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THE LOWER BOUNDS FOR THE FIRST EIGENVALUES OF THE (p, q)-LAPLACIAN ON FINSLER MANIFOLDS

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Abstract. In this paper, we study the nonlinear eigenvalue problem for some of the (p, q)-Laplacian on compact Finsler manifolds with zero boundary condition, and estimate the lower bound of the first eigenvalues for (p, q)-Laplace operators on Finsler manifolds.

1. Introduction

Finsler geometry is a natural generalization of Riemannian geometry that has no quadratic restriction. Since more than twenty years ago, substantial progress has been made in Finsler geometry and has developed rapidly in its global aspects, especially in the study of the Finsler Laplacian and it has been seen that methods in Finsler geometry are closely related to other mathematical branches such as Lie groups, nonlinear analysis and have so many applications to mathematical physics, theoretical physics, and many other fields. All in all, Finsler geometry has a broader applications in natural science.

There are several definitions of Finsler Laplacian, including nonlinear Laplacian, mean-value Laplacian, and so on. Since the investigation about the first eigenvalue of Laplace in Finsler manifolds plays an important role in Finsler geometry; (for instance see [27] for a characterization of the structure of the manifold by finding it's isometric spaces), heretofore so many eigenvalue comparison theorems such as Faber-Kahn type inequality, Cheng type inequality, Cheeger type inequality, and Mckean type inequality have been established [10, 13, 19, 21]. Afterward, Yin and He in [22, 23] improved further results. Up to now, there has been some progress on the Finsler p-Laplacian which was stated in the early 2010s by Belloni et al. [6, 7]. They worked on the p-Laplace eigenvalue problem as $p \to \infty$. After that, Kawohl and Novaga studied a special p-Laplacian in a reversible Minkowski space as $p \to 1$. We refer to [14, 24, 25, 26] for more information about the first eigenvalue of Finsler p-Laplacian.

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2. Preliminaries

Let M be an n-dimensional smooth manifold and $\pi : TM \to M$ be the natural projection from the tangent bundle TM. Let (x, y) be a point of TM with $x \in M$, $y \in T_xM$, and let (x^i, y^i) be the local coordinate on TM with $y = y^i \dfrac{\partial}{\partial x^i}$. A Finsler metric on M is a function $F : TM \to [0, \infty)$ satisfying the following properties:

(i) Regularity: F is C^∞ on the entire slit tangent bundle $TM \setminus \{0\}$.

(ii) Positive homogeneity: $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$.

(iii) Strong convexity: The $n \times n$ Hessian matrix

$$(g_{ij}) := \left([\frac{1}{2}F^2]_{y^i y^j} \right)$$

is positive definite at every point of $TM \setminus \{0\}$. Let $V = V^i \dfrac{\partial}{\partial x^i}$ be a non-vanishing vector field on an open subset $\mathcal{U} \subset M$. One can introduce a Riemannian metric $\tilde{g} = g_V$ on the tangent bundle over $\mathcal{U}$ as follows:

$$g_V(X, Y) := X^i Y^j g_{ij}(x, v), \quad \forall X = X^i \frac{\partial}{\partial x^i}, \, Y = Y^i \frac{\partial}{\partial x^i}.$$

Let ∇^V be the Chern connection, and then, the Chern curvature $\mathbf{R}^V(X, Y)Z$ for vector fields X, Y, Z on $\mathcal{U}$ is defined by:

$$\mathbf{R}^V(X, Y)Z := \nabla^V_X \nabla^V_Y Z - \nabla^V_Y \nabla^V_X Z - \nabla^V_{[X,Y]} Z.$$

For a flag (V, W) consisting of non-zero tangent vectors $V, W \in T_xM$ and a 2-plane $P \subset T_xM$ with $V \in P$ the flag curvature $K(V, W)$ is defined as follows:

$$K(V, W) := \frac{g_V(R^V(V, W)W, V)}{g_V(V, V)g_V(W, W) - g_V(V, W)^2},$$

where, W is a tangent vector such that V, W span the 2-plane P and $V \in T_xM$ is extended to a geodesic field, i.e., $\nabla^V_V V = 0$ near x. The Ricci curvature of V is defined as:

$$Ric(V) = \sum_{i=1}^{n-1} K(V, e_i),$$

where $e_1, \cdots, e_{n-1}, \dfrac{V}{F(V)}$ form an orthonormal basis of T_xM with respect to g_V, namely, one has $\mathrm{Ric}(\lambda V) = \mathrm{Ric}(V)$ for any $\lambda > 0$.

The reversible function $\lambda : M \longrightarrow \mathbb{R}$ is defined by:

$$\lambda(x) = \max_{y \in T_xM \setminus 0} \frac{F(y)}{F(-y)}.$$

It is clear that $1 \le \lambda(x) < +\infty$ for any $x \in M$. Here $\lambda_F = \sup_{x \in M} \lambda(x)$ is called the reversibility of (M, F), and (M, F) is called reversible if $\lambda_F = 1$.

The gradient vector field of a differentiable function f on M by the Legendre transformation $\mathcal{L} : T_xM \to T_x^*M$ is defined as

$$\nabla f := \mathcal{L}^{-1}(df).$$

Let $\mathcal{U} = \{x \in M : \nabla f|_x \neq 0\}$. We define the Hessian $H(f)$ of f on $\mathcal{U}$ as follows:

$$H(f)(X,Y) := XY(f) - \nabla_X^{\nabla f} Y(f), \quad \forall X, Y \in \Gamma(TM|_{\mathrm{u}}).$$

For a given volume form $d\mu = \sigma(x)dx$ and a vector $V \in T_xM \setminus \{0\}$, the distortion of M is defined by

$$\tau(V) := \ln \frac{\sqrt{\det(g_{ij}(V))}}{\sigma}.$$

Considering the rate of changes of the distortion along geodesics leads to the so called S-curvature as follows:

$$S(V) := \frac{d}{dt}[\tau(\gamma(t), \dot{\gamma}(t)]_{t=0},$$

where $\gamma(t)$ is the geodesic with $\gamma(0) = x$ and $\dot{\gamma}(0) = V$. Define

$$\dot{S}(V) := F^{-2}(V)\frac{d}{dt}[S(\gamma(t), \dot{\gamma}(t)]_{t=0}.$$

Fix a volume form $d\mu = \sigma(x)dx$, the divergence $\operatorname{div}(X)$ of a smooth vector field X is defined as:

$$\operatorname{div}(X) := \Sigma_{i=1}^n \left(\frac{\partial X^i}{\partial x^i} + X^i \frac{\partial \log \sigma}{\partial x^i} \right).$$

For a given smooth function $f : M \longrightarrow \mathbb{R}$, the Laplacian Δf of f is defined by $\Delta f = \operatorname{div}(\nabla f) = \operatorname{div}(\mathcal{L}^{-1}(df))$.

2.1. Eigenvalue of (p, q)-Laplacian

The Finsler p-Laplacian of a smooth function $f : M \to \mathbb{R}$ can be defined by

$$\Delta_p f := \operatorname{div}(|\nabla f|^{p-2}\nabla f).$$

Since the gradient operator ∇ is not a linear operator in general, the Finsler p-Laplacian is greatly different from the Riemannian p-Laplacian.

Given a vector field V such that $V \neq 0$ on $M_u = \{x \in M; du(x) \neq 0\}$, the weighted gradient vector and the weighted p-Laplacian on the weighted Riemannian manifold (M, g_V) are defined by

$$\triangledown^V f := \begin{cases} g^{ij}(V)\dfrac{\partial f}{\partial x^j}\dfrac{\partial}{\partial x^i}, & on\ M_u, \\ 0, & on\ M \setminus M_u, \end{cases} \qquad \Delta_p^V f := div(|\nabla^V f|^{p-2}\nabla^V f).$$

Here we note that $\nabla f = \nabla^V f$, $\Delta_p f = \Delta_p^V f$.
In this paper, we introduce the following nonlinear system in Finsler manifolds

which had been introduced in [5] for the Riemannian case:

$$(1) \qquad \begin{cases} \Delta_p^V u = -\lambda |u|^\alpha |v|^\beta u \quad in M \\ \Delta_q^V v = -\lambda |u|^\alpha |v|^\beta v \quad in M \\ (u,v) \in W^{1,p}(M) \times W^{1,q}(M), \end{cases}$$

where $p, q > 1$ and α, β are numbers satisfying

$$\alpha > 0, \ \beta > 0, \ \frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1.$$

Here, we say that λ is the eigenvalue of (1), when for some $u \in W_0^{1,p}(M)$ and $v \in W_0^{1,q}(M)$, we have

$$\int_M (F^*(du))^{p-2} g_V(\nabla u, \nabla \phi) d\mu = \lambda \int_M |u|^\alpha |v|^\beta v \phi d\mu,$$

and

$$\int_M (F^*(dv))^{q-2} g_V(\nabla v, \nabla \psi) d\mu = \lambda \int_M |u|^\alpha |v|^\beta u \psi d\mu,$$

where $\phi \in W^{1,p}(M)$, $\psi \in W^{1,q}(M)$ and $W_0^{1,p}(M)$ is the closure of $C_0^\infty(M)$ in Sobolev space $W^{1,p}(M)$. Here (u,v) is called eigenfunctions. A first positive eigenvalue of (1) depends on fixed vector field V considered as follows

$$\lambda_{1,p,q} = \inf\{A(u,v) : (u,v) \in W_0^{1,p}(M) \times W_0^{1,q}(M), B(u,v) = 1\},$$

where

$$\begin{aligned} A(u,v) &= \frac{\alpha+1}{p} \int_M (F^*(du))^p d\mu + \frac{\beta+1}{q} \int_M (F^*(dv))^q d\mu, \\ B(u,v) &= \int_M |u|^\alpha |v|^\beta uv d\mu. \end{aligned}$$

Let (M,F) be an n-dimensional Finsler space and $\{b_i\}_{i=1}^n$ be an arbitrary basis for T_xM. Let

$$B^n := \{(y^i) \in \mathbb{R}^n, F(\Sigma_{i=1}^n y^i b_i) < 1\},$$

and

$$B^{n-1} := \{(y^a) \in \mathbb{R}^{n-1}, F(\Sigma_{a=2}^n y^a b_a) < 1\}.$$

Both B^n and B^{n-1} depend on the choice of $\{b_a\}_{a=2}^n$. Define

$$\zeta(y) := \frac{Vol(B^n)}{Vol(B^{n-1})} . \frac{Vol(B^{n-1})}{F(y) Vol(B^n)}.$$

The function ζ is independent of the choice of basis. Now, we could state the following co-area formula from [18].

Theorem 2.1. *Let (M,F) be a Finsler space and N be a hypersurface in M. Let φ be a piecewise C^1 function on M such that every $\varphi^{-1}(t)$ is compact. Then for any continuous function f on M, we have*

$$(2) \qquad \int_M f F^*(d\varphi) d\mu = \int_{-\infty}^{\infty} \left(\int_{\varphi^{-1}(t)} f dv \right) dt.$$

Where $dv := \varphi dA_F$, is called the induced volume form by $d\mu$ with respect to the normal vector field n along N, $dA_F := \zeta(n)d\mu$.

Also we may need the Hölder inequality

$$\|fg\|_1 \leq \|f\|_p \|g\|_q, \tag{3}$$

for measurable functions f, g and $p, q \in [1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

3. Main results

There are so many papers that investigate properties of the spectrum of Laplacian and estimate other geometric quantities aspects of the Riemannian manifold M (see for example [1], [11], [12], [16], [20]). The existence, simplicity, stability and some other properties of the first eigenvalue of (p, q)-Laplacian have been studied on Riemannian manifolds in [2, 3, 8, 15, 17]. In this paper, we want to study the nonlinear eigenvalue problem for some of the (p, q)-Laplacian on compact Finsler manifolds with zero boundary condition that had been studied for the Riemannian manifolds in [5]. In this section we prove our main results as follows:

Theorem 3.1. *Let $(M, F, d\mu)$ be a complete, reversible and simply connected Finsler manifold with constant flag curvature K. Take $B_r(x)$ as a geodesic ball of radius r, centered at x, such that $Vol(M) = Vol(B_r(x))$. Then the following estimate holds for the first eigenvalue of (1)*

$$\lambda_{1,p,q}(M) \geq \lambda_{1,p,q}(B_r(x)),$$

the equality holds if and only if $M = B_r(x)$.

Proof. Suppose that (u, v) be the pair of positive eigenfunctions related to $\lambda_{1,p,q}(M)$. Let $M_t = \{x \in M : u(x) > t\}$ and $\Gamma_t = \{x \in M : u(x) = t\}$. We construct geodesic balls B_t such that $Vol(B_t) = Vol(M_t)$ for each t, and $B_r := B_0$. We define a radially decreasing function $u^* : B_r \to \mathbb{R}^+$ (for all $\rho \in So(n)$, $u^* \circ \rho = u^*$) and $\Gamma_t^* = \{x \in B_r : u^*(x) = t\}$. We denote the $(n-1)$-dimensional volume element of Γ_t and Γ_t^* by $d\Gamma_t$ and $d\Gamma_t^*$ respectively. Here we have the following identities for the volume element $d\Gamma_t$ and $d\Gamma_t^*$:

$$\frac{d}{dt}Vol(M_t) = -\int_{\Gamma_t} \frac{d\Gamma_t}{F^*(du)}, \quad \frac{d}{dt}Vol(B_t) = -\int_{\Gamma_t^*} \frac{d\Gamma_t^*}{F^*(du^*)}. \tag{4}$$

Hence, by the Co-area formula (2), we have

$$\begin{aligned}
\int_M u^p d\mu &= \int_0^\infty \int_{\Gamma_t} \frac{u^p}{F^*(du)} d\Gamma_t dt = \int_0^\infty t^p \frac{d}{dt} \left(\int_{\Gamma_t} \frac{d\Gamma_t}{F^*(du)} \right) dt \\
&= -\int_0^\infty t^p \frac{d}{dt} Vol(M_t) dt = -\int_0^\infty t^p \frac{d}{dt} Vol(B_t) dt \\
&= \int_0^\infty t^p \left(\int_{\Gamma_t^*} \frac{d\Gamma_t^*}{F^*(du^*)} \right) dt = \int_{B_r} (u^*)^p d\mu.
\end{aligned}$$

Now, using Hölder inequality (3), we obtain

$$\begin{aligned}\int_{\Gamma_t} d\Gamma_t &= \int_{\Gamma_t} (F^*(du))^{\frac{p-1}{p}} (F^*(du))^{\frac{1-p}{p}} d\Gamma_t \\ &\leq \left(\int_{\Gamma_t} (F^*(du))^{p-1} d\Gamma_t\right)^{\frac{1}{p}} \left(\int_{\Gamma_t} (F^*(du))^{-1} d\Gamma_t\right)^{\frac{p-1}{p}}.\end{aligned}$$

Note that $B_r \subset M$, thus based on the definition of u^* it is obvious that

$$\int_{\Gamma_t} d\Gamma_t \geq \int_{\Gamma_t^*} d\Gamma_t^*. \tag{5}$$

So, using Co-area formula (2) for gradient, by (4) and (5), we have

$$\begin{aligned}\int_M (F^*(du))^p d\mu = \int_0^\infty \int_{\Gamma_t} (F^*(du))^{p-1} d\mu &\geq \int_0^\infty \frac{(\int_{\Gamma_t} d\Gamma_t)^p}{(\int_{\Gamma_t} F^*(du)^{-1} d\Gamma_t)^{p-1}} dt \\ &\geq \int_0^\infty \frac{(\int_{\Gamma_t^*} d\Gamma_t^*)^p}{(\int_{\Gamma_t^*} F^*(du^*)^{-1} d\Gamma_t^*)^{p-1}} dt \\ &= \int_0^\infty \left(\int_{\Gamma_t^*} (F^*(du^*))^{p-1} d\Gamma_t^*\right) dt \\ &= \int_{B_r} (F^*(du^*))^p d\mu.\end{aligned}$$

Here in the third line we have used the following equation from [26]:

$$\left(\int_{\Gamma_t^*} d\Gamma_t^*\right)^p = \left(\int_{\Gamma_t^*} F^*(du^*)^{-1} d\Gamma_t^*\right)^{p-1} \left(\int_{\Gamma_t^*} F^*(du^*)^{p-1}\right). \tag{6}$$

The same way leads to

$$\int_M (F^*(dv))^q d\mu \geq \int_{B_r} (F^*(dv^*))^q d\mu,$$

where v^* defined like u^*. Therefore

$$\begin{aligned}\lambda_{1,p,q}(M) &= \frac{\alpha+1}{p} \int_M (F^*(du))^p d\mu + \frac{\beta+1}{q} \int_M (F^*(dv))^q d\mu \\ &\geq \frac{\alpha+1}{p} \int_{B_r} (F^*(du^*))^p d\mu + \frac{\beta+1}{q} \int_{B_r} (F^*(dv^*))^q d\mu \\ &= \lambda_{1,p,q}(B_r).\end{aligned}$$

□

For the next result, we need to recall the Cheeger constant $h(\mathcal{M})$ from [27] which defines as follows:

$$h(\mathcal{M}) := \inf_{\mathcal{M}'} \frac{(\min Vol_+(\partial \mathcal{M}'), Vol_-(\partial \mathcal{M}'))}{Vol(\mathcal{M}')}.$$

Here $\mathcal{M}^{'}$ ranges over all open submanifold of $\mathcal{M}$ with compact closure in $\mathcal{M}$ and smooth boundary $\partial\mathcal{M}^{'}$, $vol(\mathcal{M}^{'})$ denotes the volume of $\mathcal{M}^{'}$, $vol_{-}(\partial\mathcal{M}^{'})$ and $vol_{+}(\partial\mathcal{M}^{'})$ denote the volume of $\partial\mathcal{M}^{'}$ with respect to outward and inward normal vector respectively.

Theorem 3.2. *Let $(M, F, d\mu)$ be a complete (forward or backward) Finsler n-manifold. For any bounded domain $\mathcal{M}$ with piecewise smooth boundary in M, the first eigenvalue of (1) satisfies:*

$$\lambda_{1,p,q}(\mathcal{M}) \geq \frac{\alpha+1}{p}\left(\frac{h(\mathcal{M})}{p}\right)^p \int_{\mathcal{M}} |u|^p d\mu + \frac{\beta+1}{q}\left(\frac{h(\mathcal{M})}{q}\right)^q \int_{\mathcal{M}} |v|^q d\mu,$$

where (u, v) be the pair of positive eigenfunctions corresponding to $\lambda_{1,p,q}(\mathcal{M})$.

Proof. First we need to take some notations just like the last theorem. Suppose that (u, v) be the pair of positive eigenfunctions related to $\lambda_{1,p,q}(\mathcal{M})$. Let $\mathcal{M}_t = \{x \in \mathcal{M} : u(x) > t\}$ and $\Gamma_t = \{x \in \mathcal{M} : u(x) = t\}$. We construct geodesic balls B_t such that $Vol(B_t) = Vol(\mathcal{M}_t)$ for each t, and $B_r := B_0$. Here $d\Gamma_t$ denotes the $(n-1)$-dimensional volume element of Γ_t. Using co-area formula for $\varphi \in C^1(\mathcal{M})$, $(\varphi|_{\mathcal{M}} > 0, \varphi|_{\partial\mathcal{M}} = 0)$, we obtain

$$\begin{aligned}
\int_{\mathcal{M}} F^*(d\varphi)d\mu &= \int_{-\infty}^{\infty}\left(\int_{\Gamma_t} d\Gamma_t\right)dt \\
&= \int_{-\infty}^{\infty} Vol(\partial\mathcal{M}_t)dt \\
&= \int_{-\infty}^{\infty} \frac{Vol(\partial\mathcal{M}_t)}{Vol(\mathcal{M}_t)} Vol(\mathcal{M}_t)dt \\
&\geq \inf_t \left(\frac{Vol(\partial\mathcal{M}_t)}{Vol(\mathcal{M}_t)}\right)\int_{-\infty}^{\infty} Vol(\mathcal{M}_t)dt \\
&\geq h(\mathcal{M})\int_{\mathcal{M}} \varphi d\mu.
\end{aligned}$$

Suppose $\varphi = u^p$, by Hölder inequality, we get

$$\begin{aligned}
h(\mathcal{M})\int_{\mathcal{M}} u^p d\mu &\leq \int_{\mathcal{M}} (F^*(du))^p d\mu \\
&= p\int_{\mathcal{M}} u^{p-1}F^*(du)d\mu \\
&\leq p\left(\int_{\mathcal{M}} |u^p| d\mu\right)^{\frac{p-1}{p}}\left(\int_{\mathcal{M}} F^*(du)^p d\mu\right)^{\frac{1}{p}}.
\end{aligned}$$

So

$$\int_{\mathcal{M}} F^*(du)^p d\mu \geq \left(\frac{h(\mathcal{M})}{p}\right)^p \int_{\mathcal{M}} |u|^p d\mu.$$

Similarly

$$\int_{\mathcal{M}} F^*(dv)^q d\mu \geq \left(\frac{h(\mathcal{M})}{q}\right)^q \int_{\mathcal{M}} |v|^q d\mu.$$

This completes the proof. □

In the following, we remind another class of (p,q)-Laplacian which defines in the same way for finsler manifold like Riemannian manifolds [8]:

$$\Delta_p^{\nabla u} u + \Delta_q^{\nabla u} u = \operatorname{div}(((F^*(du))^{p-2} + (F^*(du))^{q-2})\nabla u), \tag{7}$$

where $u \in W = W_0^{1,p}(M) \cap W_0^{1,q}(M)$ and $1 < q < p < \infty$. We call $\lambda \in \mathbb{R}$ as an eigenvalue of (7), if there is $u \in W$, $u \neq 0$ such that $-\Delta_p u - \Delta_q u = \lambda |u|^{p-2} u$ or equivalently

$$\int_M (F^*(du))^{p-2} g_V(\nabla u, \nabla v) d\mu + \int_M (F^*(du))^{q-2} g_V(\nabla u, \nabla v) d\mu$$
$$= \lambda \int_M |u|^{p-2} uv d\mu,$$

for any $v \in W^{1,p}(M) \cap W^{1,q}(M)$.

The first positive eigenvalue $\lambda_{1,p,q}(M)$ of (7) is obtained as follows:

$$\lambda_{1,p,q}(M) = \inf \left\{ \int_M (F^*(du))^p d\mu + \int_M (F^*(du))^q d\mu : \int_M |u|^p d\mu = 1 \right\}. \tag{8}$$

Immediately, we could prove the next result just like Theorem 3.1. Therefore the proof is omitted.

Corollary 3.3. *Let $\mathcal{M}$ be a domain in a complete, simply connected Finsler manifold M of constant flag curvature and $B_r(x)$ be a geodesic ball of radius r in M such that $Vol(\mathcal{M}) = Vol(B_r(x))$. Then for the first eigenvalue of $\mathcal{M}$ and $B_r(x)$ depends on (8), we have*

$$\lambda_{1,p,q}(\mathcal{M}) \geq \lambda_{1,p,q}(B_r(x)).$$

The equality holds if and only if $\mathcal{M} = B_r(x)$.

Theorem 3.4. *Let $\mathcal{M}$ be the compact manifold with smooth boundary in a complete Finsler manifold. Then for (8), considering (u,v) as the pair of eigenfunctions corresponding to $\lambda_{1,p,q}(\mathcal{M})$, we have*

$$\lambda_{1,p,q}(\mathcal{M}) \geq \left(\frac{h(\mathcal{M})}{p}\right)^p + \left(\frac{h(\mathcal{M})}{q}\right)^q.$$

Proof. From the Theorem 3.2 for eigenfunctions $(u,v) > 0$ corresponding to $\lambda_{1,p,q}(\mathcal{M})$, we have

$$\int_{\mathcal{M}} (F^*(du))^p d\mu \geq \left(\frac{h(\mathcal{M})}{p}\right)^p \int_{\mathcal{M}} u^p d\mu \geq \left(\frac{h(\mathcal{M})}{p}\right)^p,$$

and

$$\int_{\mathcal{M}} (F^*(dv))^q d\mu \geq \left(\frac{h(\mathcal{M})}{q}\right)^q \int_{\mathcal{M}} v^q d\mu \geq \left(\frac{h(\mathcal{M})}{q}\right)^q.$$

These complete the proof. □

For a compact Finsler manifold M with negative flag curvature $K < 0$, from [4] we know that M is a Riemannian manifold. Therefore we have the following result:

Corollary 3.5. *Let M be an n-dimensional complete (forward or backward), simply connected Finsler manifold with negative constant flag curvature $-k$. Suppose $\mathcal{M}$ as a domain in M. Then for (8), the following inequality holds*

$$\lambda_{1,p,q}(\mathcal{M}) \geq \left(\frac{(n-1)\sqrt{-k}}{p}\right)^p + \left(\frac{(n-1)\sqrt{-k}}{q}\right)^q.$$

Since M is Riemannian manifold, the proof is similar to that of [5, Theorem 1.5].

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