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COMMUTATIVITY OF MULTIPLICATIVE b-GENERALIZED DERIVATIONS OF PRIME RINGS

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Abstract. Consider $\mathscr{R}$ to be an associative prime ring and $\mathscr{K}$ to be a nonzero dense ideal of $\mathscr{R}$. A mapping (need not be additive) $\mathscr{F}:\mathscr{R}\to\mathscr{Q}_{mr}$ associated with derivation $d:\mathscr{R}\to\mathscr{R}$ is called a multiplicative b-generalized derivation if $\mathscr{F}(\alpha\delta)=\mathscr{F}(\alpha)\delta+b\alpha d(\delta)$ holds for all $\alpha,\delta\in\mathscr{R}$ and for any fixed $(0\neq)b\in\mathscr{Q}_s\subseteq\mathscr{Q}_{mr}$. In this manuscript, we study the commutativity of prime rings when the map b-generalized derivation satisfies the strong commutativity preserving condition and moreover, we investigate the commutativity of prime rings that admit multiplicative b-generalized derivation, which improves many results in the literature.

1. Introduction

The algebra of derivation and generalized derivation play a crucial role in studying functional identities and their applications. There are many generalizations of derivation, viz. generalized derivation, multiplicative generalized derivation, skew generalized derivation, b-generalized derivation, etc. The notion of b-generalized derivation was first introduced by Koşan and Lee [15]. The most important and systematic research on the b-generalized derivations has been accomplished in [9, 15–17] and references therein. In this manuscript, we present multiplicative b-generalized derivation on some suitable subset of ring $\mathscr{R}$ and discuss certain differential/functional identities having multiplicative b-generalized derivation.

Throughout, unless otherwise mentioned, $\mathscr{R}$ always denotes an associative prime ring with center $\mathscr{Z}(\mathscr{R})$ but not necessarily with an identity element. The Martindale right ring of quotients and the Martindale right symmetric ring of quotients of $\mathscr{R}$ are denoted by $\mathscr{Q}_{mr}$ and $\mathscr{Q}_s$, respectively. $\mathscr{C}=\mathscr{Z}(\mathscr{Q}_{mr})=\mathscr{Z}(\mathscr{Q}_s)$ is the extended centroid of $\mathscr{R}$ and is also known as the center of $\mathscr{Q}_{mr}$ and $\mathscr{Q}_s$. It is known that $\mathscr{R}\subseteq\mathscr{Q}_s\subseteq\mathscr{Q}_{mr}$, and the overrings $\mathscr{Q}_{mr}$ is prime if $\mathscr{R}$ is prime with the same center $\mathscr{C}$. Also, $\mathscr{R}$ is a prime ring if and only if $\mathscr{C}$ is a field. We refer the reader to the book [3] for details. A ring $\mathscr{R}$ is prime if $a\mathscr{R}b=(0)$, specifies that either $a=0$ or

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$b = 0$ for any $a, b \in \mathscr{R}$, and is considered as a semiprime if $a\mathscr{R}a = (0)$, implies $a = 0$ for any $a \in \mathscr{R}$. A right ideal $\mathscr{I}$ of $\mathscr{R}$ is said to be a dense right ideal if given any $0 \neq r_1 \in \mathscr{R}, r_2 \in \mathscr{R}$, there exists $r \in \mathscr{R}$ such that $r_1 r \neq 0$ and $r_2 r \in \mathscr{I}$. In a similar way, we define the dense left ideal, and if an ideal is both right and left dense ideal then it is called the dense ideal of $\mathscr{R}$. An additive mapping $d : \mathscr{R} \to \mathscr{R}$ is known as a derivation if $d(\alpha\delta) = d(\alpha)\delta + \alpha d(\delta)$ holds, for all $\alpha, \delta \in \mathscr{R}$. In particular, for a fixed $a \in \mathscr{R}$, the mapping $\mathscr{I}_a : \mathscr{R} \to \mathscr{R}$ given by $\mathscr{I}_a(\alpha) = [a, \alpha]$ is a derivation that is said to be an inner derivation of $\mathscr{R}$ induced by a. Let $\mathscr{F} : \mathscr{R} \to \mathscr{R}$ be a map associated with a derivation d such that $\mathscr{F}(\alpha\delta) = \mathscr{F}(\alpha)\delta + \alpha d(\delta)$ holds, for all $\alpha, \delta \in \mathscr{R}$. If $\mathscr{F}$ is additive, then $\mathscr{F}$ is known as a generalized derivation. However, if $\mathscr{F}$ is not necessarily additive, then $\mathscr{F}$ is said to be a multiplicative generalized derivation.

In 1991, Daif [7] introduced the notion of multiplicative derivation and provided an affirmative answer to the question raised by Martindale: When is a multiplicative derivation additive? Further, in continuation of this study, Daif and Tammam El-Sayiad [8] presented the concept of multiplicative generalized derivation and discussed a similar situation for additivity of multiplicative generalized derivation. Precisely, they proved that a multiplicative generalized derivation is an additive if $\mathscr{R}$ is a ring having an idempotent element e $(e \neq 0, 1)$, which satisfies the conditions- (i) $\alpha\mathscr{R}e = (0)$ implies $\alpha = 0$, (ii) $e\alpha e\mathscr{R}(1-e) = (0)$, implies $e\alpha e = 0$ and (iii) $(1-e)\alpha e\mathscr{R}(1-e) = (0)$ implies $(1-e)\alpha e = 0$.

In 2015, Dhara and Ali [10] investigated multiplicative generalized derivation on semiprime rings. Precisely, they stated that let $\mathscr{R}$ be a semiprime ring and $\mathscr{F} : \mathscr{R} \to \mathscr{R}$ be a multiplicative (generalized)-derivation associated with the map $g : \mathscr{R} \to \mathscr{R}$, such that if $\mathscr{F}(\alpha\delta) \pm \alpha\delta \in \mathscr{Z}(\mathscr{R})$ for all $\alpha, \delta \in \mathscr{R}$, then $[g(\alpha), \alpha] = 0$ for all $\alpha \in \mathscr{R}$. In 2016, Gölbaşi [11] studied certain identities having multiplicative generalized derivation $\mathscr{F}$ on a nonzero ideal $\mathscr{I}$ of a semiprime ring $\mathscr{R}$ and showed that $\mathscr{R}$ contains a nonzero central ideal. In the same study, it was also reported that a prime ring $\mathscr{R}$ must be commutative if $\mathscr{F}([\alpha, \delta]) = 0$, for all $\alpha, \delta \in \mathscr{I}$. In 2018, Koç and Gölbaşi [14] described the study of strong commutativity preserving (SCP) maps having multiplicative generalized derivations $\mathscr{F}$ associated with a nonzero additive map d and they established that, for a semiprime ring $\mathscr{R}$ it contains a nonzero central ideal if $\mathscr{F}$ is SCP on $\mathscr{I}$, where $\mathscr{I}$ a nonzero ideal of $\mathscr{R}$. Similar studies of derivation/generalized derivation/multiplicative generalized derivation can be seen in [1, 2, 4, 5, 18, 21–23] and references therein.

In this manuscript, we have presented a map $\mathscr{F} : \mathscr{R} \to \mathscr{Q}_{mr}$ associated with derivation (need not be additive) $d : \mathscr{R} \to \mathscr{R}$ such that $\mathscr{F}(\alpha\delta) = \mathscr{F}(\alpha)\delta + b\alpha d(\delta)$ holds for all $\alpha, \delta \in \mathscr{R}$ and any fixed $0 \neq b \in \mathscr{Q}_s \subseteq \mathscr{Q}_{mr}$. If $\mathscr{F}$ is additive (not necessarily additive), then $\mathscr{F}$ is called b-generalized derivation (multiplicative b-generalized derivation). Also, if b is unity, then we see that the map $\mathscr{F}$ from $\mathscr{R}$ to $\mathscr{Q}_{mr}$ is given by $\mathscr{F}(\alpha\delta) = \mathscr{F}(\alpha)\delta + \alpha d(\delta)$ for all $\alpha, \delta \in \mathscr{R}$ is considered as a $1-$generalized derivation (multiplicative $1-$generalized derivation) provided that $\mathscr{F}$ is additive (not necessarily additive). So we can say that b-generalized derivation (multiplicative b-generalized derivation) is a generalization of generalized derivation (multiplicative generalized derivation). Here, we present some related examples

Example 1.1. Let $\mathscr{F} : \mathscr{R} \to \mathscr{R}$ is a map defined by $\mathscr{F}(\alpha) = c\alpha + bd(\alpha)$, where $d : \mathscr{R} \to \mathscr{R}$ is not necessarily additive and $0 \neq b \in \mathscr{R}$ and for all $c, \alpha \in \mathscr{R}$, then clearly we can observe that the map $\mathscr{F}(\alpha\delta) = \mathscr{F}(\alpha)\delta + b\alpha d(\delta)$ for all $\alpha, \delta \in \mathscr{R}$, is a multiplicative b-generalized derivation associated with a multiplicative derivation d.

Example 1.2. Let $\mathscr{R} = \left\{ \begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix} \middle| x, y, z \in \mathbb{Z} \right\}$, where $\mathbb{Z}$ is the set of integers and $\mathscr{F}$ and d is a map from $\mathscr{R} \to \mathscr{R}$ such that $\mathscr{F}\left(\begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & yz \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $d\left(\begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then it is easy to verify that $\mathscr{F}$ is a multiplicative b-generalized derivation associated with derivation d and for any fixed $b \in \mathscr{R}$.

2. Preliminaries

Throughout the paper, we use one of the properties of Martindale right symmetric ring of quotients which states as follows: for any $q \in \mathscr{Q}_s$, there exists a dense right ideal $\mathscr{K}$ such that $q\mathscr{K} \cup \mathscr{K}q \subseteq \mathscr{R}$ and if $q\mathscr{K} = 0$ (or $\mathscr{K}q = 0$) if and only if $q = 0$. In our case, for $0 \neq b \in \mathscr{Q}_s \subseteq \mathscr{Q}_{mr}$, we assume that there exists a dense right ideal $\mathscr{K}$ such that $b\mathscr{K} \cup \mathscr{K}b \subseteq \mathscr{R}$, i.e. bx or $xb \in \mathscr{R}$ for all $x \in \mathscr{K}$ and $b\mathscr{K} = 0$ (or $\mathscr{K}b = 0$) if and only if $b = 0$. In this section, we give some well-known basic identities, which will be used extensively in the forthcoming sections.

(*i*) $[x, yz] = y[x, z] + [x, y]z$
(*ii*) $[xy, z] = x[y, z] + [x, z]y$
(*ii*) $(x \circ yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$
(*iv*) $(xy \circ z) = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$.

Prior to commencing our investigation, we will present a few notable findings that will be frequently used throughout the paper.

Lemma 2.1. [19, Lemma 3] *If a prime ring $\mathscr{R}$ contains a commutative nonzero right ideal $\mathscr{I}$, then $\mathscr{R}$ is commutative.*

Lemma 2.2. [4, Theorem 4] *Let $\mathscr{R}$ be a prime ring and $\mathscr{I}$ be a nonzero left ideal of $\mathscr{R}$. If $\mathscr{R}$ admits a nonzero derivation d which is centralizing on $\mathscr{I}$, then $\mathscr{R}$ is commutative.*

Lemma 2.3. *Let $\mathscr{R}$ be a prime ring. If $[[\alpha, \beta], \delta] = 0$ for all $\alpha, \beta, \delta \in \mathscr{R}$, then $[\alpha, \beta] = 0$.*

Proof. We have given

(1) $$[[\alpha, \beta], \delta] = 0 \text{ for all } \alpha, \beta, \delta \in \mathscr{R}.$$

Replacing α by $\gamma\alpha$ in (1) for all $\gamma \in \mathscr{R}$, and using the identity of commutator, we have

(2) $$[\gamma[\alpha, \beta], \delta] + [[\gamma, \beta]\alpha, \delta] = 0 \text{ for all } \alpha, \beta, \delta, \gamma \in \mathscr{R}.$$

Again we use the identity of commutator in (2), we find that

(3) $$\gamma[[\alpha, \beta], \delta] + [\gamma, \delta][\alpha, \beta] + [\gamma, \beta][\alpha, \delta] + [[\gamma, \beta], \delta]\alpha = 0 \text{ for all } \alpha, \beta, \delta, \gamma \in \mathscr{R}.$$

Using our hypothesis in (3), we obtain

(4) $$[\gamma,\delta][\alpha,\beta]+[\gamma,\beta[\alpha,\delta]=0 \text{ for all } \alpha,\beta,\delta,\gamma\in\mathscr{R}.$$

In particular $\delta=\alpha$, the above equation yields

(5) $$[\gamma,\alpha][\alpha,\beta]=0 \text{ for all } \alpha,\beta,\gamma\in\mathscr{R}.$$

Substituting γ by $\beta\gamma$ in (5), we get

(6) $$\beta[\gamma,\alpha][\alpha,\beta]+[\beta,\alpha]\gamma[\alpha,\beta]=0 \text{ for all } \alpha,\beta,\gamma\in\mathscr{R}.$$

Using (5) in (6), we have

(7) $$[\beta,\alpha]\gamma[\alpha,\beta]=0 \text{ for all } \alpha,\beta,\gamma\in\mathscr{R}.$$

This implies that,

(8) $$[\beta,\alpha]\mathscr{R}[\alpha,\beta]=(0) \text{ for all } \alpha,\beta\in\mathscr{R}.$$

Since $\mathscr{R}$ is a prime ring, then we get either $[\beta,\alpha]=0$ or $[\alpha,\beta]=0$. Consequently, in any cases, it follows that $[\alpha,\beta]=0$ for all $\alpha,\beta\in\mathscr{R}$. □

3. Main Results

THEOREM 3.1. *Let $\mathscr{R}$ be a prime ring and $\mathscr{K}$ be a nonzero dense ideal of $\mathscr{R}$. Next, let $\mathscr{F}:\mathscr{R}\rightarrow\mathscr{Q}_{mr}$ be a multiplicative b-generalized derivation associated with derivation $d:\mathscr{R}\rightarrow\mathscr{R}$ satisfying the condition $[\mathscr{F}(\alpha),\beta]\in\mathscr{Z}(\mathscr{R})$ for all $\alpha,\beta\in\mathscr{K}$ and any $0\neq b\in\mathscr{Q}_s\subseteq\mathscr{Q}_{mr}$. Then $\mathscr{R}$ is commutative.*

Proof. We have given $[\mathscr{F}(\alpha),\beta]\in\mathscr{Z}(\mathscr{R})$ for all $\alpha,\beta\in\mathscr{K}$. Replacing α by $\alpha\beta$ in our hypothesis and using the definition of multiplicative b-generalized derivation, we get

(9) $$[\mathscr{F}(\alpha)\beta,\beta]+[b\alpha d(\beta),\beta]\in\mathscr{Z}(\mathscr{R}) \text{ for all } \alpha,\beta\in\mathscr{K}.$$

Above equation can be rewritten as

(10) $$[\mathscr{F}(\alpha),\beta]\beta+[b\alpha d(\beta),\beta]\in\mathscr{Z}(\mathscr{R}) \text{ for all } \alpha,\beta\in\mathscr{K}.$$

Commuting (10) with $\beta\in\mathscr{R}$, then we have

(11) $$[\mathscr{F}(\alpha),\beta]\beta\beta+[b\alpha d(\beta),\beta]\beta=\beta[\mathscr{F}(\alpha),\beta]\beta+\beta[b\alpha d(\beta),\beta] \text{ for all } \alpha,\beta\in\mathscr{K}.$$

Using our hypothesis in (11), we have

$$[\mathscr{F}(\alpha),\beta]\beta\beta+[b\alpha d(\beta),\beta]\beta=[\mathscr{F}(\alpha),\beta]\beta\beta+\beta[b\alpha d(\beta),\beta] \text{ for all } \alpha,\beta\in\mathscr{K}.$$

This gives us

(12) $$[[b\alpha d(\beta),\beta],\beta]=0 \text{ for all } \alpha,\beta\in\mathscr{K}.$$

By using Lemma 2.3, we have

(13) $$[b\alpha d(\beta),\beta]=0 \text{ for all } \alpha,\beta\in\mathscr{K}.$$

By the mention property of Martindale right symmetric ring of quotients, we substitute α by $\gamma b\alpha$ in (13) for all $\gamma\in\mathscr{K}$ and then using it, we obtain

(14) $$[b\gamma,\beta]b\alpha d(\beta)=0 \text{ for all } \alpha,\beta,\gamma\in\mathscr{K}.$$

Replacing α by $\alpha\delta$ in (14) for all $\delta\in\mathscr{K}$, we obtain

(15) $$[b\gamma,\beta]b\alpha\delta d(\beta)=0 \text{ for all } \alpha,\beta,\delta,\gamma\in\mathscr{K}.$$

Multiplying (14) by δ from the right, we find that

$$[b\gamma, \beta]b\alpha d(\beta)\delta = 0 \text{ for all } \alpha, \beta, \delta, \gamma \in \mathscr{K}. \tag{16}$$

Subtracting (15) from (16), we get

$$[b\gamma, \beta]b\alpha[d(\beta), \delta] = 0 \text{ for all } \alpha, \beta, \delta, \gamma \in \mathscr{K}. \tag{17}$$

In the above equation, we substitute α by $r\alpha$ for all $r \in \mathscr{R}$, we find that

$$[b\gamma, \beta]b\mathscr{R}\alpha[d(\beta), \delta] = (0) \text{ for all } \alpha, \beta, \delta, \gamma \in \mathscr{K}. \tag{18}$$

Since $\mathscr{R}$ is prime, then for each $\beta \in \mathscr{K} \subseteq \mathscr{R}$, we have either $[b\gamma, \beta]b = 0$ or $\alpha[d(\beta), \delta] = 0$ for all $\alpha, \gamma, \delta \in \mathscr{K}$. As a result, $\mathscr{R}$ is the union of two additive subgroups $\mathscr{A}$ and $\mathscr{B}$, where

$$\mathscr{A} = \{\beta \in \mathscr{K} \mid [b\gamma, \beta]b = 0\} \text{ and } \mathscr{B} = \{\beta \in \mathscr{K} \mid \alpha[d(\beta), \delta] = 0\}.$$

We must conclude that either $\mathscr{R} = \mathscr{A}$ or $\mathscr{R} = \mathscr{B}$ because a group cannot be a union of its proper subgroups. If $\mathscr{R} = \mathscr{A}$, then $[b\gamma, \beta]b = 0$ for all $\gamma \in \mathscr{K}$, now we replace β by $\alpha\beta$ in $[b\gamma, \beta]b = 0$ for all $\alpha \in \mathscr{K}$, we get

$$\alpha[b\gamma, \beta]b + [b\gamma, \alpha]\beta b = 0 \text{ for all } \alpha, \beta, \gamma \in \mathscr{K}. \tag{19}$$

Since $[b\gamma, \beta]b = 0$, then by using these in (19), we have

$$[b\gamma, \alpha]\beta b = 0 \text{ for all } \alpha, \beta, \gamma \in \mathscr{K}. \tag{20}$$

Again by replacing β by $r\beta$ for all $r \in \mathscr{R}$ and using the primeness of $\mathscr{R}$, then we have either $\beta b = 0$ or $[b\gamma, \alpha] = 0$ for all $\alpha, \beta, \gamma \in \mathscr{K}$. If $\beta b = 0$, then by the property of Martindale's right symmetric ring of quotients we have $b = 0$, which is a contradiction. Then we have $[b\gamma, \alpha] = 0$, now we substitute γ by $\gamma\beta$ for all $\beta \in \mathscr{K}$ and then using it, we obtain

$$b\gamma[\beta, \alpha] = 0 \text{ for all } \alpha, \beta, \gamma \in \mathscr{K}. \tag{21}$$

Again we replace γ by γs for all $s \in \mathscr{R}$ in the above equation and using the primeness of $\mathscr{R}$, gives us either $b\gamma = 0$ or $[\beta, \alpha] = 0$. By the above explanation, we get $b\gamma \neq 0$, so we have $[\beta, \alpha] = 0$, i.e. $\mathscr{K}$ is commuting. By using Lemma 2.1, we get $\mathscr{R}$ is commutative.

Next, if we take $\mathscr{R} = \mathscr{B}$, then for each $\beta \in \mathscr{K}$ we have $\alpha[d(\beta), \delta] = 0$ and $[b\gamma, \beta]b \neq 0$ for all $\alpha, \beta, \gamma \in \mathscr{K}$. After replacing α by αr for all $r \in \mathscr{R}$ and using the primeness of $\mathscr{R}$, we have $[d(\beta), \delta] = 0$ since $\mathscr{K}$ is nonzero. In particular, for $\delta = \beta$ and using the Lemma 2.2, we have $\mathscr{R}$ is commutative. □

Corollary 3.2. *Let $\mathscr{R}$ be a prime ring and $\mathscr{K}$ be a nonzero dense ideal of $\mathscr{R}$. Next, let $\mathscr{F} : \mathscr{R} \to \mathscr{Q}_{mr}$ be a multiplicative b-generalized derivation associated with derivation $d : \mathscr{R} \to \mathscr{R}$ satisfying the condition $[\mathscr{F}(\alpha), \beta] = 0$ for all $\alpha, \beta \in \mathscr{K}$ and any $0 \neq b \in \mathscr{Q}_s \subseteq \mathscr{Q}_{mr}$. Then $\mathscr{R}$ is commutative.*

Proof. In our hypothesis, replacing α by $\alpha\beta$ for all $\beta \in \mathscr{K}$ and using the hypothesis, we obtain

$$[\mathscr{F}(\alpha)\beta, \beta] + [b\alpha d(\beta), \beta] = 0 \text{ for all } \alpha, \beta \in \mathscr{K}. \tag{22}$$

Using the property of commutator, above relation yields that

$$[b\alpha d(\beta), \beta] = 0 \text{ for all } \alpha, \beta \in \mathscr{K}. \tag{23}$$

The above relation is the same as the relation (13). So, the above argument is true for this corollary, then we get $\mathscr{R}$ is commutative. □

THEOREM 3.3. *Let $\mathscr{R}$ be a prime ring and $\mathscr{K}$ be a nonzero dense ideal of $\mathscr{R}$. Next, let $\mathscr{F}:\mathscr{R}\rightarrow\mathscr{Q}_{mr}$ be a multiplicative b-generalized derivation associated with derivation $d:\mathscr{R}\rightarrow\mathscr{R}$ such that $\mathscr{F}(\alpha\delta)-\alpha\delta\in\mathscr{Z}(\mathscr{R})$ for all $\alpha,\delta\in\mathscr{K}$ and any $0\neq b\in\mathscr{Q}_s\subseteq\mathscr{Q}_{mr}$. Then $\mathscr{R}$ is commutative.*

Proof. We have given that $\mathscr{F}(\alpha\delta)-\alpha\delta\in\mathscr{Z}(\mathscr{R})$, for all $\alpha,\delta\in\mathscr{K}$. Replace δ by $\delta\theta$ for all $\theta\in\mathscr{K}$, our hypothesis reduces to

$$(\mathscr{F}(\alpha\delta)-\alpha\delta)\theta+b\alpha\delta d(\theta)\in\mathscr{Z}(\mathscr{R}),\ \text{for all }\alpha,\delta,\theta\in\mathscr{K}. \tag{24}$$

Commuting both sides of (24) with $\theta\in\mathscr{Z}(\mathscr{R})$, on simplifying, we obtain that

$$b\alpha\delta d(\theta)\theta-\theta b\alpha\delta d(\theta)=0,\ \text{for all }\alpha,\delta,\theta\in\mathscr{K}. \tag{25}$$

This implies that

$$[b\alpha\delta d(\theta),\theta]=0,\ \text{for all }\alpha,\delta,\theta\in\mathscr{K}. \tag{26}$$

Replace α by $\beta b\alpha$ in (26) and using the property of commutator, we find that

$$b\beta[b\alpha\delta d(\theta),\theta]+[b\beta,\theta]b\alpha\delta d(\theta)=0,\ \text{for all }\alpha,\beta,\delta,\theta\in\mathscr{K}. \tag{27}$$

Using (26) in (27), we obtain

$$[b\beta,\theta]b\alpha\delta d(\theta)=0,\ \text{for all }\alpha,\beta,\delta,\theta\in\mathscr{K}. \tag{28}$$

Replacing δ by $\delta\gamma$ for all $\gamma\in\mathscr{K}$ in (28), we have

$$[b\beta,\theta]b\alpha\delta\gamma d(\theta)=0,\ \text{for all }\alpha,\beta,\delta,\theta\in\mathscr{K}. \tag{29}$$

On multiplying (28) by γ from right, we see that

$$[b\beta,\theta]b\alpha\delta d(\theta)\gamma=0,\ \text{for all }\alpha,\beta,\delta,\theta\in\mathscr{K}. \tag{30}$$

On combining (29) and (30), we have

$$[b\beta,\theta]b\alpha\delta[d(\theta),\gamma]=0,\ \text{for all }\alpha,\beta,\delta,\theta\in\mathscr{K}. \tag{31}$$

Replacing δ by $r\delta$ for all $r\in\mathscr{R}$ and using the primeness condition of $\mathscr{R}$, for each $\theta\in\mathscr{K}$ we have either $[b\beta,\theta]b\alpha=0$ or $\delta[d(\theta),\gamma]=0$ for all $\alpha,\beta,\delta,\theta,\gamma\in\mathscr{K}$. This yields two additive subgroups of $\mathscr{R}$ (say, $\mathscr{P}$ and $\mathscr{Q}$), where

$$\mathscr{P}=\{\theta\in\mathscr{K}\mid[b\beta,\theta]b\alpha=0\mid\alpha,\beta,\in\mathscr{K}\}$$

and

$$\mathscr{Q}=\{\theta\in\mathscr{K}\mid\delta[d(\theta),\gamma]=0\mid\delta,\gamma\in\mathscr{K}\}.$$

Consequently, $\mathscr{R}$ is a union of two additive subgroups $\mathscr{P}$ and $\mathscr{Q}$, but a group cannot be a union of two of its proper subgroups, so we are forced to conclude that either $\mathscr{R}=\mathscr{P}$ or $\mathscr{R}=\mathscr{Q}$. Now, first we assume $\mathscr{R}=\mathscr{P}$, then we have $[b\beta,\theta]b\alpha=0$ for all $\alpha,\beta,\theta\in\mathscr{K}$. Replacing θ by $\theta\alpha$ in $[b\beta,\theta]b\alpha=0$ and using it also, we obtain that $[b\beta,\theta]\alpha b\alpha=0$. Again we substitute α by $r\alpha$ for all $r\in\mathscr{R}$, then the previous relation yields that $[b\beta,\theta]r\alpha br\alpha=0$. Since, $\mathscr{R}$ is a prime ring then from last relation we obtain that $[b\beta,\theta]=0$ or $\alpha b=0$ or $\alpha=0$, since $\mathscr{K}\neq0$ implies that $\alpha\neq0$ and if $\alpha b=0$, then by the property of Martindale right symmetric ring of quotients we get $b=0$, which contradict our hypothesis. So, finally, we have $[b\beta,\theta]=0$. Taking $\beta\alpha$ in place of β and applying it, gives us $b\beta[\alpha,\theta]=0$. Considering β as βr for all $r\in\mathscr{R}$ and using the primeness of $\mathscr{R}$, we find that either $b\beta=0$ (not possible by above argument) or $[\alpha,\theta]=0$. By using Lemma 2.1, we get $\mathscr{R}$ is commutative.

Next, if we consider $\mathscr{R} = \mathscr{Q}$, we have $\delta[d(\theta), \gamma] = 0$ for all $\theta, \delta, \gamma \in \mathscr{K}$. Taking δr for all $r \in \mathscr{R}$ in place of δ and using the primeness of $\mathscr{R}$, we obtain $[d(\theta), \gamma] = 0$ (since, $\mathscr{K} \neq 0$). In particular, for $\gamma = \theta$ we get $[d(\theta), \theta] = 0$, i.e. d is a commuting on $\mathscr{K}$. Then by Lemma 2.2, $\mathscr{R}$ is commutative. □

Corollary 3.4. *Let $\mathscr{R}$ be a prime ring and $0 \neq \mathscr{K}$ be a dense ideal of $\mathscr{R}$. Next, let $\mathscr{F} : \mathscr{R} \to \mathscr{Q}_{mr}$ be a multiplicative b-generalized derivation associated with derivation $d : \mathscr{R} \to \mathscr{R}$ such that $\mathscr{F}(\alpha\delta) - \alpha\delta = 0$ for all $\alpha, \delta \in \mathscr{K}$ and for any $0 \neq b \in \mathscr{Q}_s \subseteq \mathscr{Q}_{mr}$. Then $\mathscr{R}$ is commutative.*

Proof. We have given that

$$\mathscr{F}(\alpha\delta) - \alpha\delta = 0, \text{ for all } \alpha, \delta \in \mathscr{K}. \tag{32}$$

Replace δ by $\delta\theta$ for all $\theta \in \mathscr{K}$ in (32), we can see that

$$(\mathscr{F}(\alpha\delta) - \alpha\delta)\theta + b\alpha\delta d(\theta) = 0, \text{ for all } \alpha, \delta, \theta \in \mathscr{K}. \tag{33}$$

In view of (32), (33) reduces to

$$b\alpha\delta d(\theta) = 0, \text{ for all } \alpha, \delta, \theta \in \mathscr{K}. \tag{34}$$

Replace θ by $l\theta$ for all $l \in \mathscr{K}$ in (34), we get

$$b\alpha\delta d(l)\theta + b\alpha\delta l d(\theta) = 0, \text{ for all } \alpha, \delta, \theta, l \in \mathscr{K}. \tag{35}$$

By using (34) in (35), we see that

$$b\alpha\delta l d(\theta) = 0, \text{ for all } \alpha, \delta, \theta, l \in \mathscr{K}. \tag{36}$$

On multiplying (34) from the right by l, then we get

$$b\alpha\delta d(\theta) l = 0, \text{ for all } \alpha, \delta, \theta, l \in \mathscr{K}. \tag{37}$$

Subtracting (36) from (37), we get

$$b\alpha\delta[d(\theta), l] = 0, \text{ for all } \alpha, \delta, \theta, l \in \mathscr{K}. \tag{38}$$

Since $\mathscr{K}$ is dense ideal of $\mathscr{R}$ which is an ideal of $\mathscr{R}$ also, so we replace δ by δr in (38), we see that

$$b\alpha\delta r[d(\theta), l] = 0 \text{ for all } \alpha, \delta, \theta, l \in \mathscr{K} \text{ this implies that } b\alpha\delta\mathscr{R}[d(\theta), l] = (0).$$

Therefore, either $b\alpha\delta = 0$ or $[d(\theta), l] = 0$ by primeness of $\mathscr{R}$.

Now consider if $b\alpha\delta = 0$, then we replace δ by $r\delta$ and we get $b\alpha r\delta = 0$, for all $\alpha, \delta \in \mathscr{K}, r \in \mathscr{R}$. In particular, it follows that $b\alpha\mathscr{R}\delta = (0)$. Thus either $b\alpha = 0$ or $\delta = 0$. Since $\mathscr{K} \neq 0$, so we can not take $\delta = 0$ for all $\delta \in \mathscr{K}$. Hence we have $b\alpha = 0$, which is a contradiction, as we have discussed earlier. Therefore, we get $b\alpha\delta \neq 0$. Hence, the only possibility is $[d(\theta), l] = 0$ for all $\theta, l \in \mathscr{K}$. In particular for $\theta = l$, we have $[d(l), l] = 0$ and thus by Lemma 2.2, $\mathscr{R}$ is commutative. □

Theorem 3.5. *Let $\mathscr{R}$ be a prime ring and $\mathscr{K}$ be a nonzero dense ideal of $\mathscr{R}$. Next, let $\mathscr{F} : \mathscr{R} \to \mathscr{Q}_{mr}$ be a multiplicative b-generalized derivation associated with derivation $d : \mathscr{R} \to \mathscr{R}$ such that $\mathscr{F}([\alpha, \delta]) = 0$ for all $\alpha, \delta \in \mathscr{K}$ and any $0 \neq b \in \mathscr{Q}_s \subseteq \mathscr{Q}_{mr}$. Then $\mathscr{R}$ is commutative.*

Proof. We have given that $\mathscr{F}([\alpha,\delta])=0$ for all $\alpha,\delta\in\mathscr{K}$. Substituting $\delta\alpha$ for δ in our hypothesis, we get

$$\mathscr{F}([\alpha,\delta]\alpha)=0, \text{ for all } \alpha,\delta\in\mathscr{K}. \tag{39}$$

This can be rewritten as

$$\mathscr{F}([\alpha,\delta])\alpha+b[\alpha,\delta]d(\alpha)=0, \text{ for all } \alpha,\delta\in\mathscr{K}. \tag{40}$$

By using our hypothesis in (40), we get

$$b[\alpha,\delta]d(\alpha)=0, \text{ for all } \alpha,\delta\in\mathscr{K}. \tag{41}$$

Using the mentioned property of Martindale right symmetric ring of quotients, we substitute δ by $\delta b\theta$ for all $\theta\in\mathscr{K}$ in (41) and then using it, we obtain

$$b[\alpha,\delta b]\theta d(\alpha)=0, \text{ for all } \alpha,\delta\in\mathscr{K}. \tag{42}$$

Now substituting θ by $\theta\beta$ for all $\beta\in\mathscr{K}$ in (42), gives us

$$b[\alpha,\delta b]\theta\beta d(\alpha)=0, \text{ for all } \alpha,\delta\in\mathscr{K}. \tag{43}$$

Multiplying (42) by β from right, this gives

$$b[\alpha,\delta b]\theta d(\alpha)\beta=0, \text{ for all } \alpha,\delta\in\mathscr{K}. \tag{44}$$

Subtracting (43) from (44), yields that

$$b[\alpha,\delta b]\theta[d(\alpha),\beta]=0, \text{ for all } \alpha,\beta,\delta,\theta\in\mathscr{K}. \tag{45}$$

By taking $r\theta$ for θ for all $r\in\mathscr{R}$ in the above equation, we see that

$$b[\alpha,\delta b]r\theta[d(\alpha),\beta]=0, \text{ for all } \alpha,\delta\in\mathscr{K}, r\in\mathscr{R}.$$

This implies that

$$b[\alpha,\delta b]\mathscr{R}\theta[d(\alpha),\beta]=(0), \text{ for all } \alpha,\delta,\theta\in\mathscr{K}. \tag{46}$$

Since $\mathscr{R}$ is prime, for each $\alpha\in\mathscr{K}$, either $b[\alpha,\delta b]=0$ or $\theta[d(\alpha),\beta]=0$, for all $\beta,\delta,\theta\in\mathscr{K}$. We get two additive proper subgroups of $\mathscr{K}$ (say $\mathscr{M}$ and $\mathscr{N}$), such that

$$\mathscr{M}=\{\alpha\in\mathscr{K}\mid b[\alpha,\delta b]=0,\forall\ \delta\in\mathscr{K}\}$$

and

$$\mathscr{N}=\{\alpha\in\mathscr{K}\mid \theta[d(\alpha),\beta]=0,\forall\ \beta,\theta\in\mathscr{K}\}.$$

Given that $\mathscr{K}$ is a set-theoretic union of $\mathscr{M}$ and $\mathscr{N}$, but that a group cannot be a set-theoretic union of two proper subgroups, hence either $\mathscr{M}=\mathscr{K}$ or $\mathscr{N}=\mathscr{K}$. If we consider $\mathscr{N}=\mathscr{K}$, then we have $\theta[d(\alpha),\beta]=0$, for all $\alpha,\beta,\theta\in\mathscr{K}$. Now, we substituting θ by θr for all $r\in\mathscr{R}$, we find that

$$\theta r[d(\alpha),\beta]=0, \text{ for all } \alpha,\beta,\theta\in\mathscr{K}, r\in\mathscr{R}.$$

This implies that

$$\theta\mathscr{R}[d(\alpha),\beta]=(0).$$

Since $\mathscr{R}$ is prime ring then we get either $\theta=0$ (not possible, since $\mathscr{K}\neq(0)$) or $[d(\alpha),\beta]=0$ for all $\alpha,\beta\in\mathscr{K}$. In particular, for $\beta=\alpha$ we have $[d(\alpha),\alpha]=0$. So, d is a commuting map over $\mathscr{K}$, therefore, by Lemma 2.2, $\mathscr{R}$ is commutative.

Next, we have $\mathscr{M}=\mathscr{K}$, i.e. $b[\alpha,\delta b]=0$ for all $\alpha,\delta\in\mathscr{K}$. Now we replace α by $\alpha\beta$ in $b[\alpha,\delta b]=0$ for all $\beta\in\mathscr{K}$ and using it, we have

$$b\alpha[\beta,\delta b]=0, \text{ for all } \alpha,\beta,\delta\in\mathscr{K}. \tag{47}$$

Replacing α by αr for all $r \in \mathscr{R}$ in (47) and using the primeness of $\mathscr{R}$, then we get either $b\alpha = 0$, which is a contradiction (by using the property of Martindale right symmetric ring of quotients) or $[\beta, \delta b] = 0$ for all $\beta, \delta \in \mathscr{K}$. Now replacing δ by $\alpha\delta$ for all $\alpha \in \mathscr{K}$ in $[\beta, \delta b] = 0$ and using it, we obtain

$$[\beta, \alpha]\delta b = 0, \text{ for all } \alpha, \beta, \delta \in \mathscr{K}. \tag{48}$$

Again we substitute δ by $r\delta$ for all $r \in \mathscr{R}$ in (48), then we get either $\delta b = 0$ (not possible by above argument) or $[\beta, \alpha] = 0$ for all $\alpha, \beta \in \mathscr{K}$. So, $\mathscr{K}$ is commutative, then by Lemma 2.1, we get $\mathscr{R}$ is commutative. □

In the following theorem, we will study the strong commutativity preserving map having b-generalized derivation in place of multiplicative b-generalized derivation acting on prime ring $\mathscr{R}$. Now, we have assume that for $b \in \mathscr{Q}_s \subseteq \mathscr{Q}_{mr}$, there exist a dense right ideal $\mathscr{R}$ such that $b\mathscr{R} \cup \mathscr{R}b \subseteq \mathscr{R}$. More precisely, here is the theorem

THEOREM 3.6. *Let $\mathscr{R}$ be a prime ring and $\mathscr{F} : \mathscr{R} \to \mathscr{R}$ be a b-generalized derivation associated with nonzero derivation $d : \mathscr{R} \to \mathscr{R}$ such that $[\mathscr{F}(\alpha), \mathscr{F}(\delta)] - [\alpha, \delta] = 0$ for all $\alpha, \delta \in \mathscr{R}$ and for fixed $0 \neq b \in \mathscr{Q}_s \subseteq \mathscr{Q}_{mr}$, then one of the following assertions holds:*
(i) *$\mathscr{R}$ is commutative*
(ii) $[\mathscr{F}(\mathscr{R}), b]b = 0$.

Proof. (i) We have given that

$$[\mathscr{F}(\alpha), \mathscr{F}(\delta)] - [\alpha, \delta] = 0, \text{ for all } \alpha, \delta \in \mathscr{R}. \tag{49}$$

Substituting δr for δ for all $r \in \mathscr{R}$ in (49), we have

$$[\mathscr{F}(\alpha), \mathscr{F}(\delta)r] + [\mathscr{F}(\alpha), b\delta d(r)] - \delta[\alpha, r] - [\alpha, \delta]r = 0 \text{ for all } \alpha, \delta, r \in \mathscr{R}.$$

Using the property of commutator, the above relation can be expressed as

$$\begin{aligned} &\mathscr{F}(\delta)[\mathscr{F}(\alpha), r] + [\mathscr{F}(\alpha), \mathscr{F}(\delta)]r + b\delta[\mathscr{F}(\alpha), d(r)] \\ &+[\mathscr{F}(\alpha), b\delta]d(r) - \delta[\alpha, r] - [\alpha, \delta]r = 0 \text{ for all } \alpha, \delta, r \in \mathscr{R}. \end{aligned} \tag{50}$$

Using the hypothesis in (50), we find that

$$\begin{aligned} &\mathscr{F}(\delta)[\mathscr{F}(\alpha), r] + b\delta[\mathscr{F}(\alpha), d(r)] \\ &+[\mathscr{F}(\alpha), b\delta]d(r) - \delta[\alpha, r] = 0 \text{ for all } \alpha, \delta, r \in \mathscr{R}. \end{aligned} \tag{51}$$

Now, substituting $\mathscr{F}(\alpha)$ for r in (51), we get

$$\begin{aligned} &b\delta[\mathscr{F}(\alpha), d(\mathscr{F}(\alpha))] + [\mathscr{F}(\alpha), b\delta]d(\mathscr{F}(\alpha)) \\ &-\delta[\alpha, \mathscr{F}(\alpha)] = 0 \text{ for all } \alpha, \delta \in \mathscr{R}. \end{aligned} \tag{52}$$

On replacing r by $\theta\mathscr{F}(\alpha)$ for all $\theta \in \mathscr{R}$ in (51), then we have

$$\begin{aligned} &\mathscr{F}(\delta)[\mathscr{F}(\alpha), \theta\mathscr{F}(\alpha)] + b\delta[\mathscr{F}(\alpha), d(\theta\mathscr{F}(\alpha))] \\ &+[\mathscr{F}(\alpha), b\delta]d(\theta\mathscr{F}(\alpha)) - \delta[\alpha, \theta\mathscr{F}(\alpha)] = 0 \text{ for all } \alpha, \delta, \theta \in \mathscr{R}. \end{aligned}$$

This implies that,

$$\begin{aligned} &\mathscr{F}(\delta)[\mathscr{F}(\alpha), \theta]\mathscr{F}(\alpha) + b\delta[\mathscr{F}(\alpha), d(\theta)\mathscr{F}(\alpha)] + b\delta[\mathscr{F}(\alpha), \theta d(\mathscr{F}(\alpha))] \\ &+ [\mathscr{F}(\alpha), b\delta]d(\theta)\mathscr{F}(\alpha) + [\mathscr{F}(\alpha), b\delta]\theta d(\mathscr{F}(\alpha)) - \delta\theta[\alpha, \mathscr{F}(\alpha)] \\ &- \delta[\alpha, \theta]\mathscr{F}(\alpha) = 0 \text{ for all } \alpha, \delta, \theta \in \mathscr{R}. \end{aligned} \tag{53}$$

for all $\alpha, \delta, \theta \in \mathscr{R}$, using the identities of commutator in (53), we have

$$(\mathscr{F}(\delta)[\mathscr{F}(\alpha),\theta] + b\delta[\mathscr{F}(\alpha), d(\theta)] + [\mathscr{F}(\alpha), b\delta]d(\theta) - \delta[\alpha,\theta])\mathscr{F}(\alpha) + b\delta[\mathscr{F}(\alpha), \theta d(\mathscr{F}(\alpha))] + [\mathscr{F}(\alpha), b\delta]\theta d(\mathscr{F}(\alpha)) - \delta\theta[\alpha, \mathscr{F}(\alpha)] = 0. \tag{54}$$

Substituting r by θ for all $\theta \in \mathscr{R}$ in (51) and then we use it in (54), we get

$$b\delta[\mathscr{F}(\alpha), \theta d(\mathscr{F}(\alpha))] + [\mathscr{F}(\alpha), b\delta]\theta d(\mathscr{F}(\alpha)) - \delta\theta[\alpha, \mathscr{F}(\alpha)] = 0 \text{ for all } \alpha, \delta, \theta \in \mathscr{R}.$$

Above relation yields

$$[\mathscr{F}(\alpha), b\delta\theta d(\mathscr{F}(\alpha))] - \delta\theta[\alpha, \mathscr{F}(\alpha)] = 0 \text{ for all } \alpha, \delta, \theta \in \mathscr{R}. \tag{55}$$

Using the property of Martindale right symmetric ring of quotients, we replace δ by $b\delta$ in (55), we can obtain

$$b([\mathscr{F}(\alpha), b\delta\theta d(\mathscr{F}(\alpha))] - \delta\theta[\alpha, \mathscr{F}(\alpha)]) + [\mathscr{F}(\alpha), b]b\delta\theta d(\mathscr{F}(\alpha)) = 0 \text{ for all } \alpha, \delta, \theta \in \mathscr{R}.$$

Using (55) in our last equation, we get

$$[\mathscr{F}(\alpha), b]b\delta\theta d(\mathscr{F}(\alpha)) = 0 \text{ for all } \alpha, \delta, \theta \in \mathscr{R}. \tag{56}$$

This implies that

$$[\mathscr{F}(\alpha), b]b\delta\mathscr{R}d(\mathscr{F}(\alpha)) = (0) \text{ for all } \alpha, \delta \in \mathscr{R}.$$

Since $\mathscr{R}$ is a prime ring, for each $\alpha \in \mathscr{R}$ the above relation yields $[\mathscr{F}(\alpha), b]b\delta = 0$ or $d(\mathscr{F}(\alpha)) = 0$. Let us set $\mathscr{H} = \{\alpha \in \mathscr{R} \mid [\mathscr{F}(\alpha), b]b\delta = 0\}$ and $\mathscr{I} = \{\alpha \in \mathscr{R} \mid d(\mathscr{F}(\alpha)) = 0\}$. Clearly, $\mathscr{H}$ and $\mathscr{I}$ are additive subgroups of $\mathscr{R}$ such that $\mathscr{R} = \mathscr{H} \cup \mathscr{I}$, then either $\mathscr{R} = \mathscr{H}$ or $\mathscr{R} = \mathscr{I}$. Assume that $\mathscr{R} = \mathscr{I}$, then $d(\mathscr{F}(\alpha)) = 0$ and $[\mathscr{F}(\alpha), b]b\delta \neq 0$ for all $\alpha, \delta, \in \mathscr{R}$. Then, in view of (52), we have $-\delta[\alpha, \mathscr{F}(\alpha)] = 0$ this implies that $\delta[\mathscr{F}(\alpha), \alpha] = 0$, for all $\alpha, \delta \in \mathscr{R}$. Again, by using the similar argument as presented previously, we get

$$[\mathscr{F}(\alpha), \alpha] = 0 \text{ for all } \alpha \in \mathscr{R}, \tag{57}$$

for all $\alpha \in \mathscr{R}$. Substituting r by $\mathscr{F}(\alpha)\theta$ in (51), we have

$$\mathscr{F}(\delta)[\mathscr{F}(\alpha), \mathscr{F}(\alpha)\theta] + b\delta[\mathscr{F}(\alpha), d(\mathscr{F}(\alpha)\theta)] + [\mathscr{F}(\alpha), b\delta]d(\mathscr{F}(\alpha)\theta) - \delta[\alpha, \mathscr{F}(\alpha)\theta)] = 0 \text{ for all } \alpha, \delta, \theta \in \mathscr{R}. \tag{58}$$

This implies that,

$$\begin{aligned} &\mathscr{F}(\delta)\mathscr{F}(\alpha)[\mathscr{F}(\alpha), \theta] + b\delta d(\mathscr{F}(\alpha))[\mathscr{F}(\alpha), \theta] + b\delta[\mathscr{F}(\alpha), d(\mathscr{F}(\alpha))]\theta \\ + \; & b\delta\mathscr{F}(\alpha)[\mathscr{F}(\alpha), d(\theta)] + [\mathscr{F}(\alpha), b\delta]d(\mathscr{F}(\alpha))\theta + [\mathscr{F}(\alpha), b\delta]\mathscr{F}(\alpha)d(\theta) \\ - \; & \delta\mathscr{F}(\alpha)[\alpha, \theta] - \delta[\alpha, \mathscr{F}(\alpha)]\theta = 0 \text{ for all } \alpha, \delta, \theta \in \mathscr{R}. \end{aligned} \tag{59}$$

Multiplying (52) by θ from the right side and applying it in (59), we see that

$$\mathscr{F}(\delta)\mathscr{F}(\alpha)[\mathscr{F}(\alpha), \theta] + b\delta d(\mathscr{F}(\alpha))[\mathscr{F}(\alpha), \theta] + b\delta\mathscr{F}(\alpha)[\mathscr{F}(\alpha), d(\theta)] + [\mathscr{F}(\alpha), b\delta]\mathscr{F}(\alpha)d(\theta) - \delta\mathscr{F}(\alpha)[\alpha, \theta] = 0 \text{ for all } \alpha, \delta, \theta \in \mathscr{R}. \tag{60}$$

On replacing θ by $\theta\alpha$ in (60), we can obtain

$$\begin{aligned} &\mathscr{F}(\delta)\mathscr{F}(\alpha)\theta[\mathscr{F}(\alpha), \alpha] \\ + \; & \mathscr{F}(\delta)\mathscr{F}(\alpha)[\mathscr{F}(\alpha), \theta]\alpha + b\delta d(\mathscr{F}(\alpha))\theta[\mathscr{F}(\alpha), \alpha] \\ + \; & b\delta d(\mathscr{F}(\alpha))[\mathscr{F}(\alpha), \theta]\alpha + b\delta\mathscr{F}(\alpha)[\mathscr{F}(\alpha), d(\theta)\alpha] \\ + \; & b\delta\mathscr{F}(\alpha)[\mathscr{F}(\alpha), \theta d(\alpha)] + [\mathscr{F}(\alpha), b\delta]\mathscr{F}(\alpha)d(\theta)\alpha \\ + \; & [\mathscr{F}(\alpha), b\delta]\mathscr{F}(\alpha)\theta d(\alpha) - \delta\mathscr{F}(\alpha)[\alpha, \theta]\alpha = 0 \text{ for all } \alpha, \delta, \theta \in \mathscr{R}. \end{aligned} \tag{61}$$

In view of (57), (61) reduces to

$$\begin{aligned} & \mathscr{F}(\delta)\mathscr{F}(\alpha)[\mathscr{F}(\alpha),\theta]\alpha \\ + \;& b\delta d(\mathscr{F}(\alpha))[\mathscr{F}(\alpha),\theta]\alpha + b\delta\mathscr{F}(\alpha)[\mathscr{F}(\alpha),d(\theta)]\alpha \\ + \;& b\delta\mathscr{F}(\alpha)[\mathscr{F}(\alpha),\theta d(\alpha)] + [\mathscr{F}(\alpha),b\delta]\mathscr{F}(\alpha)d(\theta)\alpha \\ + \;& [\mathscr{F}(\alpha),b\delta]\mathscr{F}(\alpha)\theta d(\alpha) - \delta\mathscr{F}(\alpha)[\alpha,\theta]\alpha = 0 \text{ for all } \alpha,\delta,\theta\in\mathscr{R}. \end{aligned} \tag{62}$$

This implies that

$$\begin{aligned} & (\mathscr{F}(\delta)\mathscr{F}(\alpha)[\mathscr{F}(\alpha),\theta] + b\delta d(\mathscr{F}(\alpha))[\mathscr{F}(\alpha),\theta] + b\delta\mathscr{F}(\alpha)[\mathscr{F}(\alpha),d(\theta)] \\ + \;& [\mathscr{F}(\alpha),b\delta]\mathscr{F}(\alpha)d(\theta) - \delta\mathscr{F}(\alpha)[\alpha,\theta])\alpha + b\delta\mathscr{F}(\alpha)[\mathscr{F}(\alpha),\theta d(\alpha)] \\ & +[\mathscr{F}(\alpha),b\delta]\mathscr{F}(\alpha)\theta d(\alpha) = 0 \text{ for all } \alpha,\delta,\theta\in\mathscr{R}. \end{aligned} \tag{63}$$

In view of (60), (63) reduces to

$$b\delta\mathscr{F}(\alpha)[\mathscr{F}(\alpha),\theta d(\alpha)] + [\mathscr{F}(\alpha),b\delta]\mathscr{F}(\alpha)\theta d(\alpha) = 0 \text{ for all } \alpha,\delta,\theta\in\mathscr{R}. \tag{64}$$

for all $\alpha,\delta,\theta\in\mathscr{R}$ and further we can see that

$$\begin{aligned} & b\delta\mathscr{F}(\alpha)\mathscr{F}(\alpha)\theta d(\alpha) - b\delta\mathscr{F}(\alpha)\theta d(\alpha)\mathscr{F}(\alpha) \\ + \;& \mathscr{F}(\alpha)b\delta\mathscr{F}(\alpha)\theta d(\alpha) - b\delta\mathscr{F}(\alpha)\mathscr{F}(\alpha)\theta d(\alpha) = 0. \end{aligned}$$

This gives

$$[\mathscr{F}(\alpha), b\delta\mathscr{F}(\alpha)\theta d(\alpha)] = 0 \text{ for all } \alpha,\delta,\theta\in\mathscr{R}. \tag{65}$$

Substitute δ by $b\delta$ in (65), we have

$$b[\mathscr{F}(\alpha), b\delta\mathscr{F}(\alpha)\theta d(\alpha)] + [\mathscr{F}(\alpha),b]b\delta\mathscr{F}(\alpha)\theta d(\alpha) = 0 \text{ for all } \alpha,\delta,\theta\in\mathscr{R}. \tag{66}$$

Using (65) in (66), we get

$$[\mathscr{F}(\alpha),b]b\delta\mathscr{F}(\alpha)\theta d(\alpha) = 0 \text{ for all } \alpha,\delta,\theta\in\mathscr{R}. \tag{67}$$

for all $\alpha,\delta,\theta\in\mathscr{R}$. On replacing δ by δr in (67), one can see that

$$[\mathscr{F}(\alpha),b]b\delta r\mathscr{F}(\alpha)\theta d(\alpha) = 0, \tag{68}$$

for all $\alpha,\delta,\theta\in\mathscr{R}$. This implies that $[\mathscr{F}(\alpha),b]b\delta\mathscr{R}\mathscr{F}(\alpha)\theta d(\alpha) = (0)$. Since $\mathscr{R}$ is prime, therefore either $[\mathscr{F}(\alpha),b]b\delta = 0$ or $\mathscr{F}(\alpha)\theta d(\alpha) = 0$ as we have discussed earlier, for each $\alpha,\delta,\theta\in\mathscr{R}$. If $[\mathscr{F}(\alpha),b]b\delta = 0$, then we get a contradiction, so we have $\mathscr{F}(\alpha)\theta d(\alpha) = 0$. Using the primeness of $\mathscr{R}$, we can see that either $\mathscr{F}(\alpha) = 0$ or $d(\alpha) = 0$ for all $\alpha\in\mathscr{R}$ which implies that $\mathscr{F}(\mathscr{R}) = (0)$ or $d(\mathscr{R}) = (0)$. If $\mathscr{F}(\mathscr{R}) = (0)$, then by using our hypothesis we have $-[\alpha,\delta] = 0$, i.e. $[\alpha,\delta] = 0$. So, $\mathscr{R}$ is commutative. Next, if $d(\mathscr{R}) = (0)$, then by hypothesis we get contradiction.

(ii) Next, we consider $\mathscr{R} = \mathscr{H}$, then we have $[\mathscr{F}(\alpha),b]b\delta = 0$ and $d(\mathscr{F}(\alpha)) \neq 0$. On replacing δ and by $r\delta$ for all $r\in\mathscr{R}$ and using the primeness of $\mathscr{R}$, we get either $\delta = 0$ or $[\mathscr{F}(\alpha),b]b = 0$. Since $\mathscr{R}\neq(0)$, so δ should not be 0, which implies that we have only possibility is $[\mathscr{F}(\alpha),b]b = 0$ for all $\alpha\in\mathscr{R}$. Which implies that $[\mathscr{F}(\mathscr{R}),b]b = 0$. □

Example 3.7. Let $\mathscr{S}$ be any ring with characteristic 2, now we define a ring $\mathscr{R}$ given by $\mathscr{R} = \left\{\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \middle| x,y,z\in\mathscr{S}\right\}$. For any $0\neq y,z\in\mathscr{S}$, $\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\mathscr{R}\begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix} = (0)$, then $\mathscr{R}$ is not a prime ring. Now we define a map $\mathscr{F}$ and d from $\mathscr{R}\to\mathscr{R}$, such that $\mathscr{F}\left(\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}\right) = \begin{bmatrix} x & -y \\ 0 & z \end{bmatrix}$ and $d\left(\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}\right) = \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}$, if y is nonzero then $d(\mathscr{R}) \neq 0$. Now for $b = \begin{bmatrix} 0 & m \\ 0 & n \end{bmatrix}$ for all $m, n \in \mathscr{S}$, then it is straightforward to verify that $\mathscr{F}$ is b-generalized derivation associated with a derivation d and $[\mathscr{F}(\alpha), \mathscr{F}(\beta)] = [\alpha, \beta]$ for all $\alpha, \beta \in \mathscr{R}$ but neither $\mathscr{R}$ is commutative nor $[\mathscr{F}(\mathscr{R}), b]b = 0$. So, in this example we see that the primeness of hypothesis is essential in Theorem 3.6.

Example 3.8. Let $\mathscr{R} = \left\{ \begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix} \middle| x, y, z \in \mathbb{Z} \right\}$, where $\mathbb{Z}$ is the set of integers and $\mathscr{F}$ and d is a map from $\mathscr{R} \to \mathscr{R}$ such that $\mathscr{F}\left(\begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & yz \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $d\left(\begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then it is easy to verify that $\mathscr{F}$ is a multiplicative b-generalized derivation associated with derivation d and for any fixed $0 \neq b \in \mathscr{R}$. Let $\mathscr{K} = \left\{ \begin{bmatrix} 0 & 0 & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix} \middle| y, z \in \mathbb{Z} \right\}$. Here we see that $\mathscr{K}$ is a dense ideal of $\mathscr{R}$ and satisfies the following conditions; (i) $[\mathscr{F}(\alpha), \delta] \in \mathscr{Z}(\mathscr{R})$, (ii) $[\mathscr{F}(\alpha), \delta] = 0$, (iii) $\mathscr{F}(\alpha\delta) - \alpha\delta \in \mathscr{Z}(\mathscr{R})$, (iv) $\mathscr{F}(\alpha\delta) - \alpha\delta = 0$ and (v) $\mathscr{F}([\alpha, \delta]) = 0$ for all $\alpha, \delta \in \mathscr{K}$, but $\mathscr{R}$ is non-commutative. Hence primeness of hypothesis is essential in Theorems 3.1, 3.3, 3.5 and corollary 3.2 and 3.4.

Question 3.9. In Theorems 3.1, 3.3, 3.5 and 3.6, If we consider semiprime rings instead of prime rings, then what can we say about the validity of those results?

Question 3.10. In Theorem 3.6, If we consider multiplicative b-generalized derivation instead of b-generalized derivation, then what can we say about the existence of the result?

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