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THE THIRD HERMITIAN-TOEPLITZ AND HANKEL DETERMINANTS FOR PARABOLIC STARLIKE FUNCTIONS

Rosihan M. Ali, Sushil Kumar, and Vaithiyanathan Ravichandran

In memoriam QMH Professor M. Ataharul Islam

Abstract. A normalized analytic function f is parabolic starlike if $w(z) := zf'(z)/f(z)$ maps the unit disk into the parabolic region $\{w : \operatorname{Re} w > |w-1|\}$. Sharp estimates on the third Hermitian-Toeplitz determinant are obtained for parabolic starlike functions. In addition, upper bounds on the third Hankel determinants are also determined.

1. Introduction and main theorems

Let $\mathcal{A}$ be the class consisting of normalized analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the unit disk $\mathbb{D} := \{z \in \mathbb{C}\colon |z| < 1\}$. For two natural numbers q and n, the q^{th} Hankel determinant $H_q(n)$ associated with f is defined by $H_q(n) := \det\{a_{n+i+j-2}\}_{i,j}^{q}$, $1 \le i, j \le q$, $a_1 = 1$. Thus

$$H_q(n) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}. \tag{1}$$

Another determinant associated with the function f is the q^{th} Hermitian-Toeplitz determinant given by $T_q(n) := [a_{ij}]$, where $a_{ij} = a_{n+j-i}$ for $j \ge i$ and $a_{ij} = \overline{a_{ji}}$ for $j < i$. Thus

$$T_q(n) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ \overline{a}_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ \overline{a}_{n+q-1} & \overline{a}_{n+q-2} & \cdots & a_n \end{vmatrix}.$$

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It is readily seen that the third Hermitian-Toeplitz determinant $T_3(1)$ associated with the initial coefficients of functions $f \in \mathcal{A}$ is given by

$$T_3(1) := \begin{vmatrix} 1 & a_2 & a_3 \\ \overline{a}_2 & 1 & a_2 \\ \overline{a}_3 & \overline{a}_2 & 1 \end{vmatrix} = 2\operatorname{Re}(a_2^2\,\overline{a_3}) - 2|a_2|^2 - |a_3|^2 + 1. \tag{2}$$

The Hankel determinants are closely associated with the Hermitian-Toeplitz determinants [9, 13]. Pommerenke [21, 22] investigated estimates on Hankel determinants for univalent and starlike functions. For univalent functions $f \in \mathcal{A}$, Hayman [10] determined the sharp bound $|H_2(n)| \leq \kappa n^{1/2}$, where κ is an absolute constant. In 2010, Babalola [3] obtained estimates on the third Hankel determinant for the well-known classes of bounded-turning, starlike and convex functions. Additional details related to the third Hankel determinant may be found in [4, 5, 15–17]. In 2018, Kowalczyk *et al.* [12] obtained a sharp bound of $4/135$ on the third Hankel determinant $H_3(1)$ for the class of convex functions.

In a recent 2020 paper, Cudna *et al.* [6] obtained best lower and upper bounds on the second and third Hermitian-Toeplitz determinants for the classes of starlike and convex functions of order α. In another 2020 paper, Lecko *et al.* [18] computed the best possible estimate on the fourth Hermitian-Toeplitz determinant for convex functions. Recently, Jastrzębski *et al.* [11] obtained sharp estimates on the second and third Hermitian-Toeplitz determinants for close-to-star functions.

In this paper, we examine functions related to the class $\mathcal{UCV}$ of uniformly convex functions introduced by Goodman [7]. A function $f \in \mathcal{A}$ belongs to $\mathcal{UCV}$ if it possesses the property that for every circular arc ν contained in $\mathbb{D}$ with centre η also in $\mathbb{D}$, the arc $f(\nu)$ is convex. A function f in the class $\mathcal{UCV}$ is necessarily convex. Ma and Minda [20], as well as Rønning [25], independently found a one-variable characterization of functions $f \in \mathcal{UCV}$. From this one-variable characterization, Rønning [25] introduced the class $\mathcal{S}_P^*$ of parabolic starlike functions via the Alexander's relation: $f \in \mathcal{UCV}$ if and only if $zf'(z) \in \mathcal{S}_P^*$. Analytically,

$$f \in \mathcal{S}_P^* \quad \text{if and only if} \quad \left|\frac{zf'(z)}{f(z)} - 1\right| < \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right), \; z \in \mathbb{D}.$$

The above analytic characterization could also be expressed in terms of subordination. For two analytic functions g and h in $\mathbb{D}$, the function g is said to be subordinate to h, written $g \prec h$, if there exists a function w in $\mathbb{D}$ such that $g = h(w(z))$ for $z \in \mathbb{D}$. Here the function w satisfies the condition $w(0) = 0$ and $|w(z)| \leq 1$ in $\mathbb{D}$. Thus, a function $f \in \mathcal{S}_P^*$ if and only if $zf'(z)/f(z) \prec \varphi_{\mathcal{PAR}}(z)$ for all $z \in \mathbb{D}$, where the function $\varphi_{\mathcal{PAR}} : \mathbb{D} \to \Omega$ is given by

$$\varphi_{\mathcal{PAR}}(z) = 1 + \frac{2}{\pi^2}\left(\log\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)^2.$$

In [1], Ali found the best possible estimates on the first four coefficients as well as the Fekete-Szego functional for functions $f \in \mathcal{S}_P^*$. Additional studies regarding this class may be found in [2, 8, 23, 26].

The main results in this paper are the following two theorems which give sharp bounds on the third Hermitian-Toeplitz determinant $T_3(1)$ for functions $f \in \mathcal{S}_P^*$, and an upper bound on their third Hankel determinants $H_3(1)$ and $H_3(2)$. The proofs of these results are given in the next section.

Theorem 1.1. *The third Hermitian-Toeplitz determinant $T_3(1)$ for every function $f \in \mathcal{S}_P^*$ satisfies*

$$1 + \frac{64}{9\pi^8}(432 + 24\pi^2 - 19\pi^4) \leq T_3(1) \leq 1.$$

These bounds are best possible.

The lower bound $1 + \frac{64}{9\pi^8}(432 + 24\pi^2 - 19\pi^4) \approx 0.114232$ is best possible for the function f_0 given by

$$\frac{zf_0'(z)}{f_0(z)} = \varphi_{\mathcal{PAR}}(z).$$

From its Taylor series expansion, the second and third coefficients of f_0 are

$$a_2 = \frac{8}{\pi^2} \quad \text{and } a_3 = \frac{8}{3\pi^4}\left(\pi^2 + 12\right).$$

This readily shows that

$$T_3(1) := 2\operatorname{Re}(a_2^2\,\bar{a}_3) - 2|a_2|^2 - |a_3|^2 + 1 = 1 + \frac{64}{9\pi^8}(432 + 24\pi^2 - 19\pi^4),$$

illustrating sharpness of the lower bound. The upper bound is attained when $a_2 = 0$ and $a_3 = 0$. This happens for the function f_1 given by

$$\frac{zf_1'(z)}{f_1(z)} = \varphi_{\mathcal{PAR}}(z^4).$$

Theorem 1.2. *The third Hankel determinants $H_3(1)$ and $H_3(2)$ for every function $f \in \mathcal{S}_P^*$ satisfy*

$$|H_3(1)| \leq \frac{16}{8505\pi^{10}}\left(\frac{45360\sqrt{5}\pi^8}{\sqrt{739\pi^4 - 5760 - 360\pi^2}} + 8221\pi^6 + 68040\pi^4 - 10080\pi^2 - 362880\right) \approx 0.35985$$

and

$$|H_3(2)| \leq \frac{1}{\pi^6}\left(\frac{288189248}{1913625} + \frac{4096000}{159\sqrt{3441}}\right) + \frac{14075392}{14175\pi^{10}} + \frac{458752}{135\pi^{14}} \approx 0.624418.$$

2. Proofs

Let $\mathcal{P}$ denote the class of all analytic functions $p : \mathbb{D} \to \mathbb{C}$ with $p(0) = 1$ and $\operatorname{Re} p(z) > 0$. If $f \in \mathcal{S}^*_P$, then there is a function $w : \mathbb{D} \to \mathbb{D}$ with $|w(z)| \leq |z|$ satisfying $zf'(z)/f(z) = \varphi_{PAR}(w(z))$. Thus $p(z) = (1 + w(z))/(1 - w(z)) \in \mathcal{P}$, which gives

$$\frac{zf'(z)}{f(z)} = 1 + \frac{2}{\pi^2}\left(\log\left(p(z) + \sqrt{p^2(z) - 1}\right)\right)^2. \tag{3}$$

Since $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, it follows that

$$\begin{aligned}\frac{zf'(z)}{f(z)} &= 1 + a_2 z + (2a_3 - a_2^2)z^2 + (a_2^3 - 3a_2a_3 + 3a_4)z^3 \\ &\quad + (-a_2^4 + 4a_2^2a_3 - 4a_2a_4 - 2a_3^2 + 4a_5)z^4 \\ &\quad + (a_2^5 - 5a_2^3a_3 + 5a_2^2a_4 + 5a_2a_3^2 - 5a_2a_5 - 5a_3a_4 + 5a_6)z^5 + \cdots .\end{aligned} \tag{4}$$

Writing $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \in \mathcal{P}$ results in

$$\begin{aligned}&1 + \frac{2}{\pi^2}\left(\log\left(p(z) + \sqrt{p^2(z) - 1}\right)\right)^2 \\ &= 1 + \frac{4}{\pi^2}p_1 z - \frac{2}{3\pi^2}(p_1^2 - 6p_2)z^2 + \frac{4}{45\pi^2}(2p_1^3 - 15p_1p_2 + 45p_3)z^3 \\ &\quad - \frac{2}{105\pi^2}(3p_1^4 - 28p_1^2p_2 + 70p_1p_3 + 35p_2^2 - 210p_4)z^4 \\ &\quad + \frac{4}{1575\pi^2}(8p_1^5 - 90p_1^3p_2 + 210p_1^2p_3 + 210p_1p_2^2 \\ &\quad - 525p_1p_4 - 525p_2p_3 + 1575p_5)z^5 + \cdots .\end{aligned} \tag{5}$$

In view of expressions (3), (4) and (5), it is readily seen that

$$a_2 = \frac{4p_1}{\pi^2}, \tag{6}$$

$$a_3 = \frac{1}{3\pi^4}((24 - \pi^2)p_1^2 + 6\pi^2 p_2), \tag{7}$$

$$a_4 = \frac{4}{135\pi^6}(45\pi^4 p_3 - 15\pi^2(\pi^2 - 18)p_1p_2 + (360 - 45\pi^2 + 2\pi^4)p_1^3), \tag{8}$$

$$\begin{aligned}a_5 = \frac{1}{1890\pi^8}\Big(&- 630\pi^4(\pi^2 - 16)p_1p_3 + 84\pi^2(360 - 55\pi^2 + 3\pi^4)p_1^2p_2 \\ &+ (20160 - 5040\pi^2 + 553\pi^4 - 27\pi^6)p_1^4 \\ &- 315(-6\pi^6 p_4 + \pi^4(-12 + \pi^2)p_2^2)\Big),\end{aligned} \tag{9}$$

$$\begin{aligned}a_6 = \frac{2}{70875\pi^{10}}\Big(&(302400 - 126000\pi^2 + 24675\pi^4 - 2725\pi^6 + 144\pi^8)p_1^5 \\ &- 90\pi^2(-8400 + 2450\pi^2 - 315\pi^4 + 18\pi^6)p_1^3p_2 \\ &+ 1260\pi^4(300 - 50\pi^2 + 3\pi^4)p_1^2p_3\end{aligned}$$

$$+ 315p_1(\pi^4(900 - 175\pi^2 + 12\pi^4)p_2^2 - 30\pi^6(\pi^2 - 15)p_4)$$

$$- 9450(\pi^6(\pi^2 - 10)p_2p_3 - 3\pi^8 p_5)\Big). \tag{10}$$

Since the classes $\mathcal{S}_P^*$ and $\mathcal{P}$ are rotationally invariant, without loss of generality, we may assume that $0 \le p_1 \le 2$.

The following lemma due to Libera and Zlotkiewicz [19] is used to compute the lower and upper bounds of the third Hermitian-Toeplitz determinant $T_3(1)$ associated with functions $f \in \mathcal{S}_P^*$.

Lemma 2.1 ([19, Lemma 3, p. 254])**.** *If*

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \tag{11}$$

belongs to $\mathcal{P}$*, then*

$$2p_2 = p_1^2 + (4 - p_1^2)\xi$$

for some $\xi \in \overline{\mathbb{D}}$*.*

Proof of Theorem 1.1. In view of (6), (7) and Lemma 2.1, there exists $\xi \in \overline{\mathbb{D}}$ so that

$$2|a_2|^2 = \frac{32}{\pi^4}p_1^2, \tag{12}$$

$$\begin{aligned} 2\operatorname{Re}(a_2^2\,\overline{a_3}) &= 2\operatorname{Re}\big(\frac{16p_1^2}{\pi^4}\cdot\frac{1}{3\pi^4}\overline{((24-\pi^2)p_1^2 + 6\pi^2 p_2)}\big) \\ &= \frac{32}{3\pi^8}((24-\pi^2)p_1^4 + 3\pi^2 p_1^2\operatorname{Re}(2\overline{p_2})) \\ &= \frac{32}{3\pi^8}((24-\pi^2)p_1^4 + 3\pi^2 p_1^2(p_1^2 + (4-p_1^2)\operatorname{Re}(\overline{\xi}))) \\ &= \frac{32}{3\pi^8}((24+2\pi^2)p_1^4 + 3\pi^2 p_1^2(4-p_1^2)\operatorname{Re}(\overline{\xi})), \end{aligned} \tag{13}$$

and

$$\begin{aligned} |a_3|^2 &= \frac{1}{9\pi^8}\left|(24-\pi^2)p_1^2 + 6\pi^2 p_2\right|^2 \\ &= \frac{1}{9\pi^8}\big((24-\pi^2)^2 p_1^4 + 9\pi^4|2p_2|^2 + 6(24-\pi^2)\pi^2 p_1^2\operatorname{Re}(2\overline{p_2})\big) \\ &= \frac{1}{9\pi^8}\big((24-\pi^2)^2 p_1^4 + 9\pi^4|p_1^2 + (4-p_1^2)\xi|^2 \\ &\qquad + 6(24-\pi^2)\pi^2 p_1^2(p_1^2 + (4-p_1^2)\operatorname{Re}(\overline{\xi}))\big) \\ &= \frac{1}{9\pi^8}\big(4(12+\pi^2)^2 p_1^4 + 9\pi^4(4-p_1^2)^2|\xi|^2 \\ &\qquad + 12\pi^2(12+\pi^2)(4-p_1^2)p_1^2\operatorname{Re}(\overline{\xi})\big). \end{aligned} \tag{14}$$

Substituting the values of $2\operatorname{Re}(a_2^2\,\overline{a_3})$, $2|a_2|^2$ and $|a_3|^2$ from (12), (13), and (14) into the expression (2) yields

$$T_3(1) = 1 + \frac{32}{9\pi^8}\Big((54 + 3\pi^2 - \frac{\pi^4}{8})p_1^4 - 9\pi^4 p_1^2 - \frac{9}{32}\pi^4(4-p_1^2)^2|\xi|^2$$

$$-\frac{3}{8}\pi^2\left(\pi^2-12\right)(4-p_1^2)p_1^2\operatorname{Re}(\overline{\xi})\Big)$$

(15) $$=: F(p_1^2,\,|\xi|,\,\operatorname{Re}(\overline{\xi})).$$

Note that $T_3(1) \le F(p_1^2,\,|\xi|,\,|\xi|) =: \boldsymbol{\chi}_1(p_1^2,\,|\xi|)$, and $T_3(1) \ge F(p_1^2,\,|\xi|,\,-|\xi|) =: \boldsymbol{\chi}_2(p_1^2,\,|\xi|)$. With these considerations, and setting $p_1^2 = x \in [0,4]$ and $|\xi| = y \in [0,1]$, it follows that

$$\boldsymbol{\chi}_1(x,\,y) = 1+\frac{32}{9\pi^8}\Big((54+3\pi^2-\frac{\pi^4}{8})x^2-9\pi^4x-\frac{9}{32}\pi^4(4-x)^2y^2 - \frac{3}{8}\pi^2\left(\pi^2-12\right)(4-x)xy\Big),$$

and

$$\boldsymbol{\chi}_2(x,\,y) = 1+\frac{32}{9\pi^8}\Big((54+3\pi^2-\frac{\pi^4}{8})x^2-9\pi^4x-\frac{9}{32}\pi^4(4-x)^2y^2 + \frac{3}{8}\pi^2\left(\pi^2-12\right)(4-x)xy\Big).$$

In the rectangular domain $[0,4]\times[0,1]$, the maximum value of $\boldsymbol{\chi}_1(x,\,y)$ is 1 which occurs at (0, 0), while the minimum value of $\boldsymbol{\chi}_2(x,\,y)$ occurring at (4, 1/2) is $1+\frac{64}{9\pi^8}(432+24\pi^2-19\pi^4)$. □

In order to establish upper bounds on $H_3(1)$ and $H_3(2)$ for functions $f \in \mathcal{S}_P^*$, the following results are required.

Lemma 2.2 ([24, Lemma 2.3, p. 507])**.** *Let* $p(z) = 1+p_1z+p_2z^2+p_3z^3+\cdots \in \mathcal{P}$. *Then*

$$|\mu p_n p_m - p_{m+n}| \le \begin{cases} 2, & 0 \le \mu \le 1; \\ 2|2\mu-1|, & \textit{elsewhere}, \end{cases}$$

for all $n, m \in \mathbb{N}$. *If* $0 < \mu < 1$, *then equality holds for the function* $p(z) = (1+z^{m+n})/(1-z^{m+n})$. *In all other cases, equality holds for the function* $p_0(z) = (1+z)/(1-z)$.

Lemma 2.3 ([14, Lemma 2.1, p. 1055])**.** *Let* $p(z) = 1+p_1z+p_2z^2+p_3z^3+\cdots \in \mathcal{P}$. *Then*

$$|\mu p_3 - p_1^3| \le \begin{cases} 2|\mu-4|, & \mu \le \frac{4}{3}; \\ 2\mu\sqrt{\frac{\mu}{\mu-1}}, & \mu > \frac{4}{3}, \end{cases}$$

for any real number μ. *The result is sharp. If* $\mu \le \frac{4}{3}$, *equality holds for the function* $p_0(z) := (1+z)/(1-z)$, *and if* $\mu > \frac{4}{3}$, *then equality holds for the function*

$$p_1(z) := \frac{1-z^2}{z^2-2\sqrt{\frac{\mu}{\mu-1}}\,z+1}.$$

Proof of Theorem 1.2. From (1), the third Hankel determinants $H_3(1)$ and $H_3(2)$ are

(16) $$H_3(1) = a_3(a_2a_4-a_3^2)-a_4(a_4-a_2a_3)+a_5(a_3-a_2^2),$$

and

$$H_3(2) = a_2(a_4a_6 - a_5^2) - a_3(a_3a_6 - a_4a_5) + a_4(a_3a_5 - a_4^2). \tag{17}$$

(i) Since $f \in \mathcal{S}_P^*$, substituting the values of a_2, a_3, a_4 and a_5 from (6), (7), (8) and (9), respectively, into the expression (16) yield

$$\begin{aligned} &255150\pi^{12}H_3(1) \\ &= 2520\pi^4p_1^2(19p_1^4 + 60p_1^2p_2 + 1440p_1p_3 - 1620p_2^2) - 7257600p_1^6 \\ &\quad - 907200\pi^2p_1^4(p_1^2 - 6p_2) + \pi^8(319p_1^6 - 5190p_1^4p_2 - 11970p_1^3p_3 \\ &\quad + 315p_1^2(101p_2^2 - 270p_4) + 132300p_1p_2p_3 - 28350(3p_2^3 - 18p_2p_4 \\ &\quad + 16p_3^2)) + 225\pi^6(61p_1^6 - 378p_1^4p_2 + 1008p_1^3p_3 - 252p_1^2(p_2^2 + 36p_4) \\ &\quad + 12096p_1p_2p_3 - 4536p_2^3) \\ &= \lambda_1p_1^6 - \lambda_2p_1^4p_2 - \lambda_3p_1^2p_2^2 + \lambda_4p_1p_2p_3 + \lambda_5p_1^3p_3 - \lambda_6p_3^2 \\ &\quad - \lambda_7p_1^2p_4 + \lambda_8p_2p_4 - \lambda_9p_2^3, \end{aligned}$$

where

$$\begin{aligned} \lambda_1 &= 319\pi^8 + 13725\pi^6 + 47880\pi^4 - 907200\pi^2 - 7257600 \approx 4.67455 \times 10^6, \\ \lambda_2 &= 5190\pi^8 + 85050\pi^6 - 151200\pi^4 - 5443200\pi^2 \approx 6.25611 \times 10^7, \\ \lambda_3 &= 4082400\pi^4 + 56700\pi^6 - 31815\pi^8 \approx 1.50296 \times 10^8, \\ \lambda_4 &= 2721600\pi^6 + 132300\pi^8 \approx 3.87185 \times 10^9, \\ \lambda_5 &= 3628800\pi^4 + 226800\pi^6 - 11970\pi^8 \approx 4.57943 \times 10^8, \\ \lambda_6 &= 453600\pi^8 \approx 4.304 \times 10^9, \\ \lambda_7 &= 2041200\pi^6 + 85050\pi^8 \approx 2.76939 \times 10^9, \\ \lambda_8 &= 510300\pi^8 \approx 4.842 \times 10^9, \text{ and} \\ \lambda_9 &= 1020600\pi^6 + 85050\pi^8 \approx 1.78819 \times 10^9. \end{aligned}$$

A rearrangement of terms followed by an application of the triangle inequality reduces the above expression to the form

$$\begin{aligned} 255150\pi^{12}|H_3(1)| &\le \lambda_2|p_1|^4\left|\frac{\lambda_1}{\lambda_2}p_1^2 - p_2\right| + \lambda_4|p_1||p_2|\left|-\frac{\lambda_3}{\lambda_4}p_1p_2 + p_3\right| \\ &\quad + \lambda_5|p_3|\left|p_1^3 - \frac{\lambda_6}{\lambda_5}p_3\right| + \lambda_8|p_4|\left|-\frac{\lambda_7}{\lambda_8}p_1^2 + p_2\right| + \lambda_9|p_2^3|. \end{aligned} \tag{18}$$

Since $\lambda_1/\lambda_2 = 0.0747197 < 1$, $\lambda_3/\lambda_4 = 0.038817 < 1$, $\lambda_6/\lambda_5 = 9.39854 > 4/3$ and $\lambda_7/\lambda_8 = 0.571951 < 1$, then by repeated use of Lemma 2.2 and Lemma 2.3 and the fact $|p_n| \le 2$, the expression (18) becomes

$$255150\pi^{12}|H_3(1)| \le \lambda_2(2)^5 + \lambda_4(2)^3 + 4\lambda_6\sqrt{\frac{\lambda_6}{\lambda_6 - \lambda_5}} + \lambda_8(2)^2 + \lambda_9(2)^3.$$

Upon substituting the values of λ_2, λ_4, λ_6, λ_8 and λ_9, yields

$$|H_3(1)| \leq \frac{16}{8505\pi^{10}}\Bigg(\frac{45360\sqrt{5}\pi^8}{\sqrt{739\pi^4-360\pi^2-5760}} + 8221\pi^6+68040\pi^4-10080\pi^2-362880\Bigg).$$

(ii) Since $f \in \mathcal{S}_P^*$, substituting the values of a_2, a_3, a_4, a_5 and a_6 from (6), (7), (8), (9) and (10), respectively, into the expression (17) yields

$$\begin{aligned}
&602791875\pi^{18}H_3(2)\\
&= 38102400\pi^4p_1^5(17p_1^4-60p_1^2p_2+720p_1p_3-540p_2^2)-18289152000p_1^9\\
&\quad -2835\pi^8p_1(5407p_1^8-31560p_1^6p_2+231840p_1^5p_3-2520p_1^4(53p_2^2-120p_4)\\
&\quad -1209600p_1^3(p_2p_3-3p_5)+756000p_1^2(p_2^3-12p_2p_4)+9072000p_1p_2^2p_3\\
&\quad -3402000p_2^4)+\pi^{12}(31133p_1^9-318780p_1^7p_2+5322240p_1^6p_3\\
&\quad -9450p_1^5(299p_2^2-2628p_4)-396900p_1^4(199p_2p_3-153p_5)\\
&\quad +18900p_1^3(2395p_2^3-13230p_2p_4+3339p_3^2)+2381400p_1^2(121p_2^2p_3\\
&\quad -90p_2p_5+90p_3p_4)-893025p_1(163p_2^4-420p_2^2p_4\\
&\quad +560p_2p_3^2-2880p_3p_5+2700p_4^2)+35721000(3p_2^3p_3-54p_2^2p_5\\
&\quad +90p_2p_3p_4-40p_3^3))\\
&= -\kappa_1p_1^9+\kappa_2p_1^7p_2-\kappa_3p_1^5p_2^2+\kappa_4p_1^6p_3+\kappa_5p_1^3p_2^3-\kappa_6p_1p_2^4-\kappa_7p_1^4p_2p_3\\
&\quad +\kappa_8p_1^2p_2^2p_3+\kappa_9p_1^5p_4+\kappa_{10}p_1^3p_2p_4-\kappa_{11}p_1^4p_5+\kappa_{12}p_2^3p_3+\kappa_{13}p_1^3p_3^2\\
&\quad -\kappa_{14}p_1p_2p_3^2-\kappa_{15}p_3^3+\kappa_{16}p_1p_2^2p_4+\kappa_{17}p_1^2p_3p_4+\kappa_{18}p_2p_3p_4-\kappa_{19}p_1p_4^2\\
&\quad -\kappa_{20}p_1^2p_2p_5-\kappa_{21}p_2^2p_5+\kappa_{22}p_1p_3p_5, \qquad (19)
\end{aligned}$$

where

$$\begin{aligned}
\kappa_1 &= 18289152000-647740800\pi^4+15328845\pi^8-31133\pi^{12} \approx 7.18663\times 10^{10},\\
\kappa_2 &= -1260\pi^4\left(1814400-71010\pi^4+253\pi^8\right) \approx 3.31634\times 10^{11},\\
\kappa_3 &= 20575296000\pi^4-378642600\pi^8+2825550\pi^{12} \approx 1.02303\times 10^{12},\\
\kappa_4 &= 27433728000\pi^4-657266400\pi^8+5322240\pi^{12} \approx 1.35498\times 10^{12},\\
\kappa_5 &= 45265500\pi^{12}-2143260000\pi^8 \approx 2.15011\times 10^{13},\\
\kappa_6 &= -9644670000\pi^8+145563075\pi^{12} \approx 4.30257\times 10^{13},\\
\kappa_7 &= -3429216000\pi^8+78983100\pi^{12} \approx 4.04634\times 10^{13},\\
\kappa_8 &= 288149400\pi^{12}-25719120000\pi^8 \approx 2.22909\times 10^{13},\\
\kappa_9 &= 24834600\pi^{12}-857304000\pi^8 \approx 1.48193\times 10^{13},\\
\kappa_{10} &= 25719120000\pi^8-250047000\pi^{12} \approx 1.29259\times 10^{13},
\end{aligned}$$

$\kappa_{11} = -60725700\pi^{12} + 10287648000\pi^{8} \approx 4.14878 \times 10^{13}$,
$\kappa_{12} = 107163000\pi^{12} \approx 9.90475 \times 10^{13}$,
$\kappa_{13} = 63107100\pi^{12} \approx 5.83279 \times 10^{13}$,
$\kappa_{14} = 500094000\pi^{12} \approx 4.62221 \times 10^{14}$,
$\kappa_{15} = 1428840000\pi^{12} \approx 1.32063 \times 10^{15}$,
$\kappa_{16} = 375070500\pi^{12} \approx 3.46666 \times 10^{14}$,
$\kappa_{17} = 214326000\pi^{12} \approx 1.98095 \times 10^{14}$,
$\kappa_{18} = 3214890000\pi^{12} \approx 2.97142 \times 10^{15}$,
$\kappa_{19} = 2411167500\pi^{12} \approx 2.22857 \times 10^{15}$,
$\kappa_{20} = 214326000\pi^{12} \approx 1.98095 \times 10^{14}$,
$\kappa_{21} = 1928934000\pi^{12} \approx 1.78285 \times 10^{15}$,
$\kappa_{22} = 2571912000\pi^{12} \approx 2.37714 \times 10^{15}$.

After a rearrangement of terms, the above expression reduces to

$$\begin{aligned}
&602791875\pi^{18} H_3(2) \\
&= \kappa_2\left(-\frac{\kappa_1}{\kappa_2}p_1^2 + p_2\right)p_1^7 - \kappa_3 p_1^5 p_2^2 + \kappa_7 p_1^4 p_3\left(\frac{\kappa_4}{\kappa_7}p_1^2 - p_2\right) \\
&\quad + \kappa_6 p_1 p_2^3\left(\frac{\kappa_5}{\kappa_6}p_1^2 - p_2\right) + \kappa_{14} p_1 p_2 p_3\left(\frac{\kappa_8}{\kappa_{14}}p_1 p_2 - p_3\right) \\
&\quad + \kappa_{11} p_1^4\left(\frac{\kappa_9}{\kappa_{11}}p_1 p_4 - p_5\right) + \kappa_{10} p_1^3 p_2 p_4 + \kappa_{12} p_2^3 p_3 \\
&\quad + \kappa_{13} p_3^2\left(-\frac{\kappa_{15}}{\kappa_{13}}p_3 + p_1^3\right) + \kappa_{21} p_2^2\left(\frac{\kappa_{16}}{\kappa_{21}}p_1 p_4 - p_5\right) \\
&\quad + \kappa_{19} p_1 p_4\left(\frac{\kappa_{17}}{\kappa_{19}}p_1 p_3 - p_4\right) + \kappa_{18} p_2 p_3 p_4 \\
&\quad + \kappa_{22} p_1 p_5\left(-\frac{\kappa_{20}}{\kappa_{22}}p_1 p_2 + p_3\right).
\end{aligned} \tag{20}$$

Since $\kappa_1/\kappa_2 = 0.216704 < 1$, $\kappa_4/\kappa_7 = 0.0334866 < 1$, $\kappa_5/\kappa_6 = 0.499727 < 1$, $\kappa_8/\kappa_{14} = 0.0482257 < 1$, $\kappa_9/\kappa_{11} = 0.357197 < 1$, $\kappa_{15}/\kappa_{13} = 22.6415 > 4/3$, $\kappa_{16}/\kappa_{21} = 0.194444 < 1$, $\kappa_{17}/\kappa_{19} = 0.0888 < 1$ and $\kappa_{20}/\kappa_{22} = 0.08333 < 1$, then use of the triangle inequality as well as repeated use of Lemma 2.2 along with Lemma 2.3 and the fact $|p_n| \le 2$, the expression (20) leads to

$$\begin{aligned}
602791875\pi^{18}|H_3(2)| \le \pi^{12}&\left(\frac{274337280000000}{53}\sqrt{\frac{3}{1147}} + 90779613120\right) \\
&+ \pi^4(598556044800\pi^4 + 2048385024000),
\end{aligned}$$

which implies that

$$|H_3(2)| \leq \frac{1}{\pi^6}\left(\frac{288189248}{1913625} + \frac{4096000}{159\sqrt{3441}}\right) + \frac{14075392}{14175\pi^{10}} + \frac{458752}{135\pi^{14}}. \qquad \square$$

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Rosihan M. Ali
School of Mathematical Sciences
Universiti Sains Malaysia
11800 USM, Penang, Malaysia
Email address: rosihan@usm.my

Sushil Kumar
Bharati Vidyapeeth's College of Engineering
Delhi–110063, India
Email address: sushilkumar16n@gmail.com

Vaithiyanathan Ravichandran
Department of Mathematics
National Institute of Technology
Tiruchirappalli–620 015, India
Email address: vravi68@gmail.com, ravic@nitt.edu