Bull. Korean Math. Soc. **60** (2023), No. 2, pp. 339–348
https://doi.org/10.4134/BKMS.b220123
pISSN: 1015-8634 / eISSN: 2234-3016

RINGS AND MODULES WHICH ARE STABLE UNDER NILPOTENTS OF THEIR INJECTIVE HULLS

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Abstract. It is shown that every nilpotent-invariant module can be decomposed into a direct sum of a quasi-injective module and a square-free module that are relatively injective and orthogonal. This paper is also concerned with rings satisfying every cyclic right R-module is nilpotent-invariant. We prove that $R \cong R_1 \times R_2$, where R_1, R_2 are rings which satisfy R_1 is a semi-simple Artinian ring and R_2 is square-free as a right R_2-module and all idempotents of R_2 is central. The paper concludes with a structure theorem for cyclic nilpotent-invariant right R-modules. Such a module is shown to have isomorphic simple modules eR and fR, where e, f are orthogonal primitive idempotents such that $eRf \neq 0$.

1. Introduction

Recall that a module M is called automorphism-invariant if it is invariant under any automorphism of its injective hull [12] (see also, [7, 9, 11, 19]). Some properties of automorphism-invariant modules and the structure of rings via the class of automorphism-invariant modules are studied (see [1,5,10,15,16,18]). We notice that if f is a nilpotent endomorphism of $E(M)$ of a module M with $f^n = 0$ for some n, then $1 + f$ is an automorphism of $E(M)$, where $E(M)$ denotes the injective hull of the module M. So it is easy to see that if α is a nilpotent endomorphism of a module M, then $1 + \alpha$ is an automorphism of M. By this easy fact, a submodule N of M is said to be a nilpotent-invariant submodule of M if $\alpha(N) \leq N$ for all nilpotent elements α of $\mathrm{End}(M)$. A module is called a nilpotent-invariant module (or nil-invariant module) if it is a nilpotent-invariant submodule of its injective hull [8]. All automorphism-invariant modules are nilpotent-invariant but the converse is not true, in general (see [8, Example 2.2]).

The first section deals with some decompositions of nilpotent-invariant modules. We prove that if M is a nilpotent-invariant module, then M has a decomposition $M = X \oplus Y$ such that X is quasi-injective, Y is square-free, X

Received February 15, 2022; Accepted May 4, 2022.
2020 *Mathematics Subject Classification.* Primary 16D40, 16E50, 16D90.
Key words and phrases. Nilpotent-invariant module, automorphism-invariant module, square-free module, finite exchange property, full exchange property.

and Y are relatively injective and orthogonal (Theorem 2.2). Assume that M is a nonsingular nilpotent-invariant module with a decomposition $M = X \oplus Y$ as per-above mentioned theorem. Then for any submodules U, V of Y with $U \cap V = 0$, then $\mathrm{Hom}(U, V) = 0$, and $\mathrm{Hom}(X, Y) = \mathrm{Hom}(Y, X) = 0$ (Corollary 2.3). The next section discusses the sum of nilpotent-invariant modules and the finite/full exchange property of these modules. It is shown that: (1) If M is a nilpotent-invariant, nonsingular square-free module and $\{K_i\}_I$ is a family of closed submodules of M, then $\sum_I K_i$ is a nilpotent-invariant module (Theorem 2.4); (2) Assume that M is a nilpotent-invariant module with $S = \mathrm{End}(M)$.

(i) If M has the finite exchange property, then M has the full exchange property.
(ii) If M has the finite exchange property, then every element of S is sum of two units in S if and only if no factor ring of S is isomorphic to $\mathbb{Z}_2$.
(iii) S/Δ is right (C3), where $\Delta := \{f \in S \mid \mathrm{Ker}(f) \leq^e M\}$ (Theorem 2.5).

Section 3 deals with rings R over which every cyclic right R-module is nilpotent-invariant. We prove that $R \cong R_1 \times R_2$, where R_1, R_2 are rings which satisfy R_1 is a semi-simple Artinian ring, R_2 is square-free as a right R_2-module, and all idempotents of R_2 is central (Theorem 3.2).

Section 3 proves that a module M that has a decomposition $M = X \oplus Y$, where X is a semisimple module, Y is a square-free module, and X and Y are orthogonal if M satisfies one of the following conditions: (a) M is cyclic such that all factors are nilpotent-invariant and M generates its cyclic subfactors, or (b) M is a nilpotent-invariant module such that 2-generated subfactors are nilpotent-invariant (Theorem 3.3). This section concludes the section with a structure theorem for cyclic nilpotent-invariant right R-modules. Such a module is shown to have isomorphic simple modules eR and fR, where e, f are orthogonal primitive idempotents such that $eRf \neq 0$ (Theorem 3.6).

Throughout this article all rings are associative rings with unity, and all modules are right unital modules over a ring. We use $N \leq M$ ($N < M$) to mean that N is a submodule (respectively, proper submodule) of M, and we write $N \leq^e M$ and $N \leq^{\oplus} M$ to indicate that N is an essential submodule of M and N is a direct summand of M, respectively. $E(-)$ denotes the injective envelope for a module.

2. Some decompositions of nilpotent-invariant modules

Lee and Zhou [12] showed that an automorphism-invariant module M has a decomposition $M = A \oplus B$, where A and B are relatively injective. This also holds for nilpotent-invariant modules (see [8, Theorem 2.10(1)]).

A submodule K of an R-module M is called a closed submodule in M if K has no proper essential extension in M. Moreover, if L is any submodule of M, then there exists, by Zorn's Lemma, a submodule K of M maximal with respect to the property that L is an essential submodule of K, and in this case K is a closed submodule of M. For a submodule N of the module M, a closure of N (in M) is a submodule K of M which is maximal in the collection of submodules of M containing N as an essential submodule.

Lemma 2.1 ([18, Lemma 3.1])**.** *If M is a nilpotent-invariant module, A is a closed submodule of M and B is a submodule of M with $A \cap B = 0$, then A is B-injective. Moreover, for any monomorphism $h : A \to M$ with $A \cap h(A) = 0$, $h(A)$ is a closed submodule of M.*

Proof. Let C be a complement of A in M containing B. Then $C \oplus A \leq^e M$. Let $f : H \to A$ be a homomorphism with $H \leq C$. By [8, Theorem 2.12(1)], there exist a homomorphism $g : E(C) \to E(A)$ and a nilpotent endomorphism ϕ of $E(M)$ such that $\phi(M) \leq M$, $\phi|_C = g|_C$ and $\phi|_H = f$. Now $g(C) = \phi(C) \leq M \cap E(A) = A$, which implies that A is C-injective or A is B-injective.

Assume that $h : A \to M$ is a monomorphism and $A \cap h(A) = 0$. Let K be a closure of $h(A)$. Then $A \cap K = 0$. Therefore, A is K-injective and so there exists $k : K \to A$ such that k is an extension of $h^{-1} : h(A) \to A$. For all $a \in A$, we have $a = h^{-1}h(a) = kh(a)$. It follows that $h : A \to K$ is a split monomorphism and hence $h(A) = K$ is a closed submodule of M. □

A module is called square-free if it does not contain a direct sum of two nonzero isomorphic submodules. Two modules are said to be orthogonal to each other if they do not contain nonzero isomorphic submodules.

Theorem 2.2. *If M is a nilpotent-invariant module, then M has a decomposition $M = X \oplus Y$ such that X is quasi-injective, Y is square-free, X and Y are relatively injective and orthogonal.*

Proof. Let $\Gamma = \{(A \oplus B, \gamma) \mid A, B \leq M, A \overset{\gamma}{\cong} B\}$. We consider an order relation over Γ as follows:

$$(A_1 \oplus B_1, \gamma_1) \leq (A_2 \oplus B_2, \gamma_2) \;\Leftrightarrow A_1 \leq A_2, B_1 \leq B_2, \gamma_2|_{A_1} = \gamma_1.$$

By Zorn's Lemma, there exists a maximal element, say $(A \oplus B, \gamma)$. In addition, there exists a complement C of $A \oplus B$ in M. It follows that $E(M) = E(A) \oplus E(B) \oplus E(C)$ with $E(A) \cong E(B)$ and $M = (E(A) \cap M) \oplus (E(B) \cap M) \oplus (E(C) \cap M)$ by [8, Theorem 2.14].

It is easy to see that $C = E(C) \cap M$. We now show that $A = E(A) \cap M$ and $B = E(B) \cap M$. Note that $A \leq^e E(A) \cap M$ and $B \leq^e E(B) \cap M$. By [8, Theorem 2.10 (1)], $E(B) \cap M$ is $(E(A) \cap M)$-injective, there exists a homomorphism $\overline{\gamma} : E(A) \cap M \to E(B) \cap M$ such that $\overline{\gamma}|_A = \gamma$. Since $A \leq^e E(A) \cap M$ and φ is a monomorphism, $\overline{\gamma}$ is also a monomorphism. It is easy to see that B is a submodule of $\overline{\gamma}(E(A) \cap M)$ and $\theta : E(A) \cap M \to \overline{\gamma}(E(A) \cap M)$ is an isomorphism via $\theta(x) = \overline{\gamma}(x)$ for all $x \in E(A) \cap M$. Thus

$$[(A \oplus B, \gamma) \leq [(E(A) \cap M) \oplus \overline{\gamma}(E(A) \cap M), \theta].$$

By the maximality of $(A \oplus B, \gamma)$, we have $A = E(A) \cap M$ and $B = \overline{\gamma}(E(A) \cap M)$ which implies that $B = \overline{\gamma}(A)$ is a closed submodule of M by Lemma 2.1 or that $B = E(B) \cap M$. Thus $M = A \oplus B \oplus C$.

Since A and B are isomorphic and relatively injective, then $A \oplus B$ is quasi-injective. Furthermore, assume that there are nonzero submodules U, V of C such that $U \cap V = 0$ and $\alpha : U \to V$ is an isomorphism. Then

$$(A \oplus B, \gamma) \leq ((A \oplus U) \oplus (B \oplus V), \gamma \oplus \alpha).$$

It would contradict to the maximality of $(A \oplus B, \gamma)$. Thus C is square-free.

Let $u : U \to A \oplus B$ be a maximal monomorphism from $U \leq C$ to $A \oplus B$. Then there exist a closed submodule $\overline{U}$ of C with $U \leq^e \overline{U}$ and a monomorphism $\bar{u} : \overline{U} \to A \oplus B$ such that $\bar{u}|_U = u$. It follows that $U = \overline{U}$ is a closed submodule of M (since C is a closed submodule of M). Then by Lemma 2.1, $u(U)$ is also a closed submodule of $A \oplus B$. Since $A \oplus B$ is a quasi-injective module, $u(U)$ is a direct summand of $A \oplus B$. So $U \cong u(U)$ is quasi-injective. Therefore U is a direct summand of C, taking $C = U \oplus V$. Next, we show that V and $A \oplus B \oplus U$ are orthogonal. Indeed, there exist two non-zero submodules H and K with $H \leq V$ and $K \leq A \oplus B \oplus U$. Note that $C = U \oplus V$ is square-free, and so $K \cap U = 0$. Let $\pi : A \oplus B \oplus U \to A \oplus B$ be the projection. Then $H \cong K \cong K' = \pi(K) \leq A \oplus B$. We can obtain an isomorphism $\varphi : H \to K'$. Assume that $K' \cap u(U) \neq 0$. Then U and V contain two non-zero isomorphic submodules. Since C is square-free, it is a contradiction. So K' and $u(U)$ are orthogonal. It follows that $\varphi(H) \cap u(U) = 0$.

Now we consider the following map

$$\begin{array}{cccc} \phi & : & H \oplus U & \to & A \oplus B \\ & & x + y & \mapsto & \varphi(x) + u(y). \end{array}$$

It is easy to see that ϕ is a monomorphism and $\phi|_U = u$, this is a contradiction to the maximality of $u : U \to A \oplus B$. Taking $X = A \oplus B \oplus U$ and $Y = V$. Then $M = X \oplus Y$, X is quasi-injective, Y is square-free, X and Y are relatively injective and orthogonal. □

Corollary 2.3. *Assume that M is a nonsingular nilpotent-invariant module with a decomposition $M = X \oplus Y$ as in Theorem 2.2. Then*

(1) *For any submodules U, V of Y with $U \cap V = 0$, then* $\operatorname{Hom}(U, V) = 0$.
(2) $\operatorname{Hom}(X, Y) = \operatorname{Hom}(Y, X) = 0$.

Let M be a nonsingular square-free module. If M is automorphism-invariant, then, for any family $\{K_i\}_I$ of closed submodules of M, the submodule $\sum_I K_i$ is automorphism-invariant (see [3, Theorem 6]).

Theorem 2.4. *Assume that M is a nilpotent-invariant, nonsingular square-free module and $\{K_i\}_I$ is a family of closed submodules of M. Then $\sum_I K_i$ is a nilpotent-invariant module.*

Proof. Let $A = \sum_I K_i \leq M$. There exists $B \leq M$ such that $A \oplus B \leq^e M$ and so $E(M) = E(A) \oplus E(B)$. For any nilpotent endomorphism γ of $E(A)$, the map $\overline{\gamma} : E(M) \to E(M)$ defined by $\overline{\gamma}(x+y) = \gamma(x)$ for all $x \in E(A), y \in E(B)$, is a nilpotent homomorphism. Since M is nilpotent-invariant, $\overline{\gamma}(M) \leq M$. By [3, Theorem 6(i)], we have $\overline{\gamma}(K_i) \leq K_i$ for all $i \in I$. Thus $\gamma(A) \leq A$. □

A right R-module M has the $\mathcal{N}$-exchange property, for some cardinal $\mathcal{N} \geq 2$, if whenever there are two direct sum decompositions $A = M' \oplus N = \oplus_{\mathcal{N}} A_i$ with $M' \cong M$, there exist submodules B_i of A_i such that $A = M' \oplus (\oplus_{\mathcal{N}} B_i)$.

If M has the $\mathcal{N}$-exchange property for all cardinals $\mathcal{N}$ (respectively, all finite cardinals), then we say M has the full exchange property (respectively, the finite exchange property). A finitely generated module has the full exchange property if and only if it has the finite exchange property.

For any two direct summands A, B of a module M with $A \cap B = 0$, if the sum $A + B$ is a direct summand of M, then M is called (C3). By [8, Theorem 2.7], every nilpotent-invariant module is (C3).

For a module M, let $\Delta := \{f \in S \,|\, \mathrm{Ker}(f) \leq^e M\}$.

Theorem 2.5. *Let M be a nilpotent-invariant module and $S = \mathrm{End}(M)$.*

1. *If M has the finite exchange property, then M has the full exchange property.*
2. *If M has the finite exchange property, then every element of S is a sum two units in S if and only if no factor ring of S is isomorphic to $\mathbb{Z}_2$.*
3. *$S/\Delta(S)$ is right* (C3).

Proof. (1) Assume that M is a nilpotent-invariant module. By Theorem 2.2, we have $M = X \oplus Y$, where X is quasi-injective and Y is square-free. Since Y is square-free with the finite exchange property, Y has the full exchange property by [14, Theorem 9]. Otherwise, X is quasi-injective so X has the full exchange property. Now, by [4, Lemma 2.4], M has the full exchange property.

(2) Assume that no factor ring of S is isomorphic to $\mathbb{Z}_2$. By Theorem 2.2, $M = M_1 \oplus M_2$, where M_1 is quasi-injective, M_2 is square-free and M_1, M_2 are orthogonal. Let

$$\begin{aligned}
\Delta_1 &= \{f \in S_1 = \mathrm{End}(M_1) \,|\, \mathrm{Ker}(f) \leq^e M_1\}, \\
\Delta_2 &= \{f \in S_2 = \mathrm{End}(M_2) \,|\, \mathrm{Ker}(f) \leq^e M_2\}, \\
\overline{S} &= S/\Delta, \\
\overline{S_1} &= S_1/\Delta_1, \\
\overline{S_2} &= S_2/\Delta_2.
\end{aligned}$$

By [14, Lemma 3.3], $\overline{S} \cong \overline{S_1} \oplus \overline{S_2}$. Since M_1 is quasi-injective, $\overline{S_1}$ is regular and right self-injective by [14, Theorem 3.10]. Furthermore, since M_2 is square-free, it follows that $\overline{S_2}$ is an exchange ring with no non-zero nilpotent elements by [14, Theorem 3.12(1)]. By [6, Theorem 1], each element of $\overline{S_1}$ is a sum of two units. Since $\overline{S_2}$ has no non-zero nilpotent elements, each idempotent in $\overline{S_2}$ is central. Now, if any element $a \in \overline{S_2}$ is not a sum of two units, it is easy to find an ideal, say I, of $\overline{S_2}$ such that $x = a + I \in \overline{S_2}/I$ is not a sum of two units in $\overline{S_2}/I$ and $\overline{S_2}/I$ has no central idempotents. This implies that $\overline{S_2}/I$ is an exchange ring without any non-trivial idempotents, and hence it must be local. Let $T = \overline{S_2}/I$. Then $x + J(T)$ is not a sum of two units in $T/J(T)$ which is a division ring. Therefore, $T/J(T) \cong \mathbb{Z}_2$, a contradiction. Hence, every element of $\overline{S_2}$ is also a sum of two units. Therefore, every element of $\overline{S}$ is a sum of two units. Next, we observe that $\Delta \leq J(S)$. Suppose that $\Delta \not\leq J(S)$. Then Δ contains a non-zero idempotent, say e. But as $\mathrm{Ker}(e) \leq^e M$, $\mathrm{Ker}(e) = M$ and so $e = 0$, a contradiction. Thus $\Delta \leq J(S)$. Therefore, we may conclude that every element of S is a sum of two units.

The converse is obvious.

(3) By Theorem 2.2, we have composition $M = M_1 \oplus M_2$, where M_1 is square-free, M_2 is quasi-injective and M_1, M_2 are orthogonal. By the same notations in the proof of (2), we have $\overline{S} \cong \overline{S_1} \oplus \overline{S_2}$ by [14, Lemma 3.3]. Since M_2 is quasi-injective, $\overline{S_2}$ is regular by [14, Theorem 3.10], hence $\overline{S_2}$ has (C2). Let e, f be idempotents of $\overline{S_1}$ such that $\overline{eS_1} \cap \overline{fS_1} = 0$. Since e and f are central by [13, Lemma 3.4], $\overline{ef} = \overline{fe} \in \overline{eS_1} \cap \overline{fS_1} = 0$. Thus $\overline{e}$ and $\overline{f}$ are orthogonal idempotents, and $\overline{eS_1} \oplus \overline{fS_1}$ is a summand of $\overline{S_1}$. Hence $\overline{S_1}_{\overline{S_1}}$ satisfies (C3). Therefore $\overline{S}_{\overline{S}}$ satisfies (C3). □

A module M is called purely infinite if $M \cong M \oplus M$. Assume that M is a nilpotent-invariant module. By [8, Theorem 2.18], M is a purely infinite module if and only if $E(M)$ is a purely infinite module.

Proposition 2.6. *If M is a nilpotent-invariant module, then every purely infinite submodule of M is essential in a direct summand of M.*

Proof. Assume that N is a purely infinite submodule of M. Then $N = A_1 \oplus A_2$, where $A_1 \cong A_2 \cong N$. So $E(A_1) \cong E(A_2)$. Furthermore, because $E(M) = E(A_1) \oplus E(A_2) \oplus E(N')$ and by [8, Theorem 2.14], we have

$$M = (E(A_1) \cap M) \oplus (E(A_2) \cap M) \oplus (E(N') \cap M).$$

Since $A_1 \leq^e E(A_1) \cap M$ and $A_2 \leq^e E(A_2) \cap M$, it is easy to get that $N = A_1 \oplus A_2$ is essential in $(E(A_1) \cap M) \oplus (E(A_2) \cap M)$ which is a direct summand of M. □

3. Rings over which every cyclic module is nilpotent-invariant

The section starts by dealing with rings for which each cyclic module is nilpotent-invariant.

Example 3.1. (1) The ring $\mathbb{Z}$ of integer numbers over which every cyclic module is nilpotent-invariant.

(2) (Björk's Example) Let $\mathbb{F}$ be a field and assume that $\varphi : \mathbb{F} \to \bar{\mathbb{F}} \subseteq \mathbb{F}$ is an isomorphism defined by $a \mapsto \overline{a}$, where the subfield $\bar{\mathbb{F}} \neq \mathbb{F}$. Let R denote the left vector space on basis $\{1, t\}$, and make R into an $\mathbb{F}$-algebra by defining $t^2 = 0$ and $ta = \varphi(a)t$ for all $a \in F$. Note that R is a local ring and $J(R) = Rt = \mathbb{F}t$ is the only proper left ideal of R. Clearly, every left cyclic module is nilpotent-invariant.

Theorem 3.2. *Assume that every cyclic right R-module is nilpotent-invariant. Then $R \cong R_1 \times R_2$, where R_1, R_2 are rings satisfying the following properties*:

(1) *R_1 is a semi-simple Artinian ring.*
(2) *R_2 is square-free as a right R_2-module and all idempotents of R_2 are central.*

Proof. By the proof of Theorem 2.2, we have a decomposition $R_R = A \oplus B \oplus C$, where $A \cong B$, C is square-free and $A \oplus B$ and C are orthogonal. Let N be a submodule of A. Then, $R/N \cong A/N \oplus B \oplus C$ is nilpotent-invariant by assumption. By Lemma 2.1, A/N is B-injective. Note that $A \cong B$ whence A/Z is A-injective. Similarly, C and all factor modules of B are A-injective. Now, A is a cyclic projective module and all of whose factors are A-injective. By [2, Corollary 9.3(ii)], A is a direct sum of uniform modules. We write $A = X_1 \oplus X_2 \oplus \cdots \oplus X_n$, where X_i are uniform submodules of A. Let X be an arbitrary nonzero cyclic submodule of X_i for any i. Then X contains a nonzero factor, say X', of one of the factor modules of A, B and C. Clearly, X' is A-injective, so it is X_i-injective for any i and X-injective. We deduce that $X' = X = X_i$ which implies that each X_i is simple. Thus, $A \oplus B$ is a semisimple module. Since $A \oplus B$ and C are orthogonal projective modules and the former is now semisimple, there are no nonzero homomorphisms between them. It means that $A \oplus B$ and C are ideals of R. So $R = R_1 \oplus R_2$ with $R_1 = A \oplus B$ and $R_2 = C$.

Let e be an idempotent of R_2. We show that $eR(1-e) = 0$ and $(1-e)Re = 0$. Take $X = eR$ and $Y = (1-e)R$. Let $f : X \to Y$ be any homomorphism. Call $Y' = f(X)$. Then there exists an isomorphism $\overline{f} : X/K \to Y'$ with $K = \text{Ker}(f)$. It is easy to check that X/K is a closed submodule of R_2/K. Clearly K is essential in X since $(R_2)_{R_2}$ is square-free. Let U/K be a complement of $U/K \oplus (Y' \oplus K)/K$ in R_2/K. Since R_2/K is nilpotent-invariant by the assumption and $X/K \cong Y' \cong (Y' \oplus K)/K$, we obtain $(Y' \oplus K)/K$ is closed in R_2/K by the last part of the proof of Lemma 2.1. Applying [8, Theorem 2.14], we get $R_2/K = X/K \oplus (Y' \oplus K)/K \oplus U/K$. Since $Y' \cap (X+U) \leq Y' \cap K = 0$, we have $R_2 = Y' \oplus (X + U)$. It follows that Y'_{R_2} is projective, whence the above map f splits. On the other hand, since K is essential in X, we have $f = 0$. So, $\text{Hom}(X, Y) = 0$. Similarly, we have $\text{Hom}(Y, X) = 0$. In particular, $eR(1-e) = 0$ and $(1-e)Re = 0$. It shows that e is a central idempotent of R. □

In Theorem 2.2, we obtained a decomposition for a nilpotent-invariant module M such that $M = X \oplus Y$, where X is quasi-injective, Y is square-free, X and Y are relatively injective and orthogonal.

Theorem 3.3. *A right R-module M has a decomposition $M = X \oplus Y$, where X is a semisimple module, Y is a square-free module, and X and Y are orthogonal if M satisfies one of the following conditions*:

(1) *M is cyclic such that all factors are nilpotent-invariant, and generates its cyclic subfactors, or*

(2) *M is a nilpotent-invariant module such that 2-generated subfactors are nilpotent-invariant.*

Proof. We first note that M has a decomposition $M = A \oplus B \oplus C$, where $A \cong B$ and C is square-free and orthogonal to $A \oplus B$.

(1) By the proof of Theorem 3.2, all factors of the modules B ($\cong A$) and C are A-injective. Now let A' be any factor of A and D be a cyclic submodule of A'. Since D is generated by M, $D = D_1 + \cdots + D_n$, where each D_i is a factor of B, B' or C. Since D_1 is A-injective (whence D_1 is A'-injective), we have $D_1 \oplus D_1' = A'$ for some submodule D_1' of A'. Clearly, $D = D_1 \oplus (\pi(D_2) + \cdots + \pi(D_n))$, where each $\pi : D_1 \oplus D_1' \to D_1'$ is the canonical projection. Since each $\pi(D_k)$ is again a factor of B and C, it is A-injective, whence it is D_1'-injective. By induction on n, we obtain that D is a direct sum of A-injective cyclic modules. Hence D is A-injective. Now we have shown that each cyclic subfactor of A is A-injective. By [2, Corollary 7.14], A is semisimple. Therefore, $A \oplus B$ is semisimple. Now, the claim follows if we take $X = A \oplus B \oplus B'$ and $Y = C$.

(2) Let D and L be submodules of A such that $D \leq L$ and L/D is cyclic, and let T be a cyclic submodule of B. By the assumption, $L/D \oplus T$ is nilpotent-invariant, whence L/D is T-injective. Then, cyclic subfactors of A are B-injective, hence they are A-injective. Again, by [2, Corollary 7.14], A is semisimple. The rest of the proof follows in the same way as (1). □

We get the following lemma for using the following proofs.

Lemma 3.4. *Assume that $M = A \oplus B$ is a nilpotent-invariant module. If $\varphi : A \to B$ is a monomorphism, then $\varphi(A)$ is a direct summand of B.*

Proof. Suppose that $M = A \oplus B$ is a nilpotent-invariant module and $\varphi : A \to B$ is a monomorphism. Then, $A \cong \varphi(A)$ is B-injective by Lemma 2.1. Note that $\varphi(A)$ is a submodule of B. We deduce that $\varphi(A)$ is a direct summand of B. □

Lemma 3.5. *Assume that every cyclic right R-module is nilpotent-invariant. Let e be a primitive idempotent of R. If f is an idempotent of R which is orthogonal to e and if $eaf \neq 0$ for some $a \in R$, then $eR = eafR$.*

Proof. Let $r(ea) = \{x \in R : eax = 0\}$ denote the annihilator of ea in R. Call $I = r(ea) \cap fR$. We have the following isomorphisms

$$eafR \times eR \cong fR/I \times eR \cong (eR \oplus fR)/I \cong (e+f)R/I.$$

It means that $eafR \times eR$ is a cyclic right R-module. By our assumption, $eafR \times eR$ is a nilpotent-invariant module. Note that eR is an indecomposable module and $eafR \subset eR$. Thus, we must have $eR = eafR$ by Lemma 3.4. □

Theorem 3.6. *Assume that every cyclic right R-module is nilpotent-invariant. If e, f are orthogonal primitive idempotents such that $eRf \neq 0$, then eR and fR are isomorphic simple modules.*

Proof. By the assumption and Lemma 3.5, $eafR = eR$ for some $a \in R$. Hence $eafR$ is a projective module. Note that $eafR$ is a homomorphism image of fR. Since fR is indecomposable, we get $eR = eafR \cong fR$. We now show that eR is a minimal right ideal of R. Let $ea \in eR$ and $ea \neq 0$. If $ea(1-e) \neq 0$, then $ea(1-e)R = eR$ by Lemma 3.5. Otherwise, $eae = ea \neq 0$ and we get the following isomorphism

$$eaeR \times eR \cong eaeR \oplus fR = (eae + f)R.$$

By the hypothesis, $eaeR \times eR$ is nilpotent-invariant. By Lemma 3.4, $eaeR = eR$. Thus $eaeR = eR$ which implies that eR is minimal. □

Corollary 3.7. *If R is a semiperfect ring such that every cyclic right R-module is nilpotent-invariant, then $R \cong R_1 \times R_2$ with*

(1) $R_1 \cong \mathbb{M}_{n_1}(D_1) \times \mathbb{M}_{n_2}(D_2) \times \cdots \times \mathbb{M}_{n_k}(D_k)$, *where $\mathbb{M}_{n_i}(D_i)$ are rings of $n_i \times n_i$ matrices over division rings D_i.*

(2) $R_2 \cong \begin{pmatrix} L_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & L_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & L_m \end{pmatrix}$ *with local rings L_j.*

Proof. By Theorem 3.2, $R \cong R_1 \times R_2$, where R_1 is a semi-simple Artinian ring and R_2 is square-free as a right R_2-module and all idempotents of R_2 are central. Then, there exist division rings D_i such that $R_1 \cong \mathbb{M}_{n_1}(D_1) \times \mathbb{M}_{n_2}(D_2) \times \cdots \times \mathbb{M}_{n_k}(D_k)$. On the other hand, R is semiperfect and so R_2 is semiperfect. Then, $R_2 = e_1R_2 \oplus e_2R_2 \oplus \cdots \oplus e_mR_2$, where e_i are orthogonal local central idempotents of R_2. From Theorem 3.6 and the squareness-free of R_2, we obtain that $R_2 \cong \begin{pmatrix} e_1R_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & e_2R_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & e_mR_2 \end{pmatrix}$ and $e_iR_2 \cong \mathrm{End}(e_iR_2)$ local rings. □

Acknowledgment. The author is grateful to the referee for her/his valuable comments. The author would like to thank Prof. Truong Cong Quynh for valuable comments and useful discussion.

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