\zeta_2|^2 + 2\zeta_1|\zeta_2||1-\zeta_2|(1+|\zeta_2|^2) - (1-|\zeta_2|^4) \ge 0.$$

Let

$$\zeta_1^* = \frac{-1+\sqrt{2(1+|\zeta_2|^2)}}{|\zeta_2||1-\zeta_2|}.$$

Then $\zeta_1^* \le 1$ holds for $\zeta_2 \in \overline{\mathbb{D}}$ satisfying

$$|\zeta_2|^2|1-\zeta_2|^2 + 2|\zeta_2||1-\zeta_2| - 2|\zeta_2|^2 - 1 \ge 0. \tag{16}$$

On the other hand, under the condition (15), by (11), we have

$$\begin{aligned} |H_{3,1}(f)| &\le \frac{1}{2}(1-\zeta_1^2)^2|\zeta_2|(1-|\zeta_2|^2)(|B_3|+|B_1|)\sqrt{1-\frac{B_2^2}{4B_1B_3}} \\ &= \frac{1}{4}(1-\zeta_1^2)^2(1-|\zeta_2|^4+\zeta_1^2|\zeta_2|^2|1-\zeta_2|^2)\sqrt{\frac{2}{1+|\zeta_2|^2}} \\ &=: F_3(\zeta_1,\zeta_2). \end{aligned}$$

We now show that

$$F_3(\zeta_1,\zeta_2) \le \frac{1}{4}$$

holds for $\zeta_1 \in [\zeta_1^*, 1]$ under the constraint (16).

Let $\zeta_2 = re^{i\theta}$ with $r \in (0,1]$, and $t = \cos\theta \in [-1,1]$, and let $r_0 \approx 0.37081\ldots$ be a root of the equation $r^4 + 2r^3 + r^2 + 2r - 1 = 0$, and

$$t_0 = \frac{-3 - r^2 + r^4 + 2\sqrt{2(1+r^2)}}{2r^3}.$$

Then it is easy to see that (16) holds only when

$$r_0 \le r \le 1 \quad \text{and} \quad -1 \le t \le t_0.$$

Also let

$$x_1 = \frac{-2k_1 + k_2^2}{3k_2^2},$$

where

$$k_1 = 1 - |\zeta_2|^4 \quad \text{and} \quad k_2 = |\zeta_2||1-\zeta_2|,$$

and, let $r_1 \approx 0.919585\ldots$ be a root of the equation

$$-2(1-r^4) + r^2(1+r)^2 - 3\left[1 - 2\sqrt{2(1+r^2)} + 2(1+r^2)\right] = 0.$$

Then we have the following:

(i) $(\zeta_1^*)^2 \ge x_1$ holds for $r \le r_1$, $t \in [-1,1]$ or $r_1 \le r \le 1$, $t \in [t_1, 1]$;
(ii) $(\zeta_1^*)^2 \le x_1$ holds for $r_1 \le r \le 1$ and $t \in [-1, t_1]$,

where

$$t_1 = \frac{-11 - 5r^2 + 3r^4 + 6\sqrt{2(1+r^2)}}{2r^3}.$$

Also note that $t_0 > t_1$ holds for all $r \in [0,1]$.

We now define $h_1 : [(\zeta_1^*)^2, 1] \to \mathbb{R}$ by $h_1(x) = (1-x)^2(k_1 + k_2^2 x)$. Then $h_1'(x) = 0$ occurs at $x = 1$ or $x = x_1$. Since $x_1 < 1$ and the leading coefficient of h_1 is nonnegative, we have

$$h_1(x) \le \begin{cases} h_1(x_1), & \text{when } (\zeta_1^*)^2 \le x_1, \\ h_1((\zeta_1^*)^2), & \text{when } (\zeta_1^*)^2 \ge x_1, \end{cases}$$

for $x \in [(\zeta_1^*)^2, 1]$.

B(i) For a fixed $r \in [r_0, 1]$, we consider $h_2 : [-1, t_0] \to \mathbb{R}$ defined by

$$h_2(t) = \left[1 - \frac{C}{r^2(1+r^2-2rt)}\right]^2,$$

where $C = 1 - 2\sqrt{2(1+r^2)} + 2(1+r^2)$. Then

$$h_2'(t) = \frac{4Ch_3(t)}{r^3(1+r^2-2rt)^3},$$

where $h_3(t) = C - r^2(1+r^2-2rt)$. It is easy to see that $h_3(t) \le 0$ for all $t \in [-1, t_0]$. Therefore the function h_2 is monotonically decreasing in $[-1, t_0]$.

Consider now the case $r_0 \le r \le r_1$ and $t \in [-1, t_0]$. Then $h_1(\zeta_1^2) \le h_1((\zeta_1^*)^2)$, and $h_2(t) \le h_2(-1)$ for $t \in [-1, t_0]$. So

$$\begin{aligned}|H_{3,1}(f)| &\le \frac{1}{4} h_1(\zeta_1^2) \frac{\sqrt{2}}{\sqrt{1+r^2}} \\ &\le \frac{1}{4} h_1((\zeta_1^*)^2) \frac{\sqrt{2}}{\sqrt{1+r^2}} \\ &= \frac{1}{4} h_2(t)(k_1 + k_2^2(\zeta_1^*)^2) \frac{\sqrt{2}}{\sqrt{1+r^2}} \\ &\le \frac{1}{4} h_2(-1)(k_1 + k_2^2(\zeta_1^*)^2) \frac{\sqrt{2}}{\sqrt{1+r^2}} \\ &= \frac{1}{4}\left(1 - \frac{1 - 2\sqrt{2(1+r^2)} + 2(1+r^2)}{r^2(1+r)^2}\right)^2 \\ &\qquad \times \left(2 - r^4 - 2\sqrt{2(1+r^2)} + 2(1+r^2)\right) \frac{\sqrt{2}}{\sqrt{1+r^2}}.\end{aligned}$$

A numerical calculation shows that the last expression is less than $1/4$ provided $0.274\ldots < r \le 1$ and so for $r \in [r_0, r_1]$.

Next we consider the case $r_1 \le r \le 1$ and $t \in [t_1, t_0]$. Then $h_2(t) \le h_2(t_1)$ for $t \in [t_1, t_0]$. Therefore we have

$$\begin{aligned}|H_{3,1}(f)| &\le \frac{1}{4} h_2(t_1)(k_1 + k_2^2(\zeta_1^*)^2) \frac{\sqrt{2}}{\sqrt{1+r^2}} \\ &= \frac{\sqrt{2}\left(4 + 2r^2 - r^4 - 2\sqrt{2(1+r^2)}\right)^3}{\left(11 + 6r^2 - 2r^4 - 6\sqrt{2(1+r^2)}\right)^2 \sqrt{1+r^2}}.\end{aligned}$$

A similar numerical calculation shows that the last expression is less than $1/4$ provided $0.718\ldots < r \le 1$ and so for $r \in [r_1, 1]$.

B(ii) Next we consider the case $r_1 \le r \le 1$ and $t \in [-1, t_1]$. Define $h_4 : [1 + r^2 - 2rt_1, 1 + r^2 + 2r] \to \mathbb{R}$ by

$$h_4(s) = s^{-2}(1 - r^4 + r^2 s)^3.$$

Then

$$h_4'(s) = s^{-3}(1 - r^4 + r^2 s)^2(-2 + 2r^4 + r^2 s).$$

Since $-2 + 2r^4 + r^2 s > 0$ for $s \in [1 + r^2 - 2rt_1, 1 + r^2 + 2r]$, h_4 is increasing on $[1+r^2-2rt_1, 1+r^2+2r]$, and $h_4(s) \le h_4((1+r)^2)$ for $s \in [1+r^2-2rt_1, 1+r^2+2r]$. Thus we obtain

$$\begin{aligned}|H_{3,1}(f)| &\le \frac{1}{4} h_1(x_1) \frac{\sqrt{2}}{\sqrt{1+r^2}} \\ &= \frac{\sqrt{2}}{27r^4\sqrt{1+r^2}} h_4(1 + r^2 - 2rt)\end{aligned}$$

$$\leq \frac{\sqrt{2}}{27r^4\sqrt{1+r^2}} h_4\left((1+r)^2\right)$$
$$= \frac{\sqrt{2}\left(1-r^4+r^2(1+r)^2\right)^3}{27r^4(1+r)^4\sqrt{1+r^2}}.$$

Then as above we obtain $|H_{3,1}(f)| < 1/4$ for $r \in [r_1, 1]$.

We end the proof of Theorem 2.1 by showing that (6) is sharp. Define $p_1 \in \mathcal{P}$ by

$$p_1(z) = \frac{(1+z)[1+(x_0-y_0-x_0y_0-1)z+z^2]}{(1-z)[1+(1-x_0-y_0-x_0y_0)z+z^2]}.$$

Then p_1 is of the form (4) with

$$\begin{aligned} c_1 &= 2x_0, \\ c_2 &= 2\{-y_0+x_0^2(1+y_0)\}, \\ c_3 &= 2\{1-y_0^2+x_0^3(1+y_0)^2-x_0y_0(2+y_0)+x_0^2(-1+y_0^2)\}, \\ c_4 &= 2\{2(y_0+y_0^2-y_0^3-4x_0^2y_0(1+y_0)-2x_0(-1+y_0)(1+y_0)^2 \\ &\quad +2x_0^3(-1+y_0)(1+y_0)^2+x_0^4(1+y_0)^3\}. \end{aligned}$$

Now consider $f_1 \in \mathcal{S}_S^*$ defined by

$$\frac{2zf_1'(z)}{f_1(z)-f_1(-z)} = p_1(z).$$

Equating coefficients we obtain

$$\begin{aligned} a_2 &= x_0, \\ a_3 &= -y_0+x_0^2(1+y_0), \\ a_4 &= \frac{1}{2}\{1-y_0^2-x_0y_0(3+y_0)+x_0^2(-1+y_0^2)+x_0^3(2+3y_0+y_0^2)\}, \\ a_5 &= \frac{1}{2}\{y_0+2y_0^2-y_0^3-6x_0^2y_0(1+y_0)-2x_0(-1+y_0)(1+y_0)^2 \\ &\quad +2x_0^3(-1+y_0)(1+y_0)^2-x_0^4(1+y_0)^2(2+y_0)\}. \end{aligned}$$

Then

$$H_{3,1}(f) = \frac{1}{4}(1-x_0^2)^2(1+y_0)\{-1+y_0-2x_0y_0+(-1+x_0^2)y_0^2+(1+x_0)^2y_0^3\} = -\alpha.$$

□

We next show that the inequality $|H_{3,1}(f)| \leq \alpha$ can be improved when the coefficients of $f(z)$ are real.

Theorem 2.2. *If $f \in \mathcal{S}_S^*$ and is given by* (1), *and the coefficients of f are real, then*

$$-\alpha \leq H_{3,1}(f) \leq \frac{1}{3\sqrt{3}}. \qquad (17)$$

The inequalities are sharp.

Proof. Since the terms in (9) are real, we write $\zeta_1 = t$, $\zeta_2 = x$, $\zeta_3 = y$, and $\zeta_4 = w$, so that $t \in [0,1]$ and $x, y, w \in [-1,1]$ and obtain

$$\begin{aligned} H_{3,1}(f) &= \frac{1}{4}(1-t^2)^2(1-x)\{t^2(1-x)x^2 \\ &\quad + 2tx(1-x^2)y - (1+x)(1+x^2)y^2 + 2x(1+x)(1-y^2)w\} \\ &=: \Lambda(t,x,y,w). \end{aligned} \tag{18}$$

Since the coefficients of the extreme function for the lower bound in Theorem 2.1 are real, it is enough to establish the upper bound in (17).

(a) First assume that $x \geq 0$ and for fixed $t \in [0,1]$ and $x \in [0,1]$, define

$$G_1(y) = x(2 + (2+t^2)x - t^2x^2) - 2tx(-1+x^2)y - (1+x)^3y^2.$$

Let

$$y_1 = \frac{tx(1-x)}{(1+x)^2}.$$

Then $0 \leq y_1 \leq 1$, and G_1 has its unique critical point at $y = y_1$, and so

$$G_1(y) \leq G_1(y_1) = \frac{2x\{1 + (2+t^2)x + (1-t^2)x^2\}}{1+x}, \quad y \in [0,1]. \tag{19}$$

Therefore by (18) and (19), we have

$$\begin{aligned} \Lambda(t,x,y,w) &\leq \Lambda(t,x,y,1) \\ &= \frac{1}{4}(1-t^2)^2(1-x)G_1(y) \\ &\leq \frac{2x(1-t^2)^2(1-x)\{1+(2+t^2)x+(1-t^2)x^2\}}{4(1+x)} \\ &=: F_1(t,x). \end{aligned} \tag{20}$$

It is easy to see that F_1 is decreasing with respect to $t \in [0,1]$, and so

$$F_1(t,x) \leq F_1(0,x) = \frac{1}{2}x(1-x^2) \leq \frac{1}{3\sqrt{3}}, \quad (t,x) \in [0,1] \times [0,1].$$

Thus from (18) and (20), we obtain $H_{3,1}(f) \leq \frac{1}{3\sqrt{3}}$.

(b) We next assume that $x \leq 0$.

For fixed $t \in [0,1]$ and $x \in [-1,0]$, define

$$G_2(y) = -x(2 + (2-t^2)x + t^2x^2) + 2(1-x^2)txy - (1-x)^2(1+x)y^2.$$

Let

$$y_2 = \frac{tx}{1-x}.$$

Then $-1 \leq y_2 \leq 0$ and G_2 has its unique critical point at $y = y_2$. Therefore

$$G_2(y) \leq G_2(y_2) = -2x[1 + (1-t^2)x], \quad y \in [-1,1]. \tag{21}$$

Since $x \le 0$, from (18) and (21) we have

$$\begin{aligned}\Lambda(t,x,y,w) &\le \Lambda(t,x,y,-1)\\ &= \frac{1}{4}(1-t^2)^2(1-x)G_2(y)\\ &\le -\frac{1}{2}x(1-t^2)^2(1-x)[1+(1-t^2)x]\\ &=: F_2(t,x).\end{aligned}$$

It is easy to see that F_2 does not have any critical points in $(0,1)\times(-1,0)$, and also the following hold for $t\in[0,1]$ and $x\in[-1,0]$.

$$F_2(0,x) = -\frac{1}{2}x(1-x^2) \le \frac{1}{3\sqrt{3}}, \quad F_2(1,x)=0,$$

$$F_2(t,-1) = t^2(1-t^2)^2 \le \frac{4}{27} < \frac{1}{3\sqrt{3}}, \quad F_2(t,0)=0.$$

Thus $F_2(t,x) \le \frac{1}{3\sqrt{3}}$ holds for all $(t,x)\in[0,1]\times[-1,0]$.

Hence from (a) and (b) we have $H_{3,1}(f) \le \frac{1}{3\sqrt{3}}$, which establishes the upper bound in (17).

To see that the upper bound is sharp consider $f_2 \in \mathcal{S}_S^*$ defined by

$$f_2(z) = \frac{z}{(1+z^2)^{\frac{3-\sqrt{3}}{6}}(1-z^2)^{\frac{3+\sqrt{3}}{6}}}, \quad z\in\mathbb{D}.$$

Then f_2 is given by

$$f_2(z) = z + \frac{1}{\sqrt{3}}z^3 + \frac{2}{3}z^5 + \cdots, \quad z\in\mathbb{D},$$

which gives $H_{3,1}(f_2) = \frac{1}{3\sqrt{3}}$, and so the proof of Theorem 2.2 is complete. □

We next consider $H_{2,3}(f)$, and first prove the following.

Theorem 2.3. *If $f\in\mathcal{S}_S^*$ and is given by* (1)*, then*

$$|H_{2,3}(f)| \le 1. \tag{22}$$

The inequality is sharp.

Proof. First note from (8) that

$$H_{2,3}(f) = a_3a_5 - a_4^2 = \frac{1}{64}[4(c_2c_4 - c_3^2) + 4c_2(c_4 - c_1c_3) + c_2^2(4c_2 - c_1)].$$

So, we have

$$|H_{2,3}(f)| \le \frac{1}{64}[4|c_2c_4 - c_3^2| + 4|c_2||c_4 - c_1c_3| + |c_2|^2|4c_2 - c_1|]. \tag{23}$$

Also, as in the proof of Theorem 2.1, since $H_{2,3}(f)$ is rotationally invariant, we can assume that $c_1 = c\in[0,2]$. Then Lemma 1.1 implies that

$$|4c_2 - c_1^2| = |4\zeta_1^2 + 8(1-\zeta_1^2)\zeta_2| \le 4\zeta_1^2 + 8(1-\zeta_1^2) \le 8, \tag{24}$$

since $\zeta_1 \in [0,1]$ and $\zeta_2 \in \overline{\mathbb{D}}$. And Lemma 1.2 gives us the inequalities

$$|c_2c_4 - c_3^2| \le |c_2c_4 - c_6| + |c_6 - c_3^2| \le 4 \tag{25}$$

and

$$|c_4 - c_1c_3| \le 2. \tag{26}$$

Thus it follows from $|c_2| \le 2$, (23), (24), (25) and (26) that the inequality $|H_{2,3}(f)| \le 1$ holds.

Finally consider $f_3 \in \mathcal{S}_S^*$ defined by

$$\frac{2zf_3'(z)}{f_3(z) - f_3(-z)} = \frac{1+z^2}{1-z^2}, \quad z \in \mathbb{D}.$$

Then f_3 is given by

$$f_3(z) = z + z^3 + z^5 + \cdots, \quad z \in \mathbb{D}.$$

Then $H_{2,3}(f_3) = 1$, which shows the inequality (22) is sharp, and completes the proof of Theorem 2.3. □

Let $f_4 \in \mathcal{S}_S^*$ be defined by

$$\frac{2zf_4'(z)}{f_4(z) - f_4(-z)} = \frac{1-z^2}{1+z^2}, \quad z \in \mathbb{D}.$$

Then $H_{2,3}(f_4) = -1$. Since both f_3 and f_4 are functions in $\mathcal{S}_S^*$ with real coefficients, we deduce the following.

Theorem 2.4. *If $f \in \mathcal{S}_S^*$ and is given by* (1)*, then if the coefficients of f are real*

$$-1 \le H_{2,3}(f) \le 1.$$

The inequalities are sharp.

Remark 2.5. As was pointed out above, the sharp inequality $|H_{2,2}(f)| \le 1$ was proved in [12]. Using the same method as in the proof of Theorem 2.2, the following improvement can easily be proved when the coefficients of $f(z)$ are real. We note that the lower bound is still -1.

Theorem 2.6. *If $f \in \mathcal{S}_S^*$ and is given by* (1)*, then if the coefficients of f are real*

$$-1 \le H_{2,2}(f) \le \beta. \tag{27}$$

Here,

$$\beta = \frac{t_0(1-t_0)(8+4t_0-4t_0^2+t_0^3)}{8(2-t_0)} \approx 0.196715\ldots,$$

where $t_0 \approx 0.5900527\ldots$ is a zero of a polynomial

$$q(t) = 4t^5 - 25t^4 + 56t^3 - 44t^2 - 16t + 16. \tag{28}$$

The inequalities are sharp.

Proof. It is enough to establish the upper bound in (27). As in the proof of Theorem 2.2, we write $\zeta_1 = t$, $\zeta_2 = x$, $\zeta_3 = y$, so that $t \in [0,1]$ and x, $y \in [-1,1]$, and obtain

$$H_{2,2}(f) = \frac{1}{2}(1-t^2)\{t^2(-1+x)x - 2x^2 + (1-x^2)ty\} =: F(t,x,y). \tag{29}$$

Then

$$F(t,x,y) \le F(t,x,1) = \frac{1}{2}(1-t^2)g(t,x),$$

where

$$g(t,x) = t - t^2x + (-2-t+t^2)x^2.$$

Let $t \in [0,1]$ be fixed, and $h(x) = g(t,x)$. Since h has a unique critical point at $x = x_0$, where

$$x_0 = \frac{t^2}{2(-2-t+t^2)},$$

it follows from $h''(x_0) < 0$ and $-1 < x_0 < 1$ that

$$h(x) \le h(x_0) = t + \frac{t^4}{8+4t-4t^2}, \quad x \in [-1,1].$$

Hence

$$F(t,x,y) \le \frac{t(1-t)(8+4t-4t^2+t^3)}{8(2-t)} =: l(t). \tag{30}$$

Also $l'(t) = q(t)/(8(2-t)^2)$, where q is the polynomial defined by (28). Moreover, in $(0,1)$, $q(t) = 0$ has the unique root $t_0 \approx 0.590053\ldots$ and $l''(t_0) \approx -1.73247\ldots < 0$. Therefore we obtain

$$l(t) \le l(t_0) = \beta, \tag{31}$$

and the upper bound in (27) follows from (29), (30) and (31).

Finally note that equality holds in the upper bound in (27) for $f_5 \in \mathcal{S}_S^*$ defined by

$$\frac{2zf_5'(z)}{f_5(z)-f_5(-z)} = \frac{(1+z)(1+k_1z+z^2)}{(1-z)(1+k_2z+z^2)}, \quad z \in \mathbb{D}$$

with

$$k_1 = \frac{3t_0^2 - 6t_0 + 4}{2(-2+t_0)} \quad \text{and} \quad k_2 = \frac{-t_0^2 + 6t_0 - 4}{2(-2+t_0)},$$

which completes the proof. □

Acknowledgements. The authors would like to express their thanks to the referees for their valuable comments and suggestions. The first named author (N. E. Cho) was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2019R1I1A3A01050861).

References

[1] C. Carathéodory, *Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene Werte nicht annehmen*, Math. Ann. **64** (1907), no. 1, 95–115. https://doi.org/10.1007/BF01449883

[2] L. Carlitz, *Hankel determinants and Bernoulli numbers*, Tohoku Math. J. (2) **5** (1954), 272–276. https://doi.org/10.2748/tmj/1178245272

[3] N. E. Cho, B. Kowalczyk, O. S. Kwon, A. Lecko, and Y. J. Sim, *Some coefficient inequalities related to the Hankel determinant for strongly starlike functions of order alpha*, J. Math. Inequal. **11** (2017), no. 2, 429–439. https://doi.org/10.7153/jmi-11-36

[4] N. E. Cho, B. Kowalczyk, O. S. Kwon, A. Lecko, and Y. J. Sim, *The bounds of some determinants for starlike functions of order alpha*, Bull. Malays. Math. Sci. Soc. **41** (2018), no. 1, 523–535. https://doi.org/10.1007/s40840-017-0476-x

[5] J. H. Choi, Y. C. Kim, and T. Sugawa, *A general approach to the Fekete-Szegö problem*, J. Math. Soc. Japan **59** (2007), no. 3, 707–727. http://projecteuclid.org/euclid.jmsj/1191591854

[6] A. W. Goodman, *Univalent Functions*, Mariner, Tampa, Florida, 1983.

[7] W. K. Hayman, *On the second Hankel determinant of mean univalent functions*, Proc. London Math. Soc. (3) **18** (1968), 77–94. https://doi.org/10.1112/plms/s3-18.1.77

[8] W. Kaplan, *Close-to-convex schlicht functions*, Michigan Math. J. **1** (1952), 169–185. http://projecteuclid.org/euclid.mmj/1028988895

[9] B. Kowalczyk, A. Lecko, and Y. J. Sim, *The sharp bound for the Hankel determinant of the third kind for convex functions*, Bull. Aust. Math. Soc. **97** (2018), no. 3, 435–445. https://doi.org/10.1017/S0004972717001125

[10] B. Kowalczyk, A. Lecko, and D. K. Thomas, *The sharp bound of the third Hankel determinant for starlike functions*, Submitted for publication.

[11] O. S. Kwon, A. Lecko, and Y. J. Sim, *On the fourth coefficient of functions in the Carathéodory class*, Comput. Methods Funct. Theory **18** (2018), no. 2, 307–314. https://doi.org/10.1007/s40315-017-0229-8

[12] A. K. Mishra, J. K. Prajapat, and S. Maharana, *Bounds on Hankel determinant for starlike and convex functions with respect to symmetric points*, Cogent Math. **3** (2016), Art. ID 1160557, 9 pp. https://doi.org/10.1080/23311835.2016.1160557

[13] R. Ohno and T. Sugawa, *Coefficient estimates of analytic endomorphisms of the unit disk fixing a point with applications to concave functions*, Kyoto J. Math. **58** (2018), no. 2, 227–241. https://doi.org/10.1215/21562261-2017-0015

[14] Ch. Pommerenke, *On the coefficients and Hankel determinants of univalent functions*, J. London Math. Soc. **41** (1966), 111–122. https://doi.org/10.1112/jlms/s1-41.1.111

[15] Ch. Pommerenke, *On the Hankel determinants of univalent functions*, Mathematika **14** (1967), 108–112. https://doi.org/10.1112/S002557930000807X

[16] V. Ravichandran and S. Verma, *Bound for the fifth coefficient of certain starlike functions*, C. R. Math. Acad. Sci. Paris **353** (2015), no. 6, 505–510. https://doi.org/10.1016/j.crma.2015.03.003

[17] K. Sakaguchi, *On a certain univalent mapping*, J. Math. Soc. Japan **11** (1959), 72–75. https://doi.org/10.2969/jmsj/01110072

[18] I. J. Schoenberg, *On the maxima of certain Hankel determinants and the zeros of the classical orthogonal polynomials*, Indag. Math. **21** (1959), 282–290.

Nak Eun Cho
Department of Applied Mathematics
Pukyong National University
Busan 48513, Korea
Email address: necho@pknu.ac.kr

Young Jae Sim
Department of Applied Mathematics
Kyungsung University
Busan 48434, Korea
Email address: yjsim@ks.ac.kr

Derek K. Thomas
Department of Mathematics
Swansea University Bay Campus
Swansea SA1 8EN, United Kingdom
Email address: d.k.thomas@swansea.ac.uk