

\textbf{∆-TRANSITIVITY FOR SEMIGROUP ACTIONS}

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Abstract. In this paper, we study ∆-transitivity, ∆-weak mixing and ∆-mixing for semigroup actions and give several characterizations of them, which generalize related results in the literature.

1. Introduction

By a (topological) dynamical system, we mean a pair \((X, g)\), where \(X\) is a compact metric space and \(g : X \rightarrow X\) is a continuous map.

The study of transitive systems and its classification plays a big role in topological dynamics. Many authors have done much work in classifying transitive systems by their recurrence properties. A dynamical system \((X, g)\) is ∆-transitive if for every \(d \geq 2\) there exists a residual subset \(X_0\) of \(X\) such that for every \(x \in X_0\) the diagonal \(d\)-tuple \(x^{(d)} = (x, x, \ldots, x)\) has a dense orbit under the action \(g \times g^2 \times \cdots \times g^d\). In [9], Moothathu showed that ∆-transitivity implies weakly mixing, but there exists some strongly mixing systems which are not ∆-transitive. He also pointed out that multi-transitivity, weakly mixing and ∆-transitivity were equivalent for a minimal homeomorphism. In [7], Kwietniak and Oprocha extended this result to a non-invertible case. Using a class of Furstenberg families introduced in [3], Chen, Li and Lü in [4] characterized the entering time sets of transitive points into open sets in multi-transitive and ∆-transitive systems, answering several problems proposed in [7].

Recall that a dynamical system \((X, g)\) is ∆-weakly mixing if the product system \((X \times X, g \times g)\) is ∆-transitive. In [5], the authors showed that this kind of ∆-weakly mixing is in fact equivalent to ∆-transitivity, and then ∆-weakly mixing shares similar properties of ∆-transitivity.

A dynamical system \((X, g)\) is ∆-mixing if for every \(m \in \mathbb{N}\) and infinite subset \(A \subset \mathbb{N}\), there exists a residual subset \(X_0\) of \(X\) such that for every \(x \in X_0\), \(\{(g^n x, g^{2n} x, \ldots, g^{mn} x) : n \in A\}\) is dense in \(X^m\). In [9], Moothathu gave some relations among multi-transitivity, ∆-transitivity and ∆-mixing.

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In the past years, much attention has been paid to the research of dynamical systems under general semigroup actions, see [2,6,13–15] and references therein. In [16], we studied multi-transitivity and \( \Delta \)-transitivity for semigroup actions, while giving several characterizations of them. In [16], by a \( G \)-system, we mean a triple \( (X, G, \pi) \), where \( G \) is a discrete semigroup, \( X \) is a Polish space (i.e., a complete metrizable space) and
\[
\pi : G \times X \to X, \quad (g, x) \mapsto gx
\]
is a continuous action on \( G \times X \) with the property that \( \pi(g_1, \pi(g_2, x)) = \pi(g_1 g_2, x) \) for all \( x \in X \) and \( g_1, g_2 \in G \). Usually, we write the \( G \)-system as a pair \( (X, G) \). We define \( \Delta \)-transitivity for the system \( (X, G) \) in [16] as follows: the system \( (X, G) \) is \( \Delta \)-transitive if for every \( n \in \mathbb{N} \), there exists a residual subset \( X_0 \) of \( X \) such that for each \( x \in X_0 \), \( \{gx, g^2x, \ldots, g^nx : g \in G\} \) is dense in \( X^n \).

In this paper we define a more powerful definition of \( \Delta \)-transitivity under semigroup actions and attain stronger consequences.

In [1], Blanchard and Huang defined a local version of weak mixing, so called weakly mixing set, and proved that positive topological entropy implies the existence of weakly mixing sets. In [10–12], Oprocha and Zhang also discussed local versions of weak mixing extensively. In [5], the authors studied the property of \( \Delta \)-weakly mixing and showed that a topological dynamical system with positive topological entropy has many \( \Delta \)-weakly mixing subsets. Recently in [8], Liu extended the results in [5] to countable torsion-free discrete nilpotent group actions.

Recall that a group is torsion-free if any element has infinite order except the identity element. Liu defined \( \Delta \)-transitivity in [8] as follows: Let \( (X, G) \) be a \( G \)-system, where \( G \) is a countable torsion-free discrete group with the unit \( e \) and \( |X| \geq 2 \). We say that \( (X, G) \) is \( \Delta \)-transitive provided that there is a residual subset \( A \) of \( X \) such that for any \( x \in A \), \( d \geq 1 \) and pairwise distinct \( T_1, T_2, \ldots, T_d \in G \setminus \{e\} \), the orbit closure of the \( d \)-tuple \( (x, x, \ldots, x) \) under the action \( T_1 \times T_2 \times \cdots \times T_d \) contains \( X_d \), i.e.,
\[
\{(T_1^n x, T_2^n x, \ldots, T_d^n x) : n \in \mathbb{N}\} = X^d.
\]

Through the above ideas and results, we find that the definition of \( \Delta \)-transitivity in [8] is a more general characterization of this property. There is a close connection between this definition of \( \Delta \)-transitivity and other properties (such as topological entropy) in dynamical systems. In this paper, we define \( \Delta \)-transitivity in a similar way as in [8] for semigroup actions. Our aim is to give characterizations of \( \Delta \)-transitivity under semigroup actions of this notion and find some relations among \( \Delta \)-transitivity, \( \Delta \)-weakly mixing and \( \Delta \)-mixing. Besides, we study local version of \( \Delta \)-transitivity, and show that if \( G \) is abelian, a \( G \)-system is \( \Delta \)-transitive if and only if it is \( \Delta \)-weakly mixing while it is no longer true for \( \Delta \)-transitive subsets and \( \Delta \)-weakly mixing sets.
This paper is organized as follows. In Section 2, we introduce some notions and results which will be used later. In Section 3, we study $\Delta$-transitivity and $\Delta$-mixing for semigroup actions. We give some characterizations of $\Delta$-transitivity and discuss some properties of $\Delta$-mixing for general semigroup actions. We give some characterizations of $\Delta$-mixing and results which will be used later. In Section 3, we study $\Delta$-transitivity.

2. Preliminaries

In this paper, let $\mathbb{N}$, $\mathbb{Z}_+$ and $\mathbb{Z}$ denote the set of all positive integers, non-negative integers and integers, respectively. The cardinality of a set $B$ is denoted by $|B|$.

A subset $P \subset \mathbb{N}$ is thick if it contains arbitrarily long blocks of consecutive positive integers, that is, for every $n \geq 1$ there is $m_n \in \mathbb{N}$ such that $\{m_n, m_n + 1, \ldots, m_n + n\} \subset P$. A subset $P$ of $\mathbb{N}$ is syndetic if it has a bounded gap, that is, there is $N \in \mathbb{N}$ such that $\{n, n + 1, \ldots, n + N\} \cap P \neq \emptyset$ for every $n \in \mathbb{N}$. Let $\mathcal{A}$ be a collection of $\mathbb{N}$ with $\mathcal{A} \neq \emptyset$ and $\emptyset \notin \mathcal{A}$. If for every $A_1, A_2 \in \mathcal{A}$ there exists $A \in \mathcal{A}$ such that $A \subset A_1 \cap A_2$, then we say $\mathcal{A}$ is a filter base.

Let $G$ be a countable discrete semigroup with the identity $e$. By a $G$-system, we mean a pair $(X, G)$, where $X$ is a compact metric space and there exists a continuous map $\phi : G \times X \to X$, $(g, x) \mapsto gx$ such that $\phi(e, x) = x$ and $\phi(h, \phi(gx)) = \phi(hg, x)$ for all $g, h \in G$ and $x \in X$. For a $G$-system and $m \in \mathbb{N}$, $(X^m, G)$ is also a $G$-system, where $g(x_1, x_2, \ldots, x_m) := (gx_1, gx_2, \ldots, gx_m)$ for any $(x_1, x_2, \ldots, x_m) \in X^m$ and $g \in G$. A semigroup $G$ is abelian if $g_1g_2 = g_2g_1$ for all $g_1, g_2 \in G$. For a point $x \in X$, the orbit of $x$ is the set $Gx := \{gx : g \in G\}$. For a point $x \in X$ and a subset $U$ of $X$, we define the hitting time set of $x$ into $U$ by

$$N(x, U) = \{g \in G : gx \in U\}.$$  

For two subsets $U$ and $V$ of $X$, we define the hitting time set of $U$ and $V$ by

$$N(U, V) = \{g \in G : U \cap g^{-1}V \neq \emptyset\}.$$  

When $G$ is the semigroup $(\mathbb{Z}_+, +)$, let $g : X \to X, x \mapsto \phi(1, x)$. Then $\phi$ can be generated by $g$. In this case we denote the dynamical system by $(X, g)$.

A dynamical system $(X, G)$ is called transitive if for every non-empty open subsets $U$ and $V$ of $X$, there exists $g \in G$ such that

$$U \cap g^{-1}V \neq \emptyset.$$  

weakly mixing if $(X \times X, G)$ is transitive; strongly mixing if for every non-empty open subsets $U$ and $V$ of $X$, there exists a finite subset $F$ of $G$ such that

$$U \cap g^{-1}V \neq \emptyset, \quad \forall g \in G \setminus F.$$  

Let $G$ be a semigroup and $\mathcal{P}$ denote the collection of all subsets of $G$. A subset $F$ of $\mathcal{P}$ is called a Furstenberg family over $G$ (or just a family over $G$), if it is hereditary upward, i.e., $F_1 \subset F_2 \subset G$ and $F_1 \in \mathcal{F}$ imply $F_2 \in \mathcal{F}$.

Let $(X, G)$ be a dynamical system and $\mathcal{F}$ be a Furstenberg family over $G$. The system $(X, G)$ is called $\mathcal{F}$-transitive if for every two non-empty open
subsets $U$ and $V$ of $X$, the hitting time set $N(U, V) \in \mathcal{F}$. We say that a point $x \in X$ is $\mathcal{F}$-transitive if for every non-empty open subset $U \subset X$, $N(x, U) \in \mathcal{F}$. The collection of $\mathcal{F}$-transitive points is denoted by $\text{Trans}_\mathcal{F}(X, G)$.

**Convention.** Unless otherwise specified, in the statement of our results we will assume that $X$ has no isolated point and $G$ is infinite.

3. $\Delta$-transitivity

In this section, we define $\Delta$-transitivity for general semigroup actions and derive some properties of $\Delta$-transitivity.

**Definition 3.1.** Let $A$ be a subset of $\mathbb{N}$. We say that $(X, G)$ is $\Delta$-$A$-transitive if for any $d \geq 1$ and pairwise distinct $g_1, \ldots, g_d \in G \setminus \{e\}$, there exists a dense subset $Y$ of $X$ such that for any $x \in Y$

$$\{(g_1^n x, g_2^n x, \ldots, g_d^n x) : n \in A\}$$

is dense in $X^d$. If $A = \mathbb{N}$, we say that $(X, G)$ is $\Delta$-transitive briefly.

Let $(X, G)$ be a $G$-system. For any $d \in \mathbb{N}$, pairwise distinct $g_1, \ldots, g_d \in G \setminus \{e\}$, non-empty open subsets $U_0, U_1, \ldots, U_d$, we set

$$N(U_0; U_1, \ldots, U_d | g_1, \ldots, g_d) = \{n \in \mathbb{N} : U_0 \cap g_1^n U_1 \cap \cdots \cap g_d^n U_d \neq \emptyset\}.$$

The following proposition give some equivalent conditions of $\Delta$-$A$-transitivity, which extends the result in [9] for semigroup actions.

**Proposition 3.2.** Let $(X, G)$ be a $G$-system and $A \subset \mathbb{N}$. Then the following conditions are equivalent:

1. $(X, G)$ is $\Delta$-$A$-transitive;
2. for any $d \in \mathbb{N}$, pairwise distinct $g_1, \ldots, g_d \in G \setminus \{e\}$ and non-empty open subsets $U_0, U_1, \ldots, U_d$, $N(U_0; U_1, \ldots, U_d | g_1, \ldots, g_d) \cap A \neq \emptyset$.
3. for any $d \in \mathbb{N}$, pairwise distinct $g_1, \ldots, g_d \in G \setminus \{e\}$, there exists a residual subset $Y$ of $X$ such that for any $x \in Y$

$$\{(g_1^n x, g_2^n x, \ldots, g_d^n x) : n \in A\}$$

is dense in $X^d$.

**Proof.** (1) $\Rightarrow$ (2) Let $d \in \mathbb{N}$ and $g_1, \ldots, g_d$ be pairwise distinct elements of $G \setminus \{e\}$. Let $U_0, U_1, \ldots, U_d$ be non-empty open subsets of $X$. Choose $x \in U_0$ satisfying that

$$\{(g_1^n x, g_2^n x, \ldots, g_d^n x) : n \in A\}$$

is dense in $X^d$. Then there exists some $n \in A$ such that $(g_1^n x, g_2^n x, \ldots, g_d^n x) \in U_1 \times U_2 \times \cdots \times U_d$, that is

$$x \in U_0 \cap \bigcap_{i=1}^{d} g_i^{-n}(U_i),$$
and thus \( n \in N(U_0;U_1,\ldots,U_d \mid g_1,\ldots,g_d) \cap A. \)

(2) \( \Rightarrow \) (3) Let \( d \in N \) and \( g_1,\ldots,g_d \) be pairwise distinct elements of \( G \setminus \{e\}. \) Let \( \{B_k : k \in N\} \) be a countable base of open balls of \( X. \) Set

\[
Y = \bigcap_{(k_1,k_2,\ldots,k_d) \in N^d} \bigcup_{n \in N} g_i^{-n}(B_k).
\]

The set \( \bigcup_{n \in A} \bigcap_{i=1}^d g_i^{-n}(B_k) \) is clearly open, and it is dense by (2). So by Baire category theorem, \( Y \) is a dense \( G_δ \) subset of \( X. \) It is easy to see that for any \( x \in X \) the set \( \{ (g_1^n x, g_2^n x, \ldots, g_d^n x) : n \in A \} \) is dense in \( X^d \) if and only if \( x \in Y. \)

(3) \( \Rightarrow \) (1) It is obvious. \( \square \)

When \( G \) is abelian, the equivalent condition for \( \Delta \)-transitivity can be simpler, see the following lemma.

**Lemma 3.3.** Let \( (X,G) \) be a \( G \)-system with \( G \) abelian. Then \( (X,G) \) is \( \Delta \)-transitive if and only if for any \( d \in N \) and pairwise distinct \( g_1,\ldots,g_d \in G \setminus \{e\}, \) there exists a point \( x \in X \) such that

\[
\{ (g_1^n x, g_2^n x, \ldots, g_d^n x) : n \in N \}
\]

is dense in \( X^d. \)

**Proof.** The necessity is clear. Now we need to show the sufficiency. Fix \( d \in N. \) Let \( U_0, U_1,\ldots,U_d \) be non-empty open subsets of \( X. \) Choose \( h \in G \) such that \( y = hx \in U_0. \) Let \( g_1,\ldots,g_d \) be pairwise distinct elements of \( G \setminus \{e\}. \) As \( G \) is abelian, the set

\[
\{ (g_1^n y, g_2^n y, \ldots, g_d^n y) : n \in N \}
\]

is dense in \( X^d. \) Hence \( (g_1^n y, g_2^n y, \ldots, g_d^n y) \in U_1 \times \cdots \times U_d \) for some \( n \in N. \) Thus, \( y \in \bigcap_{i=1}^d g_i^{-n}(U_i). \) Then by Proposition 3.2, \( (X,G) \) is \( \Delta \)-transitive. \( \square \)

**Example 3.4.** Let \( X = \{0,1\}^\mathbb{Z} \) and \( G = \{\sigma^n : n = 0,1,2,\ldots\}. \) Then \( (X,G) \) is the full shift \( (\{0,1\}^\mathbb{Z}, \sigma) \). Since there is a point \( x \in \{0,1\}^\mathbb{Z} \) such that for every \( d \geq 2, \) the diagonal \( d \)-tuple \( (x,x,\ldots,x) \) has a dense orbit under the action \( \sigma \times \sigma^2 \times \cdots \times \sigma^d. \) Hence \( (X,G) \) is \( \Delta \)-transitive by Lemma 3.3.

We say that \( (X,G) \) is \( \Delta \)-\( A \)-weakly mixing if for every \( n \in N, \) \( (X^n,G) \) is \( \Delta \)-\( A \)-transitive. If \( A = N, \) then we say that \( (X,G) \) is \( \Delta \)-weakly mixing. Now we show that for a \( G \)-system and \( A \subset N, \) \( \Delta \)-\( A \)-transitive and \( \Delta \)-\( A \)-weakly mixing are equivalent with \( G \) abelian by proving the following proposition.

**Proposition 3.5.** Let \( (X,G) \) be a \( G \)-system with \( G \) abelian and \( A \subset N. \) Then the following conditions are equivalent:

1. \( (X,G) \) is \( \Delta \)-\( A \)-transitive.
(2) the collection of hitting time sets
\[ W := \{ N(U_0; U_1, \ldots, U_d \mid g_1, \ldots, g_d) \cap A : d \in \mathbb{N}, \]
\[ U_0, U_1, \ldots, U_d \text{ are non-empty open subsets of } X \text{ and } \]
\[ g_1, \ldots, g_d \text{ are pairwise distinct elements in } G \setminus \{e\} \}

is a filter base;
(3) \( (X,G) \) is \( \Delta\)-\( A \)-weakly mixing.

**Proof.** (1) \( \Rightarrow \) (2) By Proposition 3.2, for any \( d \in \mathbb{N} \), pairwise distinct \( g_1, \ldots, g_d \in G \setminus \{e\} \) and non-empty open subsets \( U_0, U_1, \ldots, U_d \), the set
\[ N(U_0; U_1, \ldots, U_d \mid g_1, \ldots, g_d) \cap A \neq \emptyset. \]
For any \( N(U_0; U_1, \ldots, U_d \mid g_1, \ldots, g_d) \cap A \) and \( N(V_0; V_1, \ldots, V_r | g_{d+1}, \ldots, g_{d+r}) \cap A \) in \( W \), choose \( g_0 \in G \setminus \{e\} \) such that \( g_1, \ldots, g_d, g_0, g_{d+1}g_0, \ldots, g_{d+r}g_0 \) are pairwise distinct elements in \( G \setminus \{e\} \).

Then for any \( n \in N(U_0; U_1, \ldots, U_d, V_0, V_1, \ldots, V_r \mid g_1, \ldots, g_d, g_0, g_{d+1}g_0, \ldots, g_{d+r}g_0) \cap A \),
\[ U_0 \cap g_1^{-n}U_1 \cap \cdots \cap g_d^{-n}U_d \cap g_0^{-n}V_0 \cap g_0^{-n}g_{d+1}^{-n}V_1 \cap \cdots \cap g_0^{-n}g_{d+r}^{-n}V_r \neq \emptyset. \]
Thus \( U_0 \cap g_1^{-n}U_1 \cap \cdots \cap g_d^{-n}U_d \neq \emptyset \) and \( V_0 \cap g_0^{-n}V_0 \cap \cdots \cap g_{d+r}^{-n}V_r \neq \emptyset \), which implies that \( n \in N(U_0; U_1, \ldots, U_d | g_1, \ldots, g_d) \cap N(V_0; V_1, \ldots, V_r | g_{d+1}, \ldots, g_{d+r}) \cap A \). Then we have
\[ N(U_0; U_1, \ldots, U_d, V_0, V_1, \ldots, V_r \mid g_1, \ldots, g_d, g_0, g_{d+1}g_0, \ldots, g_{d+r}g_0) \cap A \]
\[ \subset N(U_0; U_1, \ldots, U_d \mid g_1, \ldots, g_d) \cap N(V_0; V_1, \ldots, V_r | g_{d+1}, \ldots, g_{d+r}) \cap A. \]

(2) \( \Rightarrow \) (3) Let \( m, n \geq 1 \), \( g_1, g_2, \ldots, g_m \) be pairwise distinct elements of \( G \setminus \{e\} \) and \( U_{1,0}, U_{2,0}, \ldots, U_{n,0}, U_{1,1}, U_{2,1}, \ldots, U_{n,1}, \ldots, U_{1,m}, U_{2,m}, \ldots, U_{n,m} \) be non-empty open subsets of \( X \). It is easy to check that
\[ N(U_{1,0} \times \cdots \times U_{n,0}; U_{1,1} \times \cdots \times U_{n,1}, \ldots, U_{1,m} \times \cdots \times U_{n,m} | g_1, g_2, \ldots, g_m) \cap A \]
\[ = N(U_{1,0}; U_{1,1}, \ldots, U_{1,m} \mid g_1, g_2, \ldots, g_m) \cap \cdots \]
\[ \cap N(U_{n,0}; U_{n,1}, \ldots, U_{n,m} \mid g_1, g_2, \ldots, g_m) \cap A. \]

Since the collection of hitting time sets is a filter base, we know that the intersection above is not empty. By Proposition 3.2, \( (X^n,G) \) is \( \Delta\)-\( A \)-transitive.

(3) \( \Rightarrow \) (1) It is obvious. \( \square \)

We immediately have the following.

**Corollary 3.6.** Let \( (X,G) \) be a \( G \)-system with \( G \) abelian and \( A \subset \mathbb{N} \). \( (X,G) \) is \( \Delta\)-\( A \)-weakly mixing if and only if \( (X^n,G) \) is \( \Delta\)-\( A \)-weakly mixing for all \( n \in \mathbb{N} \).

We have the following observations about the hitting time sets of \( \Delta \)-transitive systems for abelian semigroup actions.
Proposition 3.7. Let \((X, G)\) be a \(G\)-system with \(G\) abelian. Then the following conditions are equivalent:

1. \((X, G)\) is \(\Delta\)-transitive;
2. for any \(d \in \mathbb{N}\), non-empty open subsets \(U_0, U_1, \ldots, U_d\) of \(X\) and pairwise distinct \(g_1, \ldots, g_d \in G \setminus \{e\}\), \(N(U_0; U_1, \ldots, U_d | g_1, \ldots, g_d)\) is thick of \(\mathbb{N}\);
3. for any syndetic subset \(A\) of \(\mathbb{N}\), \((X, G)\) is \(\Delta\)-\(A\)-transitive.

Proof. (3) \(\Rightarrow\) (1) It is obvious.

(1) \(\Rightarrow\) (2) Let \(d \in \mathbb{N}\). Let \(U_0, U_1, U_2, \ldots, U_d\) be non-empty open subsets of \(X\) and \(g_1, \ldots, g_d\) be pairwise distinct elements of \(G \setminus \{e\}\). By Proposition 3.5(2) with \(A = \mathbb{N}\), for any \(n \in \mathbb{N}\)

\[
N(U_0; U_1, \ldots, U_d | g_1, \ldots, g_d) \cap N(U_0; g_1^{-1}U_1, \ldots, g_d^{-1}U_d | g_1, \ldots, g_d) \\
\cap \cdots \cap N(U_0; g_1^{-n}U_1, \ldots, g_d^{-n}U_d | g_1, \ldots, g_d) \neq \emptyset.
\]

Thus \(N(U_0; U_1, \ldots, U_d | g_1, \ldots, g_d)\) is thick of \(\mathbb{N}\).

(2) \(\Rightarrow\) (3) Let \(A\) be a syndetic subset of \(\mathbb{N}\). Let \(d \in \mathbb{N}\), \(U_0, U_1, \ldots, U_d\) be non-empty open subsets of \(X\) and \(g_1, \ldots, g_d\) be pairwise distinct elements of \(G \setminus \{e\}\). Since the intersection of any thick set and a syndetic set is not empty, then \(N(U_0; U_1, \ldots, U_d | g_1, \ldots, g_d) \cap A \neq \emptyset\). By Proposition 3.2, this implies that \((X, G)\) is \(\Delta\)-\(A\)-transitive.

For \(n \geq 2\), we say that a non-empty subset \(F \subset G\) is an independent set for \((U_1, U_2, \ldots, U_n)\) if for every non-empty finite subset \(J \subset F\) and \(\sigma \in \{1, 2, \ldots, n\}^J\)

\[
\bigcap_{g \in J} g^{-1}U_{\sigma(g)} \neq \emptyset.
\]

We have the following characterization of \(\Delta\)-transitivity by independent sets of open sets.

Proposition 3.8. Let \((X, G)\) be a \(G\)-system with \(G\) abelian. If \((X, G)\) is \(\Delta\)-transitive, then for any \(d \geq 2\), non-empty open subsets \(U_1, \ldots, U_d\) of \(X\) and pairwise distinct \(g_1, \ldots, g_d\) of \(G \setminus \{e\}\), there exists \(n \in \mathbb{N}\) such that \(J := \{g_1^n, \ldots, g_d^n\}\) is an independent set for \((U_1, \ldots, U_d)\).

Proof. Fix \(d \geq 2\), non-empty open subsets \(V, U_1, \ldots, U_d\) of \(X\) and pairwise distinct \(g_1, \ldots, g_d\) of \(G \setminus \{e\}\), by Proposition 3.5(2) with \(A = \mathbb{N}\), there exists \(n \in \mathbb{N}\) such that

\[
n \in \bigcap \{N(V; U_{s_1}, \ldots, U_{s_d} | g_1, \ldots, g_d) : s_i \in \{1, \ldots, d\}, i = 1, \ldots, d\}.
\]

Now we show that \(J := \{g_1^n, \ldots, g_d^n\}\) is an independent set for \((U_1, \ldots, U_d)\). It is clear that for any \(\sigma \in \{1, 2, \ldots, d\}^J\)

\[
\bigcap_{i \in \{1, \ldots, d\}} (g_i^n)^{-1}U_{\sigma(g_i^n)} \neq \emptyset,
\]

which implies that \(J\) is an independent set for \((U_1, \ldots, U_d)\). □
Let \((X, G)\) be a \(G\)-system and \(a = (a_1, \ldots, a_r)\) be a vector in \(\mathbb{N}^r\). We say that the system \((X, G)\) is \(\Delta\)-\(a\)-transitive if there exists a dense subset \(X_0\) of \(X\) such that for each \(x \in X_0\), \(\{(g^{a_1}x, \ldots, g^{a_r}x) : g \in G\}\) is dense in \(X^r\). In [16], we defined a different \(\Delta\)-transitivity and characterized \(\Delta\)-\(a\)-transitivity by \(\mathcal{F}[a]\)-transitive points under semigroup actions. Here we obtain a similar result.

Let \(d \in \mathbb{N}\) and \(g_1, \ldots, g_d\) be pairwise distinct elements of \(G \setminus \{e\}\). We define the family generated by \(g = \{g_1, \ldots, g_d\}\), denoted by \(\mathcal{F}[g]\) as

\[
\{F \subseteq G : \forall h_1, h_2, \ldots, h_d \in G, \exists n \in \mathbb{N} \text{ s.t. } h_1g_1^n, h_2g_2^n, \ldots, h dg_d^n \in F\}.
\]

**Theorem 3.9.** Let \((X, G)\) be a \(G\)-system with \(G\) a group. Then \((X, G)\) is \(\Delta\)-transitive if and only if for any \(d \in \mathbb{N}\) and pairwise distinct elements \(g_1, \ldots, g_d\) of \(G \setminus \{e\}\), \(\text{Trans}_{\mathcal{F}[g]}(X, G)\) is residual in \(X\), where \(g := \{g_1, \ldots, g_d\}\).

**Proof.** Necessity. Let \(\{B_k : k \in \mathbb{N}\}\) be a countable base of open balls of \(X\). For any \(d \in \mathbb{N}\) and pairwise distinct elements \(g_1, \ldots, g_d\) of \(G \setminus \{e\}\), set

\[
Y = \bigcap_{(k_1, k_2, \ldots, k_d) \in \mathbb{N}^d} \bigcap_{n=1}^d g_i^{-n}(B_{k_i}).
\]

Since \((X, G)\) is \(\Delta\)-transitive, by Proposition 3.2 we obtain that \(Y\) is a dense \(G\_\Delta\) subset of \(X\). Now we only need to show that \(Y \subseteq \text{Trans}_{\mathcal{F}[g]}(X, G)\). Choose \(x \in Y\) and a non-empty open subset \(U\) of \(X\). Let \(H = \{h_1, h_2, \ldots, h_d\}\) be a finite subset of \(G\). Then there exists \((k_1, k_2, \ldots, k_d) \in \mathbb{N}^d\) such that

\[
B_{k_1} \times B_{k_2} \times \cdots \times B_{k_d} \subseteq h_1^{-1}U \times h_2^{-1}U \times \cdots \times h_d^{-1}U.
\]

By the construction of \(Y\), there exists \(n \in \mathbb{N}\) such that \(x \in \bigcap_{i=1}^d g_i^{-n}(B_{k_i})\), then \(g_i^n x \in B_{k_i} \subseteq h_i^{-1}U\) for \(i = 1, 2, \ldots, d\). Hence we obtain that \(\{h_1g_1^n, h_2g_2^n, \ldots, h dg_d^n\} \subseteq N(x, U)\) and \(x \in \text{Trans}_{\mathcal{F}[g]}(X, G)\).

Sufficiency. Let \(d \in \mathbb{N}\). For any non-empty open subsets \(U_0, U_1, U_2, \ldots, U_d\) of \(X\) and pairwise distinct elements \(g_1, \ldots, g_d\) of \(G \setminus \{e\}\), there exists \(x \in U_0\) which is a \(\mathcal{F}[g]\)-transitive point where \(g = \{g_1, \ldots, g_d\}\). Thus there exist \(h_1, \ldots, h_d\) such that \(h_i x \in U_i, i = 1, \ldots, d\). By the continuity of \(h_i\), there is a neighborhood \(U\) of \(x\) such that \(h_i U \subseteq U_i, i = 1, \ldots, d\). By the definition of \(\mathcal{F}[g]\)-transitive point, there is \(n \in \mathbb{N}\) such that

\[
(h_1^{-1}g_1^n x, h_2^{-1}g_2^n x, \ldots, h_d^{-1}g_d^n x) \in U \times \cdots \times U.
\]

Thus \(g_i^n x \subseteq U_i, i = 1, \ldots, d\). Therefore

\[
N(U_0; U_1, \ldots, U_d \mid g_1, \ldots, g_d) \neq \emptyset
\]

and by Proposition 3.2 \((X, G)\) is \(\Delta\)-transitive. \(\square\)

A factor map \(\pi : (X, G) \to (Y, G)\) between two \(G\)-systems is a continuous onto map satisfying that \(\pi \tau = \pi g\) for every \(g \in G\). We have the following.

**Proposition 3.10.** Let \(\pi : (X, G) \to (Y, G)\) be a factor map between two \(G\)-systems. If \((X, G)\) is \(\Delta\)-transitive, then so is \((Y, G)\).
Proof. Assume that \((X, G)\) is \(\Delta\)-transitive. Let \(d \in \mathbb{N}\). Let \(U_0, U_1, \ldots, U_d\) be non-empty open subsets of \(Y\) and \(g_1, \ldots, g_d\) be pairwise distinct elements of \(G \setminus \{e\}\). Since \((X, G)\) is \(\Delta\)-transitive, \(N(\pi^{-1}(U_0); \pi^{-1}(U_1), \ldots, \pi^{-1}(U_d) | g_1, \ldots, g_d) \neq \emptyset\). Hence \(N(U_0; U_1, \ldots, U_d | g_1, \ldots, g_d) \neq \emptyset\). That is, \((Y, G)\) is \(\Delta\)-transitive.

In [8], the author studied some properties of \(\Delta\)-transitive subsets and \(\Delta\)-weakly mixing subsets for countable torsion-free discrete group actions. If \(G\) is abelian, then by Proposition 3.5(2) with \(A = \mathbb{N}\) we know that a \(G\)-system is \(\Delta\)-transitive if and only if it is \(\Delta\)-weakly mixing. But this is no longer true for \(\Delta\)-transitive subsets and \(\Delta\)-weakly mixing sets, as the authors proved that they are not equivalent for \(\mathbb{Z}\)-system in [5]. Here we define the local notions of \(\Delta\)-transitivity and \(\Delta\)-weakly mixing for \(G\)-systems and obtain some similar results.

Definition 3.11. Let \((X, G)\) be a \(G\)-system and \(B\) be a closed subset of \(X\) with \(|B| \geq 2\).

1. we say that \(B\) is a \(\Delta\)-transitive subset of \((X, G)\) if there is a residual subset \(B_0\) of \(B\) such that for any \(x \in B_0\), \(d \geq 1\) and pairwise distinct \(g_1, \ldots, g_d \in G \setminus \{e\}\), the orbit closure of the \(d\)-tuple \((x, x, \ldots, x)\) under the action \(g_1 \times g_2 \times \cdots \times g_d\) contains \(B^d\), i.e.,

\[
\overline{\{(g_1^n x, g_2^n x, \ldots, g_d^n x) : n \in \mathbb{N}\}} \supseteq B^d.
\]

2. we say that \(B\) is a \(\Delta\)-weakly mixing subset of \((X, G)\) if \(B^m\) is a \(\Delta\)-transitive subset of \((X^m, G)\) for any \(m \in \mathbb{N}\).

We can easily see that \(X\) is a \(\Delta\)-transitive (resp. \(\Delta\)-weakly mixing) subset of \((X, G)\) if and only if the \(G\)-system \((X, G)\) is \(\Delta\)-transitive (resp. \(\Delta\)-weakly mixing).

The proof of the following two lemmas are similar with Lemma 3.3 and Proposition 3.4 of [8], we present the results without proof here.

Lemma 3.12. Let \((X, G)\) be a \(G\)-system and \(B\) be a closed subset of \(X\) with \(|B| \geq 2\). Then \(B\) is \(\Delta\)-transitive if and only if for any \(d \in \mathbb{N}\), pairwise distinct \(g_1, \ldots, g_d \in G \setminus \{e\}\) and non-empty open subsets \(U_0, U_1, \ldots, U_d\) of \(X\) with \(U_i \cap B \neq \emptyset\), \(i = 0, \ldots, d\),

\[
N(U_0 \cap B; U_1, \ldots, U_d | g_1, \ldots, g_d) \neq \emptyset.
\]

Lemma 3.13. Let \((X, G)\) be a \(G\)-system and \(B\) be a closed subset of \(X\) with \(|B| \geq 2\). Then \(B\) is a \(\Delta\)-weakly mixing subset of \(X\) if and only if for any \(d \in \mathbb{N}\), pairwise distinct \(g_1, \ldots, g_d \in G \setminus \{e\}\), non-empty open subsets \(U_1, \ldots, U_d\) and \(V_1, \ldots, V_d\) of \(X\) with \(U_i \cap B \neq \emptyset\) and \(V_i \cap B \neq \emptyset\) for \(i = 1, \ldots, d\), we have

\[
\bigcap_{\sigma \in \{1, \ldots, d\}^{d+1}} N(V_{\sigma(1)} \cap B; U_{\sigma(2)}, \ldots, U_{\sigma(d+1)} | g_1, \ldots, g_d) \neq \emptyset.
\]
We say that \((X, G)\) is \(\Delta\)-mixing if it is \(\Delta\)-\(A\)-transitive for any infinite subset \(A\) of \(\mathbb{N}\). We have the following equivalent condition about \(\Delta\)-mixing.

**Corollary 3.14.** Let \((X, G)\) be a \(G\)-system. Then \((X, G)\) is \(\Delta\)-mixing if and only if for any \(d \in \mathbb{N}\), pairwise distinct \(g_1, \ldots, g_d \in G \setminus \{e\}\), non-empty open subsets \(U_0, U_1, \ldots, U_d\)

\[\mathbb{N} \setminus N(U_0; U_1, \ldots, U_d \mid g_1, \ldots, g_d)\]

is finite.

**Proof.** Necessity. Assume that there exist \(d \in \mathbb{N}\), pairwise distinct \(g_1, \ldots, g_d \in G \setminus \{e\}\) and non-empty open subsets \(U_0, U_1, \ldots, U_d\) such that the set \(\mathbb{N} \setminus N(U_0; U_1, \ldots, U_d \mid g_1, \ldots, g_d)\) is infinite. Then there exists an infinite subset \(A := \mathbb{N} \setminus N(U_0; U_1, \ldots, U_d \mid g_1, \ldots, g_d)\) such that \(A \cap \mathbb{N}(U_0; U_1, \ldots, U_d \mid g_1, \ldots, g_d) = \emptyset\), which implies that \((X, G)\) is not \(\Delta\)-transitive by Proposition 3.2, contradiction.

Sufficiency. If there exists an infinite subset \(A\) such that \((X, G)\) is not \(\Delta\)-transitive, then by Proposition 3.2, there exist \(d \in \mathbb{N}\), pairwise distinct \(g_1, \ldots, g_d \in G \setminus \{e\}\) and non-empty open subsets \(U_0, U_1, \ldots, U_d\) such that \(A \cap \mathbb{N}(U_0; U_1, \ldots, U_d \mid g_1, \ldots, g_d) = \emptyset\), then \(\mathbb{N} \setminus N(U_0; U_1, \ldots, U_d \mid g_1, \ldots, g_d) \supset A\) is an infinite set, contradiction. \(\square\)

We immediately have the following statements.

**Corollary 3.15.** If \((X, G)\) is \(\Delta\)-mixing, then it is strongly mixing.

**Example 3.16.** Let \(X = \mathbb{R}^n/\mathbb{Z}^n\) \((n \geq 2)\) and \(G = SL(n, \mathbb{Z})\). Then it is the linear action of \(SL(n, \mathbb{Z})\) on the torus \(\mathbb{R}^n/\mathbb{Z}^n\), see Example 8 in [2]. Since the action is not strongly mixing, we obtain that \((X, G)\) is not \(\Delta\)-mixing by Corollary 3.15.

**Corollary 3.17.** \((X, G)\) is \(\Delta\)-mixing if and only if \((X^n, G)\) is \(\Delta\)-mixing for all \(n \in \mathbb{N}\).

**Corollary 3.18.** If \((X, G)\) is \(\Delta\)-mixing, then it is \(\Delta\)-\(A\)-weakly mixing for any infinite subset \(A\).

**Proof.** Suppose \((X, G)\) is \(\Delta\)-mixing, then \((X^n, G)\) is \(\Delta\)-mixing for all \(n \in \mathbb{N}\) by Corollary 3.17. Hence \((X^n, G)\) is \(\Delta\)-\(A\)-transitive for any infinite subset \(A\) of \(\mathbb{N}\). Then \((X, G)\) is \(\Delta\)-\(A\)-weakly mixing for any infinite subset \(A\). \(\square\)

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