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CONJUGACY CLASSIFICATION OF n-DIMENSIONAL MÖBIUS GROUP

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Abstract. In this paper, we study the n-dimensional Möbius transformation. We obtain several conjugacy invariants and give a conjugacy classification for n-dimensional Möbius transformation.

1. Introduction

Throughout this paper, we will adopt the same notations and definitions as in [1, 8, 10, 11] such as, complex Möbius transformations, $\mathrm{PSL}(2,\mathbb{C})$, the Möbius group $M(\overline{\mathbb{R}}^n)$, the Clifford matrix group $\mathrm{SL}(2,\Gamma_n)$, the Clifford algebra $\mathcal{C}_n$ and so on. For example, complex Möbius transformations: Any 2×2 matrix A in $\mathrm{GL}(2,\mathbb{C})$ induces complex Möbius transformations g by the formula $A \to g_A = g$, where $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $g_A = \frac{az+b}{cz+d}$; $\mathrm{PSL}(2,\mathbb{C})$: The collection of all complex Möbius transformations for which $ad - bc$ takes the value 1 forms a group which can be identified with $\mathrm{PSL}(2,\mathbb{C})$. In particular, a member f of $\mathrm{PSL}(2,\mathbb{C})$ is simple if it is conjugate in $\mathrm{PSL}(2,\mathbb{C})$ to an element of $\mathrm{PSL}(2,\mathbb{R})$. The map f is k-simple if it may be expressed as the composite of k simple transformations but no fewer. For more details, see [1, 4–6, 8–10, 12, 13] etc.

It is well known that to study the conjugacy classification of Möbius transformation is very important and there has been an active research in this area. In 1983, Beardon [3] proved that $\mathrm{trace}^2(g)$ (we often abbreviate $\mathrm{trace}^2(g)$ to $\mathrm{tr}^2(g)$ or τ_g^2) is invariant under any conjugation $g \mapsto hgh^{-1}$ and he established the conjugacy classification of $\mathrm{PSL}(2,\mathbb{C})$. Let $g \in \mathrm{PSL}(2,\mathbb{C})$, if $\tau_g^2 \geq 0$, then g is 1-simple and if $\mathrm{tr}^2(g) = 4$, then g is parabolic; if $\mathrm{tr}^2(g) \in [0, 4)$, then g is elliptic; if $\mathrm{tr}^2(g) \in (4,\infty)$, then g is hyperbolic. If $\mathrm{tr}^2(g)$ is either not real or is negative, then g is 2-simple and loxodromic. In 2004, Foreman [7] used the quaternionic formalism of Möbius transformations on $\widehat{\mathbb{R}}^4$ to derive conjugacy invariants on $\mathrm{SL}(2,\mathbb{H})$. For a matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{H})$, we define

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$\gamma_A = |a|^2 + |d|^2 + 4\mathrm{Re}(a)\mathrm{Re}(d) - 2\mathrm{Re}(bc)$ and $\delta_A = \mathrm{Re}(\tau_A)$. Foreman proved that γ_A and δ_A on $\mathrm{SL}(2,\mathbb{H})$ are conjugate invariant. In 2008, by Foreman's conjugacy invariants, Parker [11] had the following conjugacy classification which was more general than Beardon's result. Let $g \in \mathrm{PSL}(2,\mathbb{H})$. Case (a) if $\tau_g \in \mathbb{R}$, then g is 1-simple and if $\delta_g^2 \in [0,4)$, then g is elliptic; if $\delta_g^2 = 4$, then g is parabolic; if $\delta_g^2 \in (4,\infty)$, then g is loxodromic. Case (b) if $\beta_g = \delta_g$ and $\tau_g \notin \mathbb{R}$, then g is 2-simple if $\gamma_g - \delta_g^2 < 2$, then g is elliptic; if $\gamma_g - \delta_g^2 = 2$, then g is parabolic and if $\gamma_g - \delta_g^2 > 2$, then g is loxodromic. Case (c) if $\beta_g \neq \delta_g$ then g is 3-simple and loxodromic.

As the first main aim of this paper, we will study the conjugacy invariants further and prove.

Theorem 1.1. *Given $A \in \mathrm{SL}(2,\Gamma_n)$, $c \in V^n$. Then γ_A is preserved under conjugation in $\mathrm{SL}(2,\Gamma_n)$. If τ_A is real, then it is also preserved under conjugation in $\mathrm{SL}(2,\Gamma_n)$.*

Following Theorem 1.1, we have:

Corollary 1.2. *Let $f \in M(\overline{\mathbb{R}}^n)$ with $c \in V^n \setminus \{0\}$. Then f is conjugate to a real Möbius transformation if and only if $\tau_f \in \mathbb{R}$.*

As the second main aim of this paper, by using Theorem 1.1, we will discuss the conjugacy classification of n-dimensional Möbius transformation.

Theorem 1.3. *Let f be an n-dimensional Möbius transformation with $c \in V^n \setminus \{0\}$.*

(a) *If $\tau \in \mathbb{R}$, then f is 1-simple*
  - (i) *If $0 \le \tau^2 < 4$, then f is elliptic*;
  - (ii) *If $\tau^2 = 4$, then f is parabolic*;
  - (iii) *If $\tau^2 > 4$, then f is hyperbolic.*

(b) *If $\tau \notin \mathbb{R}$, then f is not parabolic.*
  - (i) *If $\gamma - \delta^2 > 2$, then f is loxodromic and f is 2-simple or 3-simple*;
  - (ii) *If $\gamma - \delta^2 = 2$, then f is elliptic and f is 2-simple*;
  - (iii) *If $\gamma - \delta^2 < 2$, then f is fixed point free.*

Remark 1.4. Theorem 1.1 is a generalization of Foreman's conjugacy invariants γ and δ in [7].

Remark 1.5. Corollary 1.1 is a generalization of Theorem 1.2 in [11].

Remark 1.6. Theorem 1.2 is a generalization of Parker's conjugacy classification in [11] into the case of $\mathrm{SL}(2,\mathbb{H})$.

2. Preliminaries

The Clifford algebra $\mathcal{C}_n$ shall be the associative algebra over the reals generated by elements $i_1, i_2, \ldots, i_{n-1}$ subject to the relations $i_h^2 = -1$ and $i_h i_t = -i_t i_h$, $i_t^2 = -1$. Every $q \in \mathcal{C}_n$ has a unique representation of the form

$$q = \sum q_I I,\ \ q_I \in \mathbb{R} \text{ and } I = i_{v_1} i_{v_2} \cdots i_{v_p} \text{ with } 0 < v_1 < \cdots < v_p < n.$$

Clifford numbers of the from $q = q_0 + q_1 i_1 + \cdots + q_{n-1} i_{n-1}$ are called vectors. Obviously, when $n = 3$, $\mathcal{C}_3 = \mathbb{H}$. The Clifford group Γ_n consists of all $q \in \mathcal{C}_n$ which can be written an products of non-zero vectors in V^n. We denote this real vector space by V^n and it is isomorphic to $\mathbb{R}^n$ as a vector space. The algebra $a \in \mathcal{C}_n$ has three important involutions:

(1) $a' = a_0 + \sum a_p I'_p$, $I'_p = (-i_1)(-i_2)\cdots(-i_p) = (-1)^p I_p$;

(2) $a^* = a_0 + \sum a_p I^*_p$, $I^*_p = i_p i_{p-1} \cdots i_1 = (-1)^{\frac{p(p-1)}{2}} I_p$;

(3) $\overline{a} = a_0 + \sum a_v \overline{I_v}$, $\overline{I_v} = (-i_p)(-i_{p-1})\cdots(-i_1) = (-1)^{\frac{p(p+1)}{2}} I_p$.

It is obvious that $(ab)' = a'b'$, $(ab)^* = b^* a^*$, $\overline{ab} = \bar{b}\bar{a}$. If $a, b \in \Gamma_n$, then $|ab| = |a|\,|b|$, $a\overline{a} = \overline{a}a = |a|^2$.

From Ahlfors [2] we have the following general definition:

Definition 2.1. The matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ belongs to the group $\mathrm{SL}(2, \Gamma_n)$ if

(i) $a, b, c, d \in \Gamma_n \cup \{0\}$;

(ii) $ad^* - bc^* = 1$;

(iii) $ab^*, cd^*, c^*a, d^*b \in V^n$.

An n-dimensional Möbius transformation f is an invertible map of $\overline{V}^n = V^n \cup \{\infty\}$ of the form $f(x) = (ax+b)(cx+d)^{-1}$ which is induced by the formula $A \to f_A = f$, where $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \Gamma_n)$. The map $A \mapsto f$ from $\mathrm{SL}(2, \Gamma_n)$ to the Möbius group $M(\overline{\mathbb{R}}^n)$ is a surjective homomorphism. In future, whenever we refer to A, f or an 'n-dimensional Möbius transformation', we refer to the quantities described above.

A real Möbius transformation in $\mathrm{PSL}(2, \Gamma_n)$ is a member of $\mathrm{PSL}(2, \Gamma_n)$ with real coefficients. A member f of $\mathrm{PSL}(2, \Gamma_n)$ is simple if it is conjugate in $\mathrm{PSL}(2, \Gamma_n)$ to an element of $\mathrm{PSL}(2, \mathbb{R})$. The map f is k-simple if it may be expressed as the composite of k simple transformations but no fewer. The group $\mathrm{PSL}(2, \Gamma_n)$ contains $\mathrm{PSL}(2, \mathbb{R})$ as subgroup and we further have the following lemma.

Lemma 2.2. *If $A \in \mathrm{SL}(2, \Gamma_n)$, $c \in V^n \setminus \{0\}$, then A is less than 4-simple.*

Proof. A has the factorization

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c^{*-1} & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & c^{-1}d \\ 0 & 1 \end{pmatrix}.$$

Since

$$\begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} r & \beta \\ 0 & r^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \beta^{-1} \end{pmatrix} = \begin{pmatrix} r & 1 \\ 0 & r^{-1} \end{pmatrix},$$

then $\left(\begin{smallmatrix} r & \beta \\ 0 & r^{-1} \end{smallmatrix}\right)$ is 1-simple, where $\beta \in V^n$, $r \in \mathbb{R} \setminus \{0\}$.

Suppose that $t \in V^n$ and $|t| = 1$. There exist real numbers x and y and purely imaginary unit vector μ such that $t = x + \mu y$, where $\mu = \mu_1 i_1 + \mu_2 i_2 + \cdots + \mu_{n-1} i_{n-1}$, $|\mu| = 1$.

From the matrix equation

$$\begin{pmatrix}\mu & 1\\ 1 & \mu\end{pmatrix}\begin{pmatrix}x+\mu y & 0\\ 0 & x-\mu y\end{pmatrix}\begin{pmatrix}\mu & 1\\ 1 & \mu\end{pmatrix}^{-1}=\begin{pmatrix}x & -y\\ y & x\end{pmatrix}$$

we see that $\left(\begin{smallmatrix}t & 0\\ 0 & \bar{t}\end{smallmatrix}\right)$ is 1-simple.

Further, we have

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}|c|^{-1} & ac^{-1}\\ 0 & |c|\end{pmatrix}\begin{pmatrix}\bar{t} & 0\\ 0 & t\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & c^{-1}d\\ 0 & 1\end{pmatrix},$$

where $c=|c|\,t$, $|t|=1$, $t\in V^n$.

From the above discussion, we have $\left(\begin{smallmatrix}|c|^{-1} & ac^{-1}\\ 0 & |c|\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & c^{-1}d\\ 0 & 1\end{smallmatrix}\right)$ are 1-simple, $\left(\begin{smallmatrix}\bar{t} & 0\\ 0 & t\end{smallmatrix}\right)$ is 1-simple, then A is less than 4-simple. □

3. The proofs of main results

Proof of Theorem 1.1. According to [1], we have $\mathrm{SL}(2,\Gamma_n)$ is generated by matrices of the from

$$\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix},\begin{pmatrix}r & 0\\ 0 & r^{-1}\end{pmatrix},\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\begin{pmatrix}\lambda & 0\\ 0 & \lambda'\end{pmatrix},$$

where $\beta\in V^n$, $r\in\mathbb{R}$, $|\lambda|=1$, $\lambda\in\Gamma_n$. Denote one of these matrices by P. Let $B=PAP^{-1}$. It suffices to show that for each choice of P, we have $\tau_A=\tau_B$ and $\gamma_A=\gamma_B$:

In the first case we have

$$\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}^{-1}=\begin{pmatrix}a+\beta c & -a\beta-\beta c\beta+b+\beta d\\ c & d-c\beta\end{pmatrix},$$

$$\tau_B=a+\beta c+(d-c\beta)^*=\tau_A,$$

$$\begin{aligned}\gamma_B=\ &|a+\beta c|^2+|d-c\beta|^2+4\mathrm{Re}(a+\beta c)\mathrm{Re}(d-c\beta)\\ &-2\mathrm{Re}[(-a\beta-\beta c\beta+b+\beta d)c]\\ =\ &|a|^2+|d|^2+2\,|\beta c|^2+2\mathrm{Re}(a\cdot\overline{\beta c})-2\mathrm{Re}(d\cdot\overline{c\beta})\\ &+4\mathrm{Re}(a)Re(d)-4\mathrm{Re}(a)Re(c\beta)+4\mathrm{Re}(\beta c)\mathrm{Re}(d)\\ &-4\mathrm{Re}(\beta c)Re(c\beta)+2\mathrm{Re}(a\beta c)+2\mathrm{Re}(\beta c\beta c)-2\mathrm{Re}(bc)-2\mathrm{Re}(\beta dc).\end{aligned}$$

Since $\beta,c\in V^n$, then $\beta c=q_0+q_1I_1+q_2I_2$. With the third involution, we have $\overline{\beta c}=q_0-q_1I_1-q_2I_2$. This shows that $\beta c+\overline{\beta c}$ is real. Then

$$\begin{aligned}\gamma_B&=\gamma_A+2\mathrm{Re}(a\cdot\overline{\beta c})-4\mathrm{Re}(a)Re(c\beta)+2\mathrm{Re}(a\beta c)+2\,|\beta c|^2\\ &\quad-4\mathrm{Re}(\beta c)\mathrm{Re}(c\beta)+2\mathrm{Re}(\beta c\beta c)-2\mathrm{Re}(d\cdot\overline{c\beta})+4\mathrm{Re}(\beta c)\mathrm{Re}(d)-2\mathrm{Re}(\beta dc)\\ &=\gamma_A+2\mathrm{Re}(a)[\mathrm{Re}(\overline{\beta c}+\beta c)-2\mathrm{Re}(c\beta)]+2\mathrm{Re}(\beta c)[\mathrm{Re}(\overline{\beta c}+\beta c)-2\mathrm{Re}(c\beta)]\\ &\quad+\mathrm{Re}(d)[-\mathrm{Re}(\overline{c\beta}+c\beta)+2\mathrm{Re}(c\beta)]=\gamma_A.\end{aligned}$$

In the second case we have

$$\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} a & r^2 b \\ r^{-2}c & d \end{pmatrix},$$
$$\tau_B = a + d^* = \tau_A,$$
$$\gamma_B = |a|^2 + |d|^2 + 4\mathrm{Re}(a)\mathrm{Re}(d) - 2\mathrm{Re}(r^2 b \cdot r^{-2} b) = \gamma_A.$$

In the third case we have

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}.$$

Since τ_A is real, then $\tau_A = a + d^* = (a + d^*)^* = \tau_B$,

$$\gamma_B = |a|^2 + |d|^2 + 4\mathrm{Re}(d)\mathrm{Re}(a) - 2\mathrm{Re}[(-c)\cdot(-b)] = \gamma_A.$$

In the forth case we have

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda' \end{pmatrix}^{-1} = \begin{pmatrix} \lambda a \overline{\lambda} & \lambda b \lambda^* \\ \lambda' \overline{\lambda} c & \lambda' d \lambda^* \end{pmatrix}.$$

Since τ_A is real, then $\tau_B = \lambda(a + d^*)\overline{\lambda} = (a + d^*)\lambda\overline{\lambda} = \tau_A$,

$$\gamma_B = |a|^2 + |d|^2 + 4\mathrm{Re}(\lambda a \overline{\lambda})\mathrm{Re}(\lambda' d \lambda^*) - 2\mathrm{Re}(\lambda b \lambda^* \cdot \lambda' c \overline{\lambda}) = \gamma_A. \qquad \square$$

Proof of Corollary 1.1. Let $f \in M(\overline{\mathbb{R}}^n)$ with $c \in V^n \setminus \{0\}$. If f is conjugate to a real Möbius transformation, then τ_f is real, by Theorem 1.1. Conversely, According to Theorem 5.5 in [5], if τ_f is real and $c \in V^n \setminus \{0\}$, then f is conjugate to a real Möbius transformation. $\square$

Now, we give a classification to the elements of $M(\overline{\mathbb{R}}^n)$ as follows. In the proof of Theorem 1.2, we will adopt the following classification [6,12–14].

Non-trivial element $f \in M(\overline{\mathbb{R}}^n)$ is called

(1) fixed-point-free if it has no fixed points in $\overline{\mathbb{R}}^n$ and f can be conjugate in $\mathrm{SL}(2, \Gamma_n)$ to $\begin{pmatrix} \lambda & -r^2 t' \\ t & \lambda' \end{pmatrix}$, $|\lambda| < 1$, $r \in \mathbb{R}$, $t \neq 0$;

(2) loxodromic if it (and its Poincare extension $\tilde{f}$) has two fixed points in $\overline{\mathbb{R}}^n$(and $\overline{\mathbb{R}}^{n+1}$) and f can be conjugate in $\mathrm{SL}(2, \Gamma_n)$ to $\begin{pmatrix} \lambda & 0 \\ 0 & r^{-1}\lambda' \end{pmatrix}$, where $r > 0$, $r \neq 1$, $\lambda \in \Gamma_n$ and $|\lambda| = 1$;

(3) parabolic if it has only one fixed point in $\overline{\mathbb{R}}^n$ and its Poincaré extension has infinitely many fixed points in $\overline{\mathbb{R}}^n$ and f can be conjugate in $\mathrm{SL}(2, \Gamma_n)$ to $\begin{pmatrix} a & b \\ 0 & a' \end{pmatrix}$, where $a, b \in \Gamma_n$, $|a| = 1$, $b \neq 0$ and $ab = ba'$;

(4) elliptic if it has at least two fixed points in $\overline{\mathbb{R}}^{n+1}$ and f can be conjugate in $\mathrm{SL}(2, \Gamma_n)$ to $\begin{pmatrix} u & 0 \\ 0 & u' \end{pmatrix}$, where $u \in \Gamma_n$, $|u| = 1$ and $u \notin \mathbb{R}$.

Proof of Theorem 1.2. Case (a), using Theorem 1.1, f is conjugate to a real Möbius transformation. Then corresponds to the usual classification for real Möbius transformations.

Case (b). We first prove that f is not parabolic. Suppose f is parabolic. Let $\beta = \frac{1}{2}(c^{-1}d + ac^{-1})$ and $\sigma = \frac{1}{2}(ac^{-1} + c^{-1}d)$, we have

$$\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sigma c & \sigma c \sigma - c^{-1} \\ c & c\sigma \end{pmatrix},$$

f is parabolic $\Leftrightarrow$ f has only one fixed point on $M(\overline{\mathbb{R}}^n) \Leftrightarrow \left(\begin{smallmatrix} \sigma c & \sigma c\sigma - c^{-1} \\ c & c\sigma \end{smallmatrix}\right)$ has only one fixed point on V^n. Suppose that v is the fixed point of $\left(\begin{smallmatrix} \sigma c & \sigma c\sigma - c^{-1} \\ c & c\sigma \end{smallmatrix}\right)$. Then v is the fixed point of $\left(\begin{smallmatrix} \sigma c & \sigma c\sigma - c^{-1} \\ c & c\sigma \end{smallmatrix}\right) \Leftrightarrow v$ satisfies the condition $c(v+\sigma)c(v-\sigma) = -1 \Leftrightarrow v$ and $-v$ are simultaneously fixed points. So A is parabolic $\Leftrightarrow v = -v \Leftrightarrow \sigma c \sigma - c^{-1} = 0 \Leftrightarrow (c\sigma)^2 = 1$.

Since $(c\sigma)^2 + |c\sigma|^2 = (c\sigma)(c\sigma + \overline{c\sigma}) = 2, c\sigma + \overline{c\sigma}$ is real, then $c\sigma$ is real. This is a contradiction since f is conjugate to a real Möbius transformation.

(i) If f is loxodromic, A is conjugate in $\mathrm{SL}(2, \Gamma_n)$ to $\left(\begin{smallmatrix} r\lambda & 0 \\ 0 & r^{-1}\lambda' \end{smallmatrix}\right)$, where $|\lambda| = 1$, $r > 0$, $r \neq 1$, $\lambda \in \Gamma_n$, the map satisfies:

$$\gamma - \delta^2 = r^2 + r^{-2} + [2 - (r^2 + r^{-2})]Re^2(\lambda) > 2,$$

since

$$\begin{pmatrix} r\lambda & 0 \\ 0 & r^{-1}\lambda' \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda' \end{pmatrix}.$$

Let u, v be fixed points of A and we make the specific choice h:

$$h = \begin{pmatrix} 1 & -u \\ (u-v)^{-1} & -(u-v)^{-1}v \end{pmatrix},$$

$$hfh^{-1} = \begin{pmatrix} (u-v)(cv+d)(u-v)^{-1} & 0 \\ 0 & cu+d \end{pmatrix}.$$

Then

$$cu + d = c(u + c^{-1}d) = r^{-1}\lambda',\ r^{-1} = \left|c(u + c^{-1}d)\right|,\ \lambda' = \lambda_1'\lambda_2', \lambda_1', \lambda_2' \in V^n,$$

$$\begin{pmatrix} r\lambda & 0 \\ 0 & r^{-1}\lambda' \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1' \end{pmatrix} \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2' \end{pmatrix}.$$

Using Theorem 2.1, we have f is 2-simple or 3-simple.

(ii) If f is elliptic, f can be conjugate in $\mathrm{SL}\,(2, \Gamma_n)$ to $\left(\begin{smallmatrix} u & 0 \\ 0 & u' \end{smallmatrix}\right)$, where $u \in \Gamma_n$, $|u| = 1$ and $u \notin \mathbb{R}$.

$$\gamma - \delta^2 = |u|^2 + |u'|^2 + 4Re(u)Re(u') - (Re(u + u'))^2 = 2.$$

Similar discussions as above, f is conjugate to $\left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_1' \end{smallmatrix}\right)\left(\begin{smallmatrix} \lambda_2 & 0 \\ 0 & \lambda_2' \end{smallmatrix}\right)$, then f is 2-simple.

(iii) If f is fixed point free, f can be conjugate in $\mathrm{SL}\,(2, \Gamma_n)$ to $\left(\begin{smallmatrix} \lambda & -r^2 t' \\ t & \lambda' \end{smallmatrix}\right)$, $|\lambda| < 1$, $r \in \mathbb{R}$, $t \neq 0$.

$$\gamma - \delta^2 = 2(|\lambda|^2 + r^2 Re(t't)) < 2.$$ □

Remark 3.1. Similarly, $c \in V^n \setminus \{0\}$ can be replaced by $b \in V^n \setminus \{0\}$.

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