Honam Mathematical J. **45** (2023), No. 1, pp. 130–145
https://doi.org/10.5831/HMJ.2023.45.1.130

# INTEGRAL CURVES CONNECTED WITH A FRAMED CURVE IN 3-SPACE

Mustafa Düldül* and Zeynep Bülbül

**Abstract.** In this paper, we define some integral curves connected with a framed curve in Euclidean 3-space. These curves include framed generalized principal-direction curve, framed generalized binormal-direction curve, framed principal-donor curve and framed Darboux-direction curve. We obtain some relations between the framed curvatures of new defined framed curves and framed curvatures of given framed curve. By using the obtained relationships we give some characterizations for such curves. We also give methods for constructing framed helix and framed slant helix from planar curves.

## 1. Introduction

Curves in Euclidean 3-space $\mathbb{E}^3$ have many applications since they may be considered as paths of moving objects. Among these curves, regular ones are the most studied. There exist in literature not only different types of curves but also several connected curves for such curves [6, 8, 10, 11]. Among regular curves, helical curves are most remarkable. Choi and Kim [2] introduce some connected curves of a given Frenet curve and they give some characterizations for these curves. Inspired by this work, Macit and Düldül [7] define some new connected curves of a Frenet curve in $\mathbb{E}^3$ and $\mathbb{E}^4$.

Given a regular curve $\alpha$ in $\mathbb{E}^3$ with non-vanishing curvature, i.e. $\alpha' \times \alpha'' \neq \mathbf{0}$, it is possible to define its Frenet frame and curvatures. However, a curve in $\mathbb{E}^3$ may have some singular points. Since tangent vector vanishes at singular points, we can not define its Frenet frame at such points. In 2016, as a generalization of regular curves, a new type of curve has been defined for studying the curves with singular points and such a curve has been called as framed curve [4]. In 2017, Fukunaga and Takahashi construct a framed curve such that the image of the framed base curve coincides with the image of a given smooth curve under a condition [3]. In 2019, to study rectifying curves

Received September 6, 2022. Accepted December 3, 2022.
2020 Mathematics Subject Classification. 53A04, 58K05.
Key words and phrases. framed curve, framed helix, framed slant helix, framed direction curve.
*Corresponding author

with singular points, Wang et al. [12] define framed rectifying curves and they give characterization for such curves. They also define framed helix and obtain the relationship between framed helices and framed rectifying curves. In 2020, Honda and Takahashi [5] define Bertrand and Mannheim curves of framed curves in $\mathbb{E}^3$. Recently, Okuyucu and Canbirdi [9] define framed slant helix and give its characterization.

In this paper, we define some integral curves connected with a framed curve in Euclidean 3-space. These curves include framed generalized principal-direction curve, framed generalized binormal-direction curve, framed principal-donor curve, and framed Darboux-direction curve. We obtain some relations between the framed curvatures of new defined framed curves and framed curvatures of given framed curve. By using the obtained relationships we give some characterizations for such curves. We also give methods for constructing framed helix and framed slant helix from planar curves.

## 2. Preliminaries

Let $\beta : I \subset \mathbb{R} \to \mathbb{E}^3$ be any curve. If the tangent vector of a regular curve $\beta$ is linearly independent with its curvature vector along the curve, we can frame it via its Frenet frame or any other adapted frames [1]. However, if $\beta$ has some singular points, then Frenet frame cannot be defined at those points. In this case, such a curve can be framed by the following method (see, e.g. [4, 12, 13]):

Let us consider the set

$$\Delta_2 = \{\mathbf{v} = (\mathfrak{m}_1, \mathfrak{m}_2) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \langle \mathfrak{m}_\ell, \mathfrak{m}_r \rangle = \delta_{\ell r},\ \ell, r = 1, 2\}.$$

It is clear that, if $\mathbf{v} = (\mathfrak{m}_1, \mathfrak{m}_2) \in \Delta_2$, then $\mathfrak{m} = \mathfrak{m}_1 \times \mathfrak{m}_2$ is a unit vector in $\mathbb{R}^3$.

**Definition 2.1.** *Let $\beta : I \subset \mathbb{R} \to \mathbb{E}^3$ be any curve, and $\mathbf{v} = (\mathfrak{m}_1, \mathfrak{m}_2) \in \Delta_2$. If $\langle \beta'(\ell), \mathfrak{m}_1(\ell) \rangle = 0$ and $\langle \beta'(\ell), \mathfrak{m}_2(\ell) \rangle = 0$, then $(\beta, \mathbf{v}) : I \to \mathbb{E}^3 \times \Delta_2$ is defined as a framed curve* [4].

Let us consider the framed curve $(\beta, \mathbf{v}) : I \to \mathbb{E}^3 \times \Delta_2$ and let $\mathfrak{m}(\ell) = \mathfrak{m}_1(\ell) \times \mathfrak{m}_2(\ell)$. In this case, the frame $\{\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}\}$ along $\beta$ has its Frenet-type formula:

$$\begin{cases} \mathfrak{m}_1'(\ell) = \mathfrak{r}_1(\ell)\mathfrak{m}_2(\ell) + \mathfrak{r}_2(\ell)\mathfrak{m}(\ell), \\ \mathfrak{m}_2'(\ell) = -\mathfrak{r}_1(\ell)\mathfrak{m}_1(\ell) + \mathfrak{r}_3(\ell)\mathfrak{m}(\ell), \\ \mathfrak{m}'(\ell) = -\mathfrak{r}_2(\ell)\mathfrak{m}_1(\ell) - \mathfrak{r}_3(\ell)\mathfrak{m}_2(\ell). \end{cases}$$

We also have

$$\beta'(\ell) = \mathfrak{f}(\ell)\mathfrak{m}(\ell) \tag{2.1}$$

with a smooth function $\mathfrak{f} : I \to \mathbb{R}$. The authors define the functions $\big(\mathfrak{r}_1(\ell), \mathfrak{r}_2(\ell), \mathfrak{r}_3(\ell), \mathfrak{f}(\ell)\big)$ as curvature of $\beta$ [4]. It is clear from (2.1) that $\ell_0$ is a singular point of $\beta \Leftrightarrow \mathfrak{f}(\ell_0) = 0$.

The existence theorem and uniqueness theorem of framed curves have been proved in [4].

Let us consider a framed curve $(\beta, \mathfrak{m}_1, \mathfrak{m}_2) : I \to \mathbb{E}^3 \times \Delta_2$ with its curvature $\big(\mathfrak{r}_1(\ell), \mathfrak{r}_2(\ell), \mathfrak{r}_3(\ell), \mathfrak{f}(\ell)\big)$. In [12], the authors define an adapted frame $\{\mathfrak{m}, \bar{\mathfrak{m}}_1, \bar{\mathfrak{m}}_2\}$ by using

$$\begin{bmatrix} \bar{\mathfrak{m}}_1(\ell) \\ \bar{\mathfrak{m}}_2(\ell) \end{bmatrix} = \begin{bmatrix} \cos\phi(\ell) & -\sin\phi(\ell) \\ \sin\phi(\ell) & \cos\phi(\ell) \end{bmatrix} \begin{bmatrix} \mathfrak{m}_1(\ell) \\ \mathfrak{m}_2(\ell) \end{bmatrix}.$$

This frame has its Frenet-type formula as

$$\begin{bmatrix} \mathfrak{m}'(\ell) \\ \bar{\mathfrak{m}}_1'(\ell) \\ \bar{\mathfrak{m}}_2'(\ell) \end{bmatrix} = \begin{bmatrix} 0 & \mathfrak{g}(\ell) & 0 \\ -\mathfrak{g}(\ell) & 0 & \mathfrak{h}(\ell) \\ 0 & -\mathfrak{h}(\ell) & 0 \end{bmatrix} \begin{bmatrix} \mathfrak{m}(\ell) \\ \bar{\mathfrak{m}}_1(\ell) \\ \bar{\mathfrak{m}}_2(\ell) \end{bmatrix}. \tag{2.2}$$

It is easy to verify that $(\beta, \bar{\mathfrak{m}}_1, \bar{\mathfrak{m}}_2) : I \to \mathbb{E}^3 \times \Delta_2$ is also a framed curve, where

$$\bar{\mathfrak{m}}(\ell) = \bar{\mathfrak{m}}_1(\ell) \times \bar{\mathfrak{m}}_2(\ell) = \mathfrak{m}_1(\ell) \times \mathfrak{m}_2(\ell) = \mathfrak{m}(\ell),$$

$\mathfrak{m}(\ell)$ is called the generalized tangent vector, $\bar{\mathfrak{m}}_1(\ell)$ is the generalized principal normal, and $\bar{\mathfrak{m}}_2(\ell)$ is the generalized binormal vector of the framed curve, respectively; and the functions $\big(\mathfrak{g}(\ell), \mathfrak{h}(\ell), \mathfrak{f}(\ell)\big)$ are defined as framed curvature of $\beta(\ell)$ with $\mathfrak{g}(\ell) = ||\mathfrak{m}'(\ell)|| > 0$ and $\mathfrak{h}(\ell) = \mathfrak{r}_1(\ell) - \phi'(\ell)$.

**Definition 2.2** (Framed helix)**.** *If the generalized tangent vector* $\mathfrak{m}$ *of a framed curve makes a constant angle with a fixed direction, then it is called as a framed helix* [12].

**Theorem 2.3.** *A necessary and sufficient condition for a framed curve to be a framed helix is*

$$\frac{\mathfrak{h}(\ell)}{\mathfrak{g}(\ell)} = \mp\cot\psi(\ell), \quad \psi = constant,$$

*where* $\big(\mathfrak{g}(\ell), \mathfrak{h}(\ell), \mathfrak{f}(\ell)\big)$ *denotes its framed curvature and* $\mathfrak{g}(\ell) > 0$. [12].

**Definition 2.4** (Framed slant helix)**.** *If the generalized principal normal vector* $\bar{\mathfrak{m}}_1$ *of a framed curve makes a constant angle with a fixed direction, then it is called as a framed slant helix* [9].

**Theorem 2.5.** *A necessary and sufficient condition for a framed curve to be a framed slant helix is*

$$\frac{H'}{\mathfrak{g}\Big(1 + H^2\Big)^{\frac{3}{2}}} = constant, \quad H(\ell) = \frac{\mathfrak{h}(\ell)}{\mathfrak{g}(\ell)},$$

*where* $\big(\mathfrak{g}(\ell), \mathfrak{h}(\ell), \mathfrak{f}(\ell)\big)$ *denotes its framed curvature and* $\mathfrak{g}(\ell) > 0$. [9].

## 3. Framed direction curves

In this section, we define some integral curves connected with a given framed curve.

Let us consider a framed curve $(\beta, \bar{\mathfrak{m}}_1, \bar{\mathfrak{m}}_2) : I \to \mathbb{E}^3 \times \Delta_2$, and let $\big(\mathfrak{g}(\ell), \mathfrak{h}(\ell), \mathfrak{f}(\ell)\big)$ denote its framed curvature, where $\mathfrak{g}(\ell) > 0$.

Let us consider the vector field

$$\mathsf{F}(\ell) = \mathfrak{c}_1(\ell)\bar{\mathfrak{m}}_1(\ell) + \mathfrak{c}_2(\ell)\bar{\mathfrak{m}}_2(\ell) + \mathfrak{c}_3(\ell)\bar{\mathfrak{m}}(\ell),$$

where $\bar{\mathfrak{m}}(\ell) = \bar{\mathfrak{m}}_1(\ell) \times \bar{\mathfrak{m}}_2(\ell) = \mathfrak{m}(\ell)$ and the functions $\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3$ defined on $I$ satisfy $\sum\limits_{i=1}^{3} \mathfrak{c}_i^2 = 1$.

Now, we define some new framed curves connected with the framed curve $\beta$ as following:

**Definition 3.1.** *An integral curve of the vector field $\mathfrak{f}(\ell)\mathsf{F}(\ell)$ is called as framed $\mathsf{F}$-direction curve of $\beta$.*

**Definition 3.2.** *An integral curve of the vector field $\mathfrak{f}(\ell)\bar{\mathfrak{m}}_1(\ell)$ is called as framed generalized principal-direction curve of $\beta$, and an integral curve of the vector field $\mathfrak{f}(\ell)\bar{\mathfrak{m}}_2(\ell)$ is called as framed generalized binormal-direction curve of $\beta$.*

**Remark 3.3.** *Note that an integral curve of the vector field $\mathfrak{f}(\ell)\mathfrak{m}(\ell)$ is nothing but $\beta(\ell)$ up to a translation. It is known that the integral curve is unique with its initial point.*

**Proposition 3.4.** *Let $(\beta, \bar{\mathfrak{m}}_1, \bar{\mathfrak{m}}_2)$ be a framed curve with its curvature $\big(\mathfrak{g}(\ell), \mathfrak{h}(\ell), \mathfrak{f}(\ell)\big)$, $\mathfrak{g}(\ell) > 0$ and $\tilde{\beta}$ be an $\mathsf{F}$-direction curve of $\beta$, where $\mathsf{F}(\ell) = \mathfrak{c}_1(\ell)\bar{\mathfrak{m}}_1(\ell) + \mathfrak{c}_2(\ell)\bar{\mathfrak{m}}_2(\ell) + \mathfrak{c}_3(\ell)\bar{\mathfrak{m}}(\ell)$. Then, $\beta$ is a framed generalized principal-direction curve of $\tilde{\beta}$ up to a translation if and only if*

$$\mathfrak{c}_1(\ell) = \cos\Big(\int \mathfrak{h}(\ell)d\ell\Big), \quad \mathfrak{c}_2(\ell) = -\sin\Big(\int \mathfrak{h}(\ell)d\ell\Big), \quad \mathfrak{c}_3(\ell) = 0. \tag{3.1}$$

*Proof.* Since $\tilde{\beta}$ is an integral curve of $\mathfrak{f}(\ell)\mathsf{F}(\ell)$, we have $\tilde{\beta}'(\ell) = \mathfrak{f}(\ell)\mathsf{F}(\ell)$. Let $\mathfrak{a}_1(\ell)$ denote the generalized principal normal of $\tilde{\beta}$. Thus $(\tilde{\beta}, \mathfrak{a}_1, \mathfrak{a}_2)$ is a framed curve with $\mathfrak{a}_2(\ell) = \mathsf{F}(\ell) \times \mathfrak{a}_1(\ell)$.

$(\Rightarrow)$ : Let us assume that $\beta$ be a framed generalized principal-direction curve of $\tilde{\beta}$. Then, according to definition 3.2, we may write $\beta'(\ell) = \mathfrak{f}(\ell)\mathfrak{a}_1(\ell)$. On the other hand, since $\beta'(\ell) = \mathfrak{f}(\ell)\mathfrak{m}(\ell)$, we obtain $\mathfrak{a}_1(\ell) = \mathfrak{m}(\ell)$ which yields $\mathfrak{c}_3(\ell) = 0$. If we use

$$\mathsf{F}(\ell) = \mathfrak{c}_1(\ell)\bar{\mathfrak{m}}_1(\ell) + \mathfrak{c}_2(\ell)\bar{\mathfrak{m}}_2(\ell), \quad \mathfrak{a}_2(\ell) = \mathfrak{c}_2(\ell)\bar{\mathfrak{m}}_1(\ell) - \mathfrak{c}_1(\ell)\bar{\mathfrak{m}}_2(\ell), \tag{3.2}$$

we find

$$\bar{\mathfrak{m}}_1(\ell) = \mathfrak{c}_1(\ell)\mathsf{F}(\ell) + \mathfrak{c}_2(\ell)\mathfrak{a}_2(\ell), \quad \bar{\mathfrak{m}}_2(\ell) = \mathfrak{c}_2(\ell)\mathsf{F}(\ell) - \mathfrak{c}_1(\ell)\mathfrak{a}_2(\ell). \tag{3.3}$$

If we differentiate the first equation of (3.2), and use $\mathfrak{c}_1^2 + \mathfrak{c}_2^2 = 1$ and (3.3), we have

$$\mathsf{F}'(\ell) = \left(\mathfrak{c}_1'(\ell)\mathfrak{c}_2(\ell) - \mathfrak{c}_1(\ell)\mathfrak{c}_2'(\ell) - \mathfrak{h}(\ell)\right)\mathfrak{a}_2(\ell) - \mathfrak{c}_1(\ell)\mathfrak{g}(\ell)\mathfrak{a}_1(\ell).$$

Thus, according to (2.2), we obtain

$$\mathfrak{c}_1'(\ell)\mathfrak{c}_2(\ell) - \mathfrak{c}_1(\ell)\mathfrak{c}_2'(\ell) - \mathfrak{h}(\ell) = 0.$$

Let $\mathfrak{c}_1(\ell) > 0$. Then, differentiating $\mathfrak{c}_1(\ell) = \sqrt{1-\mathfrak{c}_2^2(\ell)}$ and substituting the result into the last equation yields

$$\mathfrak{c}_2'(\ell) = -\mathfrak{h}(\ell)\sqrt{1-\mathfrak{c}_2^2(\ell)}$$

whose solution is obtained as $\mathfrak{c}_2(\ell) = -\sin\Big(\int \mathfrak{h}(\ell)d\ell\Big)$. Hence, we get $\mathfrak{c}_1(\ell) = \cos\Big(\int \mathfrak{h}(\ell)d\ell\Big)$.

$(\Leftarrow)$ : Let us assume that the equations in (3.1) hold. In this case, we may write

$$\tilde{\beta}'(\ell) = \mathfrak{f}(\ell)\left\{\cos\Big(\int \mathfrak{h}(\ell)d\ell\Big)\bar{\mathfrak{m}}_1(\ell) - \sin\Big(\int \mathfrak{h}(\ell)d\ell\Big)\bar{\mathfrak{m}}_2(\ell)\right\}$$

from which we have $\Big(\tilde{\beta}, \mathfrak{m}, -\sin\Big(\int \mathfrak{h}(\ell)d\ell\Big)\bar{\mathfrak{m}}_1 - \cos\Big(\int \mathfrak{h}(\ell)d\ell\Big)\bar{\mathfrak{m}}_2\Big)$ is a framed curve. Thus, $\beta$ is a framed generalized principal-direction curve of $\tilde{\beta}$ as desired. □

If we consider the proof given above, we also obtain:

**Corollary 3.5.** *The framed curvature* $\big(\mathfrak{p}(\ell), \mathfrak{q}(\ell), \mathfrak{f}(\ell)\big)$ *of* $\tilde{\beta}$ *is obtained as*

$$\mathfrak{p}(\ell) = \mathfrak{g}(\ell)\left|\cos\Big(\int \mathfrak{h}(\ell)d\ell\Big)\right|, \quad \mathfrak{q}(\ell) = -\mathfrak{g}(\ell)\sin\Big(\int \mathfrak{h}(\ell)d\ell\Big).$$

We may give the following definition for framed curves as given for Frenet curves in [2]:

**Definition 3.6.** *An integral curve of the vector field*

$$\mathfrak{f}(\ell)\left\{\cos\Big(\int \mathfrak{h}(\ell)d\ell\Big)\bar{\mathfrak{m}}_1(\ell) - \sin\Big(\int \mathfrak{h}(\ell)d\ell\Big)\bar{\mathfrak{m}}_2(\ell)\right\}$$

*is called as framed principal-donor curve of the framed curve* $(\beta, \bar{\mathfrak{m}}_1, \bar{\mathfrak{m}}_2)$.

### 3.1. Framed generalized principal-direction curve

Let us now examine the framed generalized principal-direction curves and give some properties.

Let us consider a framed curve $(\beta, \bar{\mathfrak{m}}_1, \bar{\mathfrak{m}}_2) : I \to \mathbb{E}^3 \times \Delta_2$ with its framed curvature $\big(\mathfrak{g}(\ell), \mathfrak{h}(\ell), \mathfrak{f}(\ell)\big)$, $\mathfrak{g}(\ell) > 0$. If $\tilde{\beta}_p$ denotes the framed generalized principal-direction curve of $\beta$, we have by its definition

$$\tilde{\beta}_p'(\ell) = \mathfrak{f}(\ell)\bar{\mathfrak{m}}_1(\ell).$$

Then, it is clearly seen that $(\tilde{\beta}_p, \bar{\mathfrak{m}}, \bar{\mathfrak{m}}_{\mathfrak{p}}) : I \to \mathbb{E}^3 \times \Delta_2$ denotes a framed curve, where $\bar{\mathfrak{m}}_{\mathfrak{p}} = -\bar{\mathfrak{m}}_2$. Let us define

$$\begin{bmatrix} \tilde{\bar{\mathfrak{m}}}(\ell) \\ \tilde{\bar{\mathfrak{m}}}_{\mathfrak{p}}(\ell) \end{bmatrix} = \begin{bmatrix} \cos\xi(\ell) & -\sin\xi(\ell) \\ \sin\xi(\ell) & \cos\xi(\ell) \end{bmatrix} \begin{bmatrix} \bar{\mathfrak{m}}(\ell) \\ \bar{\mathfrak{m}}_{\mathfrak{p}}(\ell) \end{bmatrix}, \tag{3.4}$$

where $\xi(\ell)$ is a smooth function. Using (3.4), we have

$$\tilde{\bar{\mathfrak{m}}}_1(\ell) = \tilde{\bar{\mathfrak{m}}}(\ell) \times \tilde{\bar{\mathfrak{m}}}_{\mathfrak{p}}(\ell) = \bar{\mathfrak{m}}_2(\ell) \times \bar{\mathfrak{m}}(\ell) = \bar{\mathfrak{m}}_1(\ell)$$

and using (2.2) we obtain

$$\tilde{\bar{\mathfrak{m}}}'(\ell) = \xi'(\ell)\cos\xi(\ell)\bar{\mathfrak{m}}_2(\ell) - \xi'(\ell)\sin\xi(\ell)\bar{\mathfrak{m}}(\ell) + (\mathfrak{g}(\ell)\cos\xi(\ell) - \mathfrak{h}(\ell)\sin\xi(\ell))\bar{\mathfrak{m}}_1(\ell)$$

and

$$\tilde{\bar{\mathfrak{m}}}'_{\mathfrak{p}}(\ell) = \xi'(\ell)\sin\xi(\ell)\bar{\mathfrak{m}}_2(\ell) + \xi'(\ell)\cos\xi(\ell)\bar{\mathfrak{m}}(\ell) + (\mathfrak{g}(\ell)\sin\xi(\ell) + \mathfrak{h}(\ell)\cos\xi(\ell))\bar{\mathfrak{m}}_1(\ell).$$

We assume that $\xi$ satisfies

$$\mathfrak{g}(\ell)\sin\xi(\ell) + \mathfrak{h}(\ell)\cos\xi(\ell) = 0, \tag{3.5}$$

i.e.

$$\mathfrak{g}(\ell) = -\rho(\ell)\cos\xi(\ell), \quad \mathfrak{h}(\ell) = \rho(\ell)\sin\xi(\ell). \tag{3.6}$$

Then we obtain

$$\tilde{\bar{\mathfrak{m}}}'_1(\ell) = \bar{\mathfrak{m}}'_1(\ell) = -\mathfrak{g}(\ell)\mathfrak{m}(\ell) + \mathfrak{h}(\ell)\bar{\mathfrak{m}}_2(\ell) = \rho(\ell)\tilde{\bar{\mathfrak{m}}}(\ell),$$

$$\tilde{\bar{\mathfrak{m}}}'_{\mathfrak{p}}(\ell) = \xi'(\ell)\tilde{\bar{\mathfrak{m}}}(\ell),$$

$$\tilde{\bar{\mathfrak{m}}}'(\ell) = -\xi'(\ell)\tilde{\bar{\mathfrak{m}}}_{\mathfrak{p}}(\ell) - \rho(\ell)\tilde{\bar{\mathfrak{m}}}_1(\ell).$$

Thus, the Frenet-type formula of the adapted frame $\{\tilde{\bar{\mathfrak{m}}}_1, \tilde{\bar{\mathfrak{m}}}, \tilde{\bar{\mathfrak{m}}}_{\mathfrak{p}}\}$ of $\tilde{\beta}_p$ is obtained as

$$\begin{bmatrix} \tilde{\bar{\mathfrak{m}}}'_1(\ell) \\ \tilde{\bar{\mathfrak{m}}}'(\ell) \\ \tilde{\bar{\mathfrak{m}}}'_{\mathfrak{p}}(\ell) \end{bmatrix} = \begin{bmatrix} 0 & \rho(\ell) & 0 \\ -\rho(\ell) & 0 & \sigma(\ell) \\ 0 & -\sigma(\ell) & 0 \end{bmatrix} \begin{bmatrix} \tilde{\bar{\mathfrak{m}}}_1(\ell) \\ \tilde{\bar{\mathfrak{m}}}(\ell) \\ \tilde{\bar{\mathfrak{m}}}_{\mathfrak{p}}(\ell) \end{bmatrix}.$$

This means that the framed curvature of $\tilde{\beta}_p$ is given by $(\rho(\ell), \sigma(\ell), \mathfrak{f}(\ell))$, where $\sigma(\ell) = -\xi'(\ell)$. Hence, we may give the following:

**Proposition 3.7.** *Let $(\beta, \bar{\mathfrak{m}}_1, \bar{\mathfrak{m}}_2) : I \to \mathbb{E}^3 \times \Delta_2$ be a framed curve, and $\big(\mathfrak{g}(\ell), \mathfrak{h}(\ell), \mathfrak{f}(\ell)\big)$ denote its framed curvature with $\mathfrak{g}(\ell) > 0$. Then the framed curvature $(\rho(\ell), \sigma(\ell), \mathfrak{f}(\ell))$ of framed generalized principal-direction curve $\tilde{\beta}_p$ of $\beta$ can be expressed with*

$$\rho(\ell) = \sqrt{\mathfrak{g}^2(\ell) + \mathfrak{h}^2(\ell)}, \quad \sigma(\ell) = \frac{\mathfrak{g}^2(\ell)}{\mathfrak{g}^2(\ell) + \mathfrak{h}^2(\ell)} \left(\frac{\mathfrak{h}(\ell)}{\mathfrak{g}(\ell)}\right)'. \tag{3.7}$$

*Proof.* If we use (3.6), we obtain $\mathfrak{g}^2(\ell)+\mathfrak{h}^2(\ell)=\rho^2$, i.e. $\rho(\ell)=\sqrt{\mathfrak{g}^2(\ell)+\mathfrak{h}^2(\ell)}$. (3.6) yields also

$$\begin{cases} \mathfrak{g}'(\ell)=-\rho'(\ell)\cos\xi(\ell)+\xi'(\ell)\rho(\ell)\sin\xi(\ell), \\ \mathfrak{h}'(\ell)=\rho'(\ell)\sin\xi(\ell)+\xi'(\ell)\rho(\ell)\cos\xi(\ell). \end{cases} \tag{3.8}$$

If we use (3.6) and (3.8), we find

$$\frac{\mathfrak{g}^2(\ell)}{\mathfrak{g}^2(\ell)+\mathfrak{h}^2(\ell)}\left(\frac{\mathfrak{h}(\ell)}{\mathfrak{g}(\ell)}\right)'=-\xi'(\ell)=\sigma(\ell)$$

which completes the proof. □

**Corollary 3.8.** *If we consider Definition 3.6, $\beta$ is a framed principal-donor curve of $\tilde{\beta}_p$. Thus, by using Corollary 3.5, we have*

$$\mathfrak{g}(\ell)=\rho(\ell)\left|\cos\Big(\int\sigma(\ell)d\ell\Big)\right|,\quad \mathfrak{h}(\ell)=-\rho(\ell)\sin\Big(\int\sigma(\ell)d\ell\Big).$$

If we consider Proposition 1 of [12], we have:

**Proposition 3.9.** *Let $(\beta,\bar{\mathfrak{m}}_1,\bar{\mathfrak{m}}_2):I\to\mathbb{E}^3\times\Delta_2$ be a framed curve, and $\big(\mathfrak{g}(\ell),\mathfrak{h}(\ell),\mathfrak{f}(\ell)\big)$ denote its framed curvature with $\mathfrak{g}(\ell)>0$. If $\beta$ is a regular curve, then the curvature $\tilde{\kappa}(\ell)$ and the torsion $\tilde{\tau}(\ell)$ of its framed generalized principal-direction curve can be given by*

$$\tilde{\kappa}(\ell)=\frac{\sqrt{\mathfrak{g}^2(\ell)+\mathfrak{h}^2(\ell)}}{|\mathfrak{f}(\ell)|},\quad \tilde{\tau}(\ell)=\frac{\mathfrak{g}^2(\ell)}{\mathfrak{f}(\ell)(\mathfrak{g}^2(\ell)+\mathfrak{h}^2(\ell))}\left(\frac{\mathfrak{h}(\ell)}{\mathfrak{g}(\ell)}\right)'.$$

### 3.2. Framed generalized binormal-direction curve

Let us consider a framed curve $(\beta,\bar{\mathfrak{m}}_1,\bar{\mathfrak{m}}_2):I\to\mathbb{E}^3\times\Delta_2$ with its framed curvature $\big(\mathfrak{g}(\ell),\mathfrak{h}(\ell),\mathfrak{f}(\ell)\big)$, $\mathfrak{g}(\ell)>0$. If $\tilde{\beta}_b$ denotes the framed generalized binormal-direction curve of $\beta$, then, we have

$$\tilde{\beta}_b'(\ell)=\mathfrak{f}(\ell)\bar{\mathfrak{m}}_2(\ell).$$

Then, it is clearly seen that $(\tilde{\beta}_b,\bar{\mathfrak{m}}_{\mathfrak{b}},\bar{\mathfrak{m}}):I\to\mathbb{E}^3\times\Delta_2$ is also a framed curve whose Frenet-type formula is expressed with

$$\begin{bmatrix}\bar{\mathfrak{m}}_2'(\ell)\\ \bar{\mathfrak{m}}_{\mathfrak{b}}'(\ell)\\ \bar{\mathfrak{m}}'(\ell)\end{bmatrix}=\begin{bmatrix}0 & \mathfrak{h}(\ell) & 0\\ -\mathfrak{h}(\ell) & 0 & \mathfrak{g}(\ell)\\ 0 & -\mathfrak{g}(\ell) & 0\end{bmatrix}\begin{bmatrix}\bar{\mathfrak{m}}_2(\ell)\\ \bar{\mathfrak{m}}_{\mathfrak{b}}(\ell)\\ \bar{\mathfrak{m}}(\ell)\end{bmatrix},$$

where $\bar{\mathfrak{m}}_{\mathfrak{b}}=-\bar{\mathfrak{m}}_1$. This means that the framed curvature of framed generalized binormal-direction curve $\tilde{\beta}_b$ is $\big(|\mathfrak{h}(\ell)|,\mathfrak{g}(\ell),\mathfrak{f}(\ell)\big)$.

### 3.3. Framed Darboux-direction curve

In this section, by defining the Darboux vector of a framed curve, we define its Darboux-direction curve and give its characterization.

Let us consider the framed curve $(\beta, \bar{\mathfrak{m}}_1, \bar{\mathfrak{m}}_2) : I \to \mathbb{E}^3 \times \Delta_2$ with its framed curvature $\big(\mathfrak{g}(\ell), \mathfrak{h}(\ell), \mathfrak{f}(\ell)\big)$ and $\mathfrak{g}(\ell) > 0$.

Let us define the vector field

$$\mathfrak{d}(\ell) = \mathfrak{h}(\ell)\bar{\mathfrak{m}}(\ell) + \mathfrak{g}(\ell)\bar{\mathfrak{m}}_2(\ell), \quad \forall \ell \in I.$$

We can easily see that $\mathfrak{d}(\ell)$ satisfies

$$\bar{\mathfrak{m}}'(\ell) = \mathfrak{d}(\ell) \times \bar{\mathfrak{m}}(\ell), \quad \bar{\mathfrak{m}}_1' = \mathfrak{d}(\ell) \times \bar{\mathfrak{m}}_1(\ell), \quad \bar{\mathfrak{m}}_2' = \mathfrak{d}(\ell) \times \bar{\mathfrak{m}}_2(\ell).$$

We call $\mathfrak{d}(\ell)$ as generalized Darboux vector of the framed curve $\beta$. Let

$$\mathfrak{D}(\ell) = \frac{\mathfrak{d}(\ell)}{||\mathfrak{d}(\ell)||} = \frac{1}{\sqrt{\mathfrak{g}^2(\ell) + \mathfrak{h}^2(\ell)}}\{\mathfrak{h}(\ell)\bar{\mathfrak{m}}(\ell) + \mathfrak{g}(\ell)\bar{\mathfrak{m}}_2(\ell)\}.$$

Now, we define a new framed curve connected with the framed curve $\beta$ as:

**Definition 3.10.** *Let $(\beta, \bar{\mathfrak{m}}_1, \bar{\mathfrak{m}}_2) : I \to \mathbb{E}^3 \times \Delta_2$ denote a framed curve with the curvature $\big(\mathfrak{g}(\ell), \mathfrak{h}(\ell), \mathfrak{f}(\ell)\big)$ and $\mathfrak{g}(\ell) > 0$. We define an integral curve of the vector field $\mathfrak{f}(\ell)\mathfrak{D}(\ell)$ as framed generalized Darboux-direction curve of $\beta$.*

**Remark 3.11.** *We may write $\mathfrak{D}(\ell)$ as*

$$\mathfrak{D}(\ell) = \frac{1}{\sqrt{1 + \left(\frac{\mathfrak{h}(\ell)}{\mathfrak{g}(\ell)}\right)^2}} \left\{\frac{\mathfrak{h}(\ell)}{\mathfrak{g}(\ell)}\bar{\mathfrak{m}}(\ell) + \bar{\mathfrak{m}}_2(\ell)\right\}.$$

*Thus, $\mathfrak{D}(\ell)$ is a constant vector for a framed helix, since $\mathfrak{D}'(\ell) = 0$, $\forall \ell \in I$.*

**Proposition 3.12.** *Let $(\beta, \bar{\mathfrak{m}}_1, \bar{\mathfrak{m}}_2) : I \to \mathbb{E}^3 \times \Delta_2$ denote a framed curve which is not a framed helix with the framed curvature $\big(\mathfrak{g}(\ell), \mathfrak{h}(\ell), \mathfrak{f}(\ell)\big)$ and $\mathfrak{g}(\ell) > 0$. The framed curvature $(\zeta(\ell), \epsilon(\ell), \mathfrak{f}(\ell))$ of framed generalized Darboux-direction curve of $\beta$ can be expressed with*

$$(3.9) \qquad \zeta(\ell) = ||\mathfrak{D}'(\ell)|| = \frac{|\mathfrak{g}(\ell)\mathfrak{h}'(\ell) - \mathfrak{g}'(\ell)\mathfrak{h}(\ell)|}{\mathfrak{g}^2(\ell) + \mathfrak{h}^2(\ell)}, \quad \epsilon(\ell) = \sqrt{\mathfrak{g}^2(\ell) + \mathfrak{h}^2(\ell)}.$$

*Proof.* Let $\beta_d$ denote the framed generalized Darboux-direction curve of $\beta$. We may write

$$\beta_d'(\ell) = \mathfrak{f}(\ell)\mathfrak{D}(\ell).$$

Let

$$\bar{\mathfrak{D}}(\ell) = \frac{1}{\mathfrak{g}^2(\ell) + \mathfrak{h}^2(\ell)}(\mathfrak{g}(\ell)\bar{\mathfrak{m}}(\ell) - \mathfrak{h}(\ell)\bar{\mathfrak{m}}_2(\ell)).$$

In this case we have $\bar{\mathfrak{D}}(\ell)\times\bar{\mathfrak{m}}_1(\ell)=\mathfrak{D}(\ell)$. This means $(\beta_d,\bar{\mathfrak{D}},\bar{\mathfrak{m}}_1):I\to\mathbb{E}^3\times\Delta_2$ is also a framed curve. Then, $\{\mathfrak{D},\bar{\mathfrak{D}},\bar{\mathfrak{m}}_1\}$ forms an adapted frame along $\beta_d$ whose derivatives can be given by using (2.2) as

$$\mathfrak{D}'(\ell)=\frac{\mathfrak{g}(\ell)\mathfrak{h}'(\ell)-\mathfrak{g}'(\ell)\mathfrak{h}(\ell)}{\mathfrak{g}^2(\ell)+\mathfrak{h}^2(\ell)}\bar{\mathfrak{D}}(\ell),\quad \bar{\mathfrak{m}}_1'(\ell)=-\sqrt{\mathfrak{g}^2(\ell)+\mathfrak{h}^2(\ell)}\bar{\mathfrak{D}}(\ell)$$

and

$$\bar{\mathfrak{D}}'=\frac{\mathfrak{h}(\ell)\mathfrak{g}'(\ell)-\mathfrak{h}'(\ell)\mathfrak{g}(\ell)}{\mathfrak{g}^2(\ell)+\mathfrak{h}^2(\ell)}\mathfrak{D}(\ell)+\sqrt{\mathfrak{g}^2(\ell)+\mathfrak{h}^2(\ell)}\bar{\mathfrak{m}}_1.$$

Then, the Frenet-type formula of $\{\mathfrak{D},\bar{\mathfrak{D}},\bar{\mathfrak{m}}_1\}$ is obtained as

$$\begin{bmatrix}\mathfrak{D}'(\ell)\\ \bar{\mathfrak{D}}'(\ell)\\ \bar{\mathfrak{m}}_1'(\ell)\end{bmatrix}=\begin{bmatrix}0&\eta(\ell)&0\\-\eta(\ell)&0&\epsilon(\ell)\\0&-\epsilon(\ell)&0\end{bmatrix}\begin{bmatrix}\mathfrak{D}(\ell)\\ \bar{\mathfrak{D}}(\ell)\\ \bar{\mathfrak{m}}_1(\ell)\end{bmatrix},$$

where

$$\eta(\ell)=\frac{\mathfrak{g}(\ell)\mathfrak{h}'(\ell)-\mathfrak{g}'(\ell)\mathfrak{h}(\ell)}{\mathfrak{g}^2(\ell)+\mathfrak{h}^2(\ell)},\quad \epsilon(\ell)=\sqrt{\mathfrak{g}^2(\ell)+\mathfrak{h}^2(\ell)}.$$

This completes the proof. □

## 4. Applications

In this section, we give some applications for framed curves by using the results obtained in previous section.

**Proposition 4.1.** *The following results are equivalent:*
**a)** *A framed curve $\beta$ is a framed helix.*
**b)** *A framed generalized principal-direction curve of $\beta$ is a planar curve.*
**c)** *$\beta$ is a framed principal-donor curve of a planar curve.*

*Proof.* If we use Proposition 3.7, the framed curvature $(\rho(\ell),\sigma(\ell),\mathfrak{f}(\ell))$ of framed generalized principal-direction curve $\tilde{\beta}_p$ of $\beta$ satisfies

$$\frac{\sigma(\ell)}{\rho(\ell)}=\frac{\mathfrak{g}^2(\ell)}{(\mathfrak{g}^2(\ell)+\mathfrak{h}^2(\ell))^{\frac{3}{2}}}\left(\frac{\mathfrak{h}(\ell)}{\mathfrak{g}(\ell)}\right)',\tag{4.1}$$

where $\big(\mathfrak{g}(\ell),\mathfrak{h}(\ell),\mathfrak{f}(\ell)\big)$, $(\mathfrak{g}(\ell)>0)$, denotes the framed curvature of $\beta$. Thus, the above equivalences can be easily seen by using (4.1). □

If we use (4.1), we can also give the following:

**Proposition 4.2.** *The following results are equivalent:*
**a)** *A framed curve $\beta$ is a framed slant helix.*
**b)** *A framed generalized principal-direction curve of $\beta$ is a framed helix.*
**c)** *$\beta$ is a framed principal donor-curve of a framed helix.*

### 4.1. Framed helix construction from a planar curve

According to the Proposition 4.1, if a planar curve is given, we can construct a framed helix as a framed principal-donor curve of the given planar curve as following:

Let us consider a planar curve $\alpha(\ell) = (\alpha_1(\ell), \alpha_2(\ell), 0)$. Let

$$\alpha'(\ell) = (\alpha_1'(\ell), \alpha_2'(\ell), 0) := \mathfrak{f}(\ell)(\mathfrak{u}_1(\ell), \mathfrak{u}_2(\ell), 0),$$

where $\mathfrak{m}(\ell) := (\mathfrak{u}_1(\ell), \mathfrak{u}_2(\ell), 0)$ is a unit vector. Then, it is easy to see that $(\alpha, \bar{\mathfrak{m}}_1, \bar{\mathfrak{m}}_2)$ is a framed curve, where

$$\bar{\mathfrak{m}}_1(\ell) = (-\mathfrak{u}_2(\ell), \mathfrak{u}_1(\ell), 0), \quad \bar{\mathfrak{m}}_2(\ell) = (0, 0, 1),$$

and the framed curvature $\big(\mathfrak{g}(\ell), \mathfrak{h}(\ell), \mathfrak{f}(\ell)\big)$ of $\alpha$ is given by

$$\mathfrak{g}(\ell) = ||\mathfrak{m}'(\ell)|| = \sqrt{\mathfrak{u}_1'(\ell) + \mathfrak{u}_2'(\ell)}, \quad \mathfrak{h}(\ell) = 0.$$

Also, it is easy to verify that $\bar{\mathfrak{m}}_1$ corresponds to the generalized principal normal vector and $\bar{\mathfrak{m}}_2$ corresponds to the generalized binormal vector of $\alpha$. Since $\mathfrak{h}(\ell) = 0$, by using Definition 3.6, a framed principal-donor curve of $\alpha$, i.e. a framed helix $\beta$, is obtained from

$$\beta'(\ell) = \mathfrak{f}(\ell)(\lambda_1\bar{\mathfrak{m}}_1 + \lambda_2\bar{\mathfrak{m}}_2) = \Big(-\lambda_1\alpha_2'(\ell), \lambda_1\alpha_1'(\ell), \lambda_2\mathfrak{f}(\ell)\Big), \quad \lambda_1, \lambda_2 = \text{constant},$$

as

$$\beta(\ell) = \Big(-\lambda_1\alpha_2(\ell),\ \lambda_1\alpha_1(\ell),\ \lambda_2\int \mathfrak{f}(\ell)d\ell\Big) \tag{4.2}$$

with $\lambda_1^2 + \lambda_2^2 = 1$ and $\lambda_1 \neq 0$. The framed curvature $\big(\mathfrak{p}(\ell), \mathfrak{q}(\ell), \mathfrak{f}(\ell)\big)$ of $\beta$ is obtained as

$$\mathfrak{p}(\ell) = \lambda_1\mathfrak{g}(\ell) = \lambda_1\sqrt{\mathfrak{u}_1'(\ell) + \mathfrak{u}_2'(\ell)}, \quad \mathfrak{q}(\ell) = -\lambda_1\mathfrak{g}(\ell) = -\lambda_1\sqrt{\mathfrak{u}_1'(\ell) + \mathfrak{u}_2'(\ell)}.$$

### 4.2. Framed slant helix construction from a planar curve

According to the Proposition 4.2, a framed slant helix is a framed principal-donor curve of a framed helix. Thus, combining this with Proposition 4.1, we can construct a framed slant helix from a planar curve as following:

Let us consider a framed helix $\beta$ given by (4.2) obtained from the planar curve $\alpha(\ell) = (\alpha_1(\ell), \alpha_2(\ell), 0)$ with

$$\alpha'(\ell) = (\alpha_1'(\ell), \alpha_2'(\ell), 0) := \mathfrak{f}(\ell)(\mathfrak{u}_1(\ell), \mathfrak{u}_2(\ell), 0).$$

It is clearly seen that the generalized tangent vector of $\beta$ is

$$\mathfrak{m}(\ell) = (-\lambda_1\mathfrak{u}_2(\ell), \lambda_1\mathfrak{u}_1(\ell), \lambda_2)$$

and the generalized principal normal vector of $\beta$ is $\bar{\mathfrak{m}}_1(\ell) := (\mathfrak{u}_1(\ell), \mathfrak{u}_2(\ell), 0)$. Thus, the generalized binormal vector of $\beta$ is $\bar{\mathfrak{m}}_2(\ell) = (-\lambda_2\mathfrak{u}_2(\ell), \lambda_2\mathfrak{u}_1(\ell), -\lambda_1)$.

If we use Definition 3.6, a framed principal-donor curve of $\beta$, i.e. a framed slant helix $\gamma$ constructed from the planar curve $\alpha$, is obtained from

$$\gamma'(\ell) = \mathfrak{f}(\ell)\left\{\cos\Big(\int \mathfrak{q}(\ell)d\ell\Big)\bar{\mathfrak{m}}_1(\ell) - \sin\Big(\int \mathfrak{q}(\ell)d\ell\Big)\bar{\mathfrak{m}}_2(\ell)\right\}$$

as $\gamma(\ell) = (\gamma_1(\ell), \gamma_2(\ell), \gamma_3(\ell))$, where

$$\gamma_1(\ell) = \int\left\{\cos\Big(\int \lambda_1\mathfrak{g}(\ell)d\ell\Big)\alpha_1'(\ell) - \lambda_2\sin\Big(\int \lambda_1\mathfrak{g}(\ell)d\ell\Big)\alpha_2'(\ell)\right\}d\ell,$$

$$\gamma_2(\ell) = \int\left\{\cos\Big(\int \lambda_1\mathfrak{g}(\ell)d\ell\Big)\alpha_2'(\ell) + \lambda_2\sin\Big(\int \lambda_1\mathfrak{g}(\ell)d\ell\Big)\alpha_1'(\ell)\right\}d\ell,$$

$$\gamma_3(\ell) = -\lambda_1\int \sin\Big(\int \lambda_1\mathfrak{g}(\ell)d\ell\Big)\mathfrak{f}(\ell)d\ell.$$

**Proposition 4.3.** *Let $(\beta, \bar{\mathfrak{m}}_1, \bar{\mathfrak{m}}_2) : I \to \mathbb{E}^3 \times \Delta_2$ denote a framed curve which is not a framed helix and $\beta_d$ denote the framed generalized Darboux-direction curve of $\beta$. Then, $\beta$ is a framed slant helix if and only if $\beta_d$ is a framed helix.*

*Proof.* Let $\big(\mathfrak{g}(\ell), \mathfrak{h}(\ell), \mathfrak{f}(\ell)\big)$ denote the framed curvature of framed curve $\beta$ which is not a framed helix. Then the framed curvature $(\zeta(\ell), \epsilon(\ell), \mathfrak{f}(\ell))$ of the framed generalized Darboux-direction curve of $\beta$ satisfies

$$\frac{\epsilon(\ell)}{\zeta(\ell)} = \frac{\big(\mathfrak{g}^2(\ell) + \mathfrak{h}^2(\ell)\big)^{\frac{3}{2}}}{\mathfrak{g}^2(\ell)\left|\Big(\frac{\mathfrak{h}(\ell)}{\mathfrak{g}(\ell)}\Big)'\right|}$$

which yields the desired result. □

## 5. Examples

**Example 5.1.** *Let us consider the planar curve $\alpha(\ell) = (\ell^2, \ell^3, 0)$. Then, we have*

$$\mathfrak{f}(\ell) = \ell\sqrt{4 + 9\ell^2}, \quad \mathfrak{g}(\ell) = \frac{6}{4 + 9\ell^2}.$$

*Thus, a framed helix $\beta$ constructed by using $\alpha$ is obtained as*

$$\beta(\ell) = \Big(-\lambda_1\ell^3, \lambda_1\ell^2, \frac{\lambda_2}{27}\big(4 + 9\ell^2\big)^{3/2}\Big).$$

*A framed slant helix $\gamma$ constructed from $\alpha$ with $\lambda_1 = 1, \lambda_2 = 0$ is obtained as*

$$\gamma(\ell) = \left(\frac{4}{9}\sqrt{4 + 9\ell^2}, \frac{1}{3}\ell\sqrt{4 + 9\ell^2} - \frac{4}{9}\ln\left(\frac{\left|3\ell + \sqrt{4 + 9\ell^2}\right|}{2}\right), \ell^3\right).$$

**Example 5.2.** *Let us consider the curve $\beta : (-2\pi, 2\pi) \to \mathbb{E}^3$ with its parametric equation*

$$\beta(\ell) = \frac{\sqrt{6}}{5}\left(\sin\left(\frac{3\ell}{5}\right) - \frac{2}{7}\sin\left(\frac{7\ell}{5}\right) - \frac{\sin\ell}{5}, \frac{\cos\ell}{5} - \cos\left(\frac{3\ell}{5}\right) + \frac{2}{7}\cos\left(\frac{7\ell}{5}\right), \frac{2\sqrt{6}\ell}{5} - \sqrt{6}\sin\left(\frac{2\ell}{5}\right)\right).$$

*It is easy to see that the tangent vector vanishes at $\beta(0)$. $(\beta, \bar{\mathfrak{m}}_1, \bar{\mathfrak{m}}_2) : (-2\pi, 2\pi) \to \mathbb{E}^3 \times \Delta_2$ is a framed curve* [9], *where*

$$\mathfrak{m}(\ell) = \left(\frac{3}{5}\sin\left(\frac{4\ell}{5}\right) + \frac{2}{5}\sin\left(\frac{6\ell}{5}\right), -\frac{3}{5}\cos\left(\frac{4\ell}{5}\right) - \frac{2}{5}\cos\left(\frac{6\ell}{5}\right), \frac{2\sqrt{6}}{5}\sin\left(\frac{\ell}{5}\right)\right),$$

$$\bar{\mathfrak{m}}_1(\ell) = \left(\frac{2\sqrt{6}}{5}\cos\ell, \frac{2\sqrt{6}}{5}\sin\ell, \frac{1}{5}\right),$$

$$\bar{\mathfrak{m}}_2(\ell) = \left(\frac{2}{5}\cos\left(\frac{6\ell}{5}\right) - \frac{3}{5}\cos\left(\frac{4\ell}{5}\right), \frac{2}{5}\sin\left(\frac{6\ell}{5}\right) - \frac{3}{5}\sin\left(\frac{4\ell}{5}\right), \frac{2\sqrt{6}}{5}\cos\left(\frac{\ell}{5}\right)\right).$$

*The framed curvature of framed curve $(\beta, \bar{\mathfrak{m}}_1, \bar{\mathfrak{m}}_2)$ is obtained as*

$$\mathfrak{g}(\ell) = \frac{2\sqrt{6}}{5}\cos\left(\frac{\ell}{5}\right), \quad \mathfrak{h}(\ell) = \frac{2\sqrt{6}}{5}\sin\left(\frac{\ell}{5}\right), \tag{5.1}$$

*where $\beta$ is a framed slant helix* [9]. *Besides, since $\beta'(\ell) = \mathfrak{f}(\ell)\mathfrak{m}(\ell)$, we obtain*

$$\mathfrak{f}(\ell) = \langle \beta'(\ell), \mathfrak{m}(\ell)\rangle = \frac{2\sqrt{6}}{5}\sin\left(\frac{\ell}{5}\right).$$

*Then, the framed generalized principal-direction curve $\tilde{\beta}_p$ of $\beta$ which satisfies $\tilde{\beta}_p(0) = \left(\frac{1}{5}, 0, -\frac{2\sqrt{6}}{5}\right)$ is obtained as (see Figure 1)*

$$\tilde{\beta}_p(\ell) = \left(\frac{3}{5}\cos\left(\frac{4\ell}{5}\right) - \frac{2}{5}\cos\left(\frac{6\ell}{5}\right), \frac{3}{5}\sin\left(\frac{4\ell}{5}\right) - \frac{2}{5}\sin\left(\frac{6\ell}{5}\right), -\frac{2\sqrt{6}}{5}\cos\left(\frac{\ell}{5}\right)\right).$$

*If we use (3.5), we get*

$$\cos\left(\frac{\ell}{5}\right)\sin\xi(\ell) + \sin\left(\frac{\ell}{5}\right)\cos\xi(\ell) = 0,$$

*i.e.* $\sin\left(\xi(\ell) + \frac{\ell}{5}\right) = 0$ *which yields $\xi(\ell) = -\frac{\ell}{5} + k\pi, k \in \mathbb{Z}$. Then, by using (3.4) we obtain*

$$\tilde{\bar{\mathfrak{m}}}(\ell) = (\sin\ell, -\cos\ell, 0) \quad \text{and} \quad \tilde{\bar{\mathfrak{m}}}_{\mathfrak{p}}(\ell) = \left(\frac{1}{5}\cos\ell, \frac{1}{5}\sin\ell, -\frac{2\sqrt{6}}{5}\right).$$

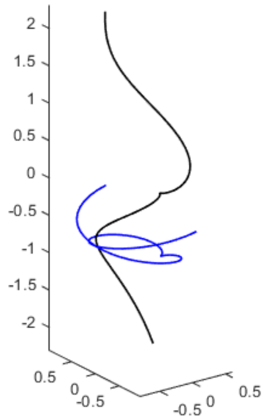


FIGURE 1. The curve $\beta$ (black) and its framed generalized principal-direction curve $\tilde{\beta}_p$ (blue)

*Since we also have* $\tilde{\bar{\mathfrak{m}}}_1(\ell) = \bar{\mathfrak{m}}_1(\ell)$, *we find*

$$\tilde{\bar{\mathfrak{m}}}_1'(\ell) = -\frac{2\sqrt{6}}{5}\tilde{\bar{\mathfrak{m}}}(\ell), \quad \tilde{\bar{\mathfrak{m}}}_{\mathfrak{p}}'(\ell) = -\frac{1}{5}\tilde{\bar{\mathfrak{m}}}(\ell).$$

*Thus, we obtain the framed curvature* $(\rho(\ell), \sigma(\ell), \mathfrak{f}(\ell))$ *of the framed curve* $(\tilde{\beta}_p, \tilde{\bar{\mathfrak{m}}}, \tilde{\bar{\mathfrak{m}}}_{\mathfrak{p}})$ *as*

$$\rho(\ell) = \frac{2\sqrt{6}}{5}, \quad \sigma(\ell) = \frac{1}{5} \tag{5.2}$$

*which yields that, according to the Theorem 2.3,* $\tilde{\beta}_p$ *is a framed helix. The framed curvature of* $\tilde{\beta}_p$ *given in (5.2) can also be obtained by using (3.7) and (5.1).*

*Furthermore, the framed generalized binormal-direction curve* $\tilde{\beta}_b$ *of* $\beta$ *which satisfies* $\tilde{\beta}_b(0) = \left(-\frac{2\sqrt{6}}{35}, 0, -\frac{6}{5}\right)$ *is obtained as (see Figure 2)*

$$\tilde{\beta}_b(\ell) = \sqrt{6}\left(\frac{\cos\ell}{5} - \frac{1}{5}\cos\left(\frac{3\ell}{5}\right) - \frac{2}{35}\cos\left(\frac{7\ell}{5}\right), \frac{\sin\ell}{5} - \frac{1}{5}\sin\left(\frac{3\ell}{5}\right) - \frac{2}{35}\sin\left(\frac{7\ell}{5}\right), -\frac{\sqrt{6}}{5}\cos\left(\frac{2\ell}{5}\right)\right).$$

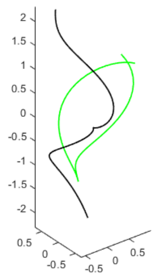


FIGURE 2. The curve $\beta$ (black) and its framed generalized binormal-direction curve $\tilde{\beta}_b$ (green)

*On the other hand, the unit generalized Darboux vector of $\gamma$ is obtained as*

$$\mathfrak{D}(\ell) = \left( -\frac{1}{5}\cos\ell, -\frac{1}{5}\sin\ell, \frac{2\sqrt{6}}{5} \right).$$

*Thus, the framed generalized Darboux-direction curve $\beta_d$ of $\beta$ which satisfies $\beta_d(0) = \left(-\frac{\sqrt{6}}{60}, 0, -\frac{24}{5}\right)$ is obtained as (see Figure 3)*

$$\beta_d(\ell) = \left( \frac{\sqrt{6}}{30}\cos\left(\frac{6\ell}{5}\right) - \frac{\sqrt{6}}{20}\cos\left(\frac{4\ell}{5}\right), \frac{\sqrt{6}}{30}\sin\left(\frac{6\ell}{5}\right) - \frac{\sqrt{6}}{20}\sin\left(\frac{4\ell}{5}\right), -\frac{24}{5}\cos\left(\frac{\ell}{5}\right) \right)$$

*and its framed curvature is obtained as $\zeta(\ell) = \frac{1}{5}$, $\epsilon(\ell) = \frac{2\sqrt{6}}{5}$ by using (3.9). It is obvious that $\beta_d$ is a framed helix.*

## 6. Conclusion

Choi and Kim [2] introduce some integral curves connected with a Frenet curve in Euclidean 3-space. They characterize general helices and slant helices in terms of their associated curves and also give methods to construct these helical curves. Motivated by their study, this paper introduces some integral curves connected with a given framed curve. Framed curve is a generalization of regular curves with linearly independent condition. We define framed generalized principal-direction curve, framed generalized binormal-direction curve, framed principal-donor curve and framed generalized Darboux-direction curve. It is shown that these new curves are related with framed helix and framed slant helix. Since helices have applications in science and nature, we also present methods to construct framed helix and framed slant helix from a planar curve. Such integral curves can be defined and studied for framed curves in higher dimensional spaces.

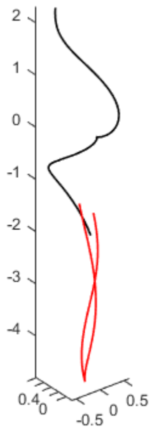


FIGURE 3. The curve $\beta$ (black) and its framed generalized Darboux-direction curve $\beta_d$ (red)

**Acknowledgment.** The authors would like to thank reviewers for their comments and suggestions.

## References

[1] R. L. Bishop, *There is more than one way to frame a curve*, American Mathematical Monthly **82** (1975), no. 3, 246–251.

[2] J. H. Choi and Y. H. Kim, *Associated curves of a Frenet curve and their applications*, Applied Mathematics and Computation **218** (2012), 9116–9124.

[3] T. Fukunaga and M. Takahashi, *Existence conditions of framed curves for smooth curves*, Journal of Geometry **108** (2017), 763–774.

[4] S. Honda and M. Takahashi, *Framed curves in the Euclidean space*, Advances in Geometry **16** (2016), no. 3, 265–276.

[5] S. Honda and M. Takahashi, *Bertrand and Mannheim curves of framed curves in the 3-dimensional Euclidean space*, Turkish Journal of Mathematics **44** (2020), 883–899.

[6] T. Körpınar, M. T. Sarıaydın, and E. Turhan, *Associated curves according to Bishop frame in Euclidean 3-space*, Advanced Modelling and Optimization **15** (2013), 713–717.

[7] N. Macit and M. Düldül, *Some new associated curves of a Frenet curve in* $\mathbb{E}^3$ *and* $\mathbb{E}^4$, Turkish Journal of Mathematics **38** (2014), 1023–1037.

[8] N. Macit and M. Düldül, *Relatively normal-slant helices lying on a surface and their characterizations*, Hacettepe Journal of Mathematics and Statistics **46** (2017), no. 3, 397–408.

[9] O. Z. Okuyucu and M. Canbirdi, *Framed slant helices in Euclidean 3-space*, Advances in Difference Equations **2021** (2021), Article number 504.

[10] B. Şahiner, *Direction curves of tangent indicatrix of a curve*, Applied Mathematics and Computation **343** (2019), 273–284.

[11] S. Şenyurt, D. Canlı, and K. H. Ayvacı, *Associated curves from a different point of view in $E^3$*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **71** (2022), no. 3, 826–845.

[12] Y. Wang, D. Pei, and R. Gao, *Generic properties of framed rectifying curves*, Mathematics **7** (2019), no. 1, 37.

[13] B. Doğan Yazıcı, S. Özkaldı Karakuş, and M. Tosun, *Framed curves and their applications based on a new differential equation*, International Electronic Journal of Geometry **15** (2022), no. 1, 47-56.

Mustafa Düldül
Department of Mathematics, Yildiz Technical University,
İstanbul 34220, Türkiye.
E-mail: mduldul@yildiz.edu.tr

Zeynep Bülbül
Department of Mathematics, Yildiz Technical University,
İstanbul 34220, Türkiye.
E-mail: zeynepbulbul3@gmail.com