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BOUND ON HANKEL DETERMINANTS $H_4^{(2)}(f)$ AND $H_4^{(3)}(f)$ FOR LEMNISCATE STARLIKE FUNCTIONS

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Abstract. We determine the upper bounds on fourth order Hankel determinants $H_4^{(2)}(f)$ and $H_4^{(3)}(f)$ for the class $\mathcal{S}_L^*$ of lemniscate starlike functions defined on the open unit disk which was introduced by Sokół and Stankiewicz in [17].

1. Preliminaries and the class $\mathcal{S}_L^*$

Let $\mathcal{A}$ be the class of all analytic functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

defined in open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = 0$ and $f'(0) = 1$. The subclass $\mathcal{S}$ of the class $\mathcal{A}$ consists of univalent functions. Let $\langle a_k \rangle_{k \geq 1}$ denote a sequence of coefficients of the functions $f \in \mathcal{A}$. The Hankel determinant of order n associated with the sequence $\langle a_k \rangle_{k \geq 1}$ is defined by

$$H_q^{(n)}(f) := |\{a_{n+i+j-2}\}|_{i,j}^q, \quad (i,\, j \in \mathbb{N}; a_1 = 1) \tag{2}$$

where q and n are positive integers. A function $f \in \mathcal{S}$ is starlike if $f(\mathbb{D})$ is starlike with respect to the origin and the class of such functions is denoted by $\mathcal{S}^*$. If f and g are analytic functions in $\mathbb{D}$, then f is subordinate to g, written as $f \prec g$, if there exists a Schwarz function w such that $f = g \circ w$. For univalent function g, the equivalence condition $f \prec g \Leftrightarrow f(0) = g(0)$ and $f(|z| < 1) \subset g(|z| < 1)$ holds [9]. A function $f \in \mathcal{S}$ is lemniscate starlike if the quantity $zf'(z)/f(z)$ lies in the region bounded by the right–half of the lemniscate of Bernoulli given by $|w^2 - 1| < 1$. The class of such functions is denoted by $\mathcal{S}_L^*$ and was introduced by Sokół and Stankiewicz [17]. In terms of subordination, a function $f \in \mathcal{S}_L^*$ if and only if $zf'(z)/f(z) \prec \sqrt{1+z}$ for all $z \in \mathbb{D}$. For various geometric properties such as the structural formula,

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growth and distortion theorems, Fekete-Szegö functionals, radius estimates, subordination relations, coefficient estimates of such functions, see [1, 2, 3, 13].

Let $\mathcal{P}$ be the class of analytic functions having the Taylor series of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots . \tag{3}$$

If the function $f \in \mathcal{S}_L^*$, then $zf'(z)/f(z) = \sqrt{1+w(z)}$ where $w(z) = c_1 z + c_2 z^2 \cdots$ is the Schwarz function defined over $\mathbb{D}$. Note that $w(z) = (p(z)-1)/(p(z)+1)$ where $p \in \mathcal{P}$. Thus if $f \in \mathcal{S}_L^*$, then

$$\frac{zf'(z)}{f(z)} = \sqrt{2}\left(\frac{p(z)}{p(z)+1}\right)^{1/2} \tag{4}$$

for some $p \in \mathcal{P}$. Using (1), (3), (4), and comparing the coefficients on both sides, we get

$$a_2 = \frac{p_1}{4}, \tag{5}$$

$$a_3 = \frac{-3p_1^2 + 8p_2}{64}, \tag{6}$$

$$a_4 = \frac{13p_1^3 - 56p_1p_2 + 64p_3}{768}, \tag{7}$$

$$a_5 = \frac{1}{6144}(-49p_1^4 + 272p_1^2p_2 - 352p_1p_3 - 192(p_2^2 - 2p_4)), \tag{8}$$

$$\begin{aligned} a_6 = \frac{1}{122880}(&543p_1^5 - 3568p_1^3p_2 + 4608p_1^2p_3 - 6400p_2p_3 \\ &+64p_1(77p_2^2 - 90p_4) + 6144p_5), \end{aligned} \tag{9}$$

(10)

$$\begin{aligned} a_7 = \frac{1}{11796480}(&-32303p_1^6 + 241688p_1^4p_2 - 301888p_1^3p_3 + 64p_1^2(-7457p_2^2 \\ &+5940p_4) + 9728p_1(85p_2p_3 - 48p_5) + 2560(57p_2^3 - 104p_3^2 - 204p_2p_4 + 192p_6)), \end{aligned}$$

(11)

$$\begin{aligned} a_8 = \frac{1}{330301440}(&607537p_1^7 - 5077864p_1^5p_2 + 6101312p_1^4p_3 + 64p_1^3(198751p_2^2 \\ &-116940p_4) + 512p_1^2(18168p_5 - 46901p_2p_3) - 512p_1(16371p_2^3 \\ &-39780p_2p_4 + 1840(12p_6 - 11p_3^2)) + 24576(445p_2^2p_3 \\ &-516p_2p_5 - 530p_3p_4 + 480p_7). \end{aligned}$$

Bounds on coefficients of the univalent functions yields information regarding the geometric characteristics of the functions. For the function $f \in \mathcal{S}_L^*$, Sokół [16] established the following sharp bounds on initial coefficients

$$|a_2| \le \frac{1}{2}, \quad |a_3| \le \frac{1}{4} \quad \text{and} \quad |a_4| \le \frac{1}{6}. \tag{12}$$

In this sequel, authors [15] obtained sharp bound on fifth coefficients that is given by

$$|a_5| \leq \frac{1}{8}. \tag{13}$$

In 2018, Sokół and Thomas [18] reported a non–sharp estimate for n^{th} coefficient. This estimate is close to the conjecture related to the bound on n^{th} coefficient in [16] that is given by

$$|a_n| \leq \frac{2-\sqrt{2}}{n-1}, \quad (n = 2, 3, 4, \ldots). \tag{14}$$

Initially, Pommerenke [14] determined estimates on the Hankel determinants for starlike functions and univalent functions. Sharp estimates on $H_2^{(2)}(f) = a_2a_4 - a_3^2$ for unified classes of Ma-Minda starlike and convex functions were estimated in [12]. For analytic functions with bounded turning, starlike and convex functions, Babalola [7] computed non-sharp estimates on $H_3^{(1)}(f)$. Further, the best possible bound on $H_3^{(1)}(f)$ for the function $f \in \mathcal{S}_L^*$ is 1/36 [8]. Authors [4, 5] investigated the estimates on the fourth and fifth order Hankel determinants for the functions with bounded turning. Authors [11] computed the bound on $H_4^{(1)}(f)$ for the certain strongly starlike functions. In 2020, Arif *et al.* [6] determined non-sharp bound on $H_4^{(1)}(f)$ for the functions $f \in \mathcal{S}_L^*$.

Motivated by the above discussed work related to estimates on Hankel determinants for subclasses of starlike functions, we determine the bounds on fourth order Hankel determinants $H_4^{(2)}(f)$ and $H_4^{(3)}(f)$ for the function $f \in \mathcal{S}_L^*$.

2. Bounds on $H_4^{(2)}(f)$ and $H_4^{(3)}(f)$

In this section, we state our main results.

Theorem 2.1. *Let the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}_L^*$. Then the following estimate on fourth order Hankel determinant holds:*

$$|H_4^{(2)}(f)| \leq 0.129167.$$

Theorem 2.2. *Let the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}_L^*$. Then the following estimate on fourth order Hankel determinant holds:*

$$|H_4^{(3)}(f)| \leq 0.163932.$$

In order to prove our results, we need the following lemmas.

Lemma 2.3. [15, Lemma 2.3, p. 507] *Let $p \in \mathcal{P}$. Then for all $n, m \in \mathbb{N}$*

$$|\mu p_n p_m - p_{m+n}| \leq \begin{cases} 2, & 0 \leq \mu \leq 1; \\ 2|2\mu - 1|, & \textit{elsewhere.} \end{cases}$$

If $0 < \mu < 1$, *then the inequality is sharp for the function* $p(z) = (1 + z^{m+n})/(1 - z^{m+n})$. *In the other cases, the inequality is sharp for the function* $p_0(z) = (1 + z)/(1 - z)$.

Lemma 2.4. [10] *Let* $p \in \mathcal{P}$. *Then for any real number* μ, *the following holds:*

$$|\mu p_3 - p_1^3| \leq \begin{cases} 2|\mu - 4|, & \mu \leq \frac{4}{3}; \\ 2\mu\sqrt{\frac{\mu}{\mu-1}}, & \mu > \frac{4}{3}. \end{cases}$$

The result is sharp. If $\mu \leq \frac{4}{3}$, *then equality holds for the function* $p_0(z) := (1 + z)/(1 - z)$ *and if* $\mu > \frac{4}{3}$, *then equality holds for the function*

$$p_1(z) := \frac{1 - z^2}{z^2 - 2\sqrt{\frac{\mu}{\mu-1}}\, z + 1}.$$

Lemma 2.5. *Let the function* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be in the class* $\mathcal{S}_L^*$. *Then the estimate on third order Hankel determinant is given by*

$$|H_3^{(2)}(f)| \leq \frac{606811}{1105920} + \frac{\sqrt{3}}{36} \approx 0.596806.$$

Proof. In view of definition (2), we have a third order Hankel determinant

$$H_3^{(2)}(f) = a_2(a_2 a_6 - a_5^2) - a_3(a_3 a_6 - a_4 a_5) + a_4(a_3 a_5 - a_4^2). \tag{15}$$

On substituting the values of a_i $(i = 2, 3, 4, 5, 6)$ from (5)–(9) in expression (15), we have

$$4529848320 H_3^{(2)}(f) = \boldsymbol{\chi}_1(p_i), \tag{16}$$

where

$$\begin{aligned}\boldsymbol{\chi}_1(p_i) = &- 80653 p_1^9 + 48 p_1^7(26064 + 18523 p_2) - 1038528 p_1^6 p_3 - 1152 p_1^5(p_2(7136 \\ &+ 2741 p_2) - 995 p_4) + 1024 p_1^3(11088 p_2^2 + 3761 p_2^3 - 3210 p_3^2 - 90(144 \\ &+ 61 p_2) p_4) - 36864 p_1(37 p_2^4 - 40 p_2 p_3^2 - 70 p_2^2 p_4 + 120 p_4^2) + 768 p_1^4((13824 \\ &+ 7627 p_2) p_3 - 648 p_5) + 16384(5 p_3(9 p_2^3 - 32 p_3^2 + 72 p_2 p_4) - 216 p_2^2 p_5) \\ &- 24576 p_1^2(p_3(p_2(600 + 223 p_2) - 240 p_4) - 36(16 + 3 p_2) p_5).\end{aligned}$$

After rearrangement of terms and applying triangle inequality in the expression of $\boldsymbol{\chi}_1(p_i)$, we have

$$\begin{aligned}|\boldsymbol{\chi}_1(p_i)| \leq & 889104 |p_1|^7 \left|\frac{80653}{889104} p_1^2 - p_2\right| + 8220672 |p_1|^5 \left|\frac{543}{3568} p_1^2 - p_2\right| \\ &+ 3851264 |p_1|^3 |p_2|^2 \left|\frac{24669}{30088} p_1^2 - p_2\right| + 1146240 |p_1|^5 \left|\frac{1803}{1990} p_1 p_3 - p_4\right| \\ &+ 13271040 |p_1|^3 \left|\frac{77}{90} p_2^2 - p_4\right| + 5621760 |p_2||p_4| \left|\frac{64}{61} p_3 - p_1^3\right|\end{aligned}$$

$$+ 2621440|p_3|^2 \left|\frac{9}{16}p_1p_2 - p_3\right| + 3538944|p_2|^2 \left|\frac{35}{48}p_1p_4 - p_5\right|$$
$$+ 4423680|p_1||p_4| \left|\frac{4}{3}p_1p_3 - p_4\right| + 737280|p_2|^3 \left|\frac{37}{20}p_1p_2 - p_3\right|$$
$$+ 497664|p_1|^4 \left|\frac{7627}{648}p_2p_3 - p_5\right| + 2654208|p_1|^2|p_2| \left|\frac{223}{108}p_2p_3 - p_5\right|$$
$$+ 14155776|p_1|^2 \left|\frac{25}{24}p_2p_3 - p_5\right| + 3287040|p_1|^3|p_3|^2 + 10616832|p_1|^4|p_3|.$$

Applying Lemmas 2.3 and 2.4 and the fact $|p_n| \leq 2$, we have

$$\begin{aligned}|\boldsymbol{\chi}_1(p_i)| \leq& 889104.2^8 + 8220672.2^6 + 3851264.2^6 + 1146240.2^6\\ &+ 13271040.2^4 + 5621760.2^3 \left(\frac{64}{61}.\frac{8}{\sqrt{3}}\right) + 10616832.2^5\\ &+ 2621440.2^3 + 3538944.2^3 + 4423680.2^3 \left(\frac{5}{3}\right) + 3287040.2^5\\ &+ 737280.2^4 \left(\frac{27}{10}\right) + 497664.2^5 \left(\frac{7303}{324}\right)\\ &+ 2654208.2^4 \left(\frac{169}{54}\right) + 14155776.2^3 \left(\frac{13}{12}\right)\\ =& 4096(606811 + 30720\sqrt{3}).\end{aligned}$$

Thus from (16), we get

$$|H_3^{(2)}(f)| \leq \frac{4096(606811 + 30720\sqrt{3})}{4529848320} = \frac{606811}{1105920} + \frac{\sqrt{3}}{36}.$$

□

Lemma 2.6. *Let the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{S}_L^*$. Then the estimate on third order Hankel determinant is*

$$|H_3^{(3)}(f)| \leq \frac{888953739851}{10422897868800} + \frac{2266967}{265420800}\sqrt{\frac{2266967}{2260394}} \approx 0.093842.$$

Proof. In view of definition (2), a third order Hankel determinant is given by

$$H_3^{(3)}(f) = a_3(a_5a_7 - a_6^2) - a_4(a_4a_7 - a_5a_6) + a_5(a_4a_6 - a_5^2).$$

On substituting the values of a_i $(i = 3, 4, 5, 6, 7)$ from (6)–(10) in expression of $H_3^{(3)}(f)$, we have

$$69578470195200 H_3^{(3)}(f) = \boldsymbol{\chi}_2(p_i),$$

where

$$\begin{aligned}\boldsymbol{\chi}_2(p_i) =& - 667801p_1^{12} + 8788448p_1^{10}p_2 + 2434208p_1^9p_3 - 960p_1^8(54925p_2^2 - 108522p_4)\\ &- 3072p_1^7(68293p_2p_3 - 85112p_5) + 2048p_1^6(132250p_2^3 - 660210p_2p_4 + 6573p_3^2\end{aligned}$$

$$+ 123600p_6) + 12288p_1^5(147737p_2^2p_3 - 194576p_2p_5 + 73980p_3p_4)$$
$$- 4096p_1^4(264943p_2^4 - 1423380p_2^2p_4 + 48p_2(8213p_3^2 + 8900p_6)$$
$$- 24(12536p_3p_5 + 2025p_4^2)) - 65536p_1^3(70843p_2^3p_3 - 115368p_2^2p_5 + 117780p_2p_3p_4$$
$$- 20(571p_3^3 - 300p_3p_6 + 108p_4p_5)) + 393216p_1^2(5461p_2^5 - 25310p_2^3p_4$$
$$+ 5p_2^2(4187p_3^2 + 2480p_6) + 4p_2(675p_4^2 - 5476p_3p_5) + 288(40p_3^2p_4 - 75p_4p_6 + 72p_5^2))$$
$$+ 131072p_1(16845p_2^4p_3 - 63072p_2^3p_5 + 96360p_2^2p_3p_4 - 40p_2(1355p_3^3 - 2760p_3p_6$$
$$+ 2376p_4p_5) + 240p_3(405p_4^2 - 448p_3p_5)) - 262144(4725p_2^6 - 22950p_2^4p_4 + 600p_2^3(29p_3^2$$
$$+ 72p_6) - 1080p_2^2(96p_3p_5 + 5p_4^2) + 48p_2(25p_4(91p_3^2 - 72p_6) + 1728p_5^2)$$
$$- 160(260p_3^4 - 480p_3^2p_6 + 864p_3p_4p_5 - 405p_4^3)).$$

On rearranging the terms in the above expression and applying triangle inequality in the expression of $\boldsymbol{\chi}_2(p_i)$, we have

$$\begin{aligned}
|\boldsymbol{\chi}_2(p_i)| \leq & 8788448|p_1|^{10}\left|\frac{667801}{8788448}p_1^2 + p_2\right| + 960(108522)|p_1|^8\left|\frac{54925}{108522}p_2^2 - p_4\right| \\
& + 3072(68293)|p_1|^7|p_2|\left|\frac{76069}{6556128}p_1^2 - p_2\right| + 4096(48)(8900)|p_1|^4|p_6| \\
& \left|\frac{103}{712}p_1^2 - p_2\right| + 2048(660210)|p_1|^6|p_2|\left|\frac{13225}{66021}p_2^2 - p_4\right| \\
& + 4096(1423380)|p_1|^4|p_2|^2\left|\frac{37849}{203340}p_2^2 - p_4\right| + 4096(24)(12536)|p_1|^4|p_3| \\
& \left|\frac{8213}{6268}p_2p_3 - p_5\right| + 2048(6573)|p_1^3p_2^2|\left|\frac{2266976}{6573}p_3 - p_1^3\right| \\
& + 12288(194576)|p_1|^5|p_2|\left|\frac{147737}{194576}p_2p_3 - p_5\right| + 393216(4)(675)|p_1^2p_2p_4^2| \\
& + 26144(600)(29)|p_2^2p_3^2|\left|\frac{2143744}{118465}p_1^2 - p_2\right| + 131072(40)(2760)|p_1p_2p_3| \\
& \left|\frac{271}{552}p_3^2 - p_6\right| + 131072(63072)|p_1||p_2|^3\left|\frac{5615}{21024}p_2p_3 - p_5\right| \\
& + 4096(24)(2025)|p_1^4p_4^2| + 131072(40)(2376)|p_1p_2p_4|\left|\frac{73}{72}p_2p_3 - p_5\right| \\
& + 65536(115368)|p_1^3p_2^2p_5| + 65536(117780)|p_1^3p_3p_4|\left|\frac{3699}{31408}p_1^2 - p_3\right| \\
& + 65536(6000)|p_1^3p_3|\left|\frac{571}{300}p_3^3 - p_6\right| + 262144(22950)|p_2|^4\left|\frac{7}{34}p_2^2 - p_4\right|
\end{aligned}$$

$$+ 26144(48)(1728)|p_2p_5|\left|\frac{5}{4}p_2p_3 - p_5\right|$$
$$+ 26144(48)(25)(72)|p_2p_4|\left|\frac{91}{72}p_3^2 - p_6\right| + 26144(160)(480)|p_3|^2\left|\frac{13}{24}p_3^2 - p_6\right|$$
$$+ 393216(25310)|p_1^2p_2^3|\left|\frac{5461}{25310}p_2^2 - p_4\right| + 26144(600)(72)|p_2^2p_6|\left|\frac{31744}{7353}p_1^2 - p_2\right|$$
$$+ 26144(160)(405)|p_4|^2\left|\frac{1}{12}p_2^2 - p_4\right| + 393216(288)(72)|p_1^2p_5|\left|\frac{1369}{1296}p_2p_3 - p_5\right|$$
$$+ 26144(160)(864)|p_3p_5|\left|\frac{28672}{7353}p_1p_3 - p_5\right| + 131072(240)(405)|p_1p_3p_4^2|$$
$$+ 393216(288)(75)|p_1^2p_4|\left|\frac{8}{15}p_3^2 - p_6\right| + 3072(85112)|p_1^7p_5| + 65536(20)(108)|p_1^3p_4p_5|.$$

Using Lemmas 2.3, 2.4 and the inequality $|p_n| \leq 2$, we have

$$|\boldsymbol{\chi}_2(p_i)| \leq \frac{43693854221156352}{7363} + 297135898624\sqrt{\frac{4533934}{1130197}}$$

so that

$$|H_3^{(3)}(f)| \leq \frac{888953739851}{10422897868800} + \frac{2266967}{265420800}\sqrt{\frac{2266967}{2260394}} \approx 0.093842.$$

□

Lemma 2.7. *Let the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{S}_L^*$. Then*

$$\begin{aligned}|\Delta_1| =& |a_2(a_4a_7 - a_5a_6) - a_3(a_3a_7 - a_5^2) + a_4(a_3a_6 - a_4a_5)|\\ \leq& \frac{893770979}{4529848320} + \frac{1}{25}\sqrt{\frac{2}{177}} \approx 0.201559.\end{aligned}$$

Proof. On substituting the values of a_i $(i = 2, 3, 4, 5, 6, 7)$ from (5)–(10) in the expression of Δ_1, we get

$$144955146240\Delta_1 = \boldsymbol{\chi}_3(p_i),$$

where

$$\begin{aligned}\boldsymbol{\chi}_3(p_i) =& -139627p_1^{10} + 1287528p_1^8p_2 - 3187776p_1^7p_3 - 384p_1^6(4795p_2^2 + 11674p_4)\\ &+ 1536p_1^5(18719p_2p3 - 1936p_5) - 1024p_1^4(12913p_2^3 - 25974p_2p_4 + 24(739p_3^2\\ &- 500p_6)) - 12288p_1^3(4905p_2^2p_3 + 224p_2p_5 + 2872p_3p_4) + 12288p_1^2(3231p_2^4\\ &- 2812p_2^2p_4 + 8p_2(1271p_3^2 - 400p_6) - 3584p_3p_5 + 6480p_4^2) - 32768p_1(1257p_2^3p_3\\ &- 2448p_2^2p_5 + 3240p_2p_3p_4 + 64(5p_3^3 - 60p_3p_6 + 54p_4p_5)) - 98304(105p_2^5\\ &- 300p_2^3p_4 - 40p_2^2(p_3^2 - 24p_6) - 48p_2(16p_3p_5 + 15p_4^2) + 640p_3^2p_4).\end{aligned}$$

After rearrangement of terms, expression of $\boldsymbol{\chi}_3(p_i)$ is written as

$$\begin{aligned}\boldsymbol{\chi}_3(p_i) =& 1287528p_1^8\left(\frac{-139627}{1287528}p_1^2+p_2\right)+1536(18719)p_1^5p_3\left(-\frac{16603}{149752}p_1^2+p_2\right)\\ &+1024(24)(500)p_1^4\left(-\frac{121}{500}p_1p_5+p_6\right)-1024(24)(739)p_1^4p_3^2\\ &+12288(244)p_2p_5\left(\frac{1536}{61}p_3-p_1^3\right)+1024(25974)p_1^4p_2\left(-\frac{349}{702}p_2^2+p_4\right)\\ &+98304(104)p_2^2\left(\frac{3231}{832}p_1^2-p_2\right)+32768(64)(54)p_1p_4\left(-\frac{45}{64}p_1p_4-p_5\right)\\ &+32768(64)(60)p_1p_6\left(-\frac{5}{16}p_1p_2+p_3\right)+98304(40)(24)p_2^2\left(\frac{5}{16}p_2p_4-p_6\right)\\ &+98304(640)p_3^2\left(\frac{1}{16}p_2^2-p_4\right)+98304(48)(15)p_2p_4\left(-\frac{3}{2}p_1p_3+p_4\right)\\ &+32768(2448)p_1p_2^2\left(-\frac{419}{816}p_2p_3+p_5\right)-12288(2812)p_1^2p_2^2p_4\\ &+12288(8)(12271)p_1^2p_2p_3\left(-\frac{4905}{98168}p_1p_2+p_3\right)-384(4795)p_1^6p_2^2\\ &-12288(3584)p_1^2p_3p_5-32768(64)(5)p_1p_3^3-384(11674)p_1^6p_4.\end{aligned}$$

On applying triangle inequality and using Lemmas 2.3, 2.4 and the inequality $|p_n|\leq 2$ in above expression, we get

$$\begin{aligned}|\boldsymbol{\chi}_3(p_i)|\leq & 1287528.2^9+28752384.2^7+12288000.2^5+120628838.2^5\\ &+2998272.2^3\left(\frac{1536}{61}\sqrt{\frac{\frac{1536}{61}}{\frac{1536}{61}-1}}\right)+26597376.2^6\\ &+10223616.2^5\left(2\times\frac{3231}{832}-1\right)+1133246208.2^3\\ &+125829120.2^3+94371840.2^3+62914560.2^4+70778880.2^4\\ &+80216064.2^4+1841280.2^8+4482816.2^7+35291136.2^5\\ &+18161664.2^6+34553856.2^5+44040192.2^4+10485760.2^4\\ =& 28600671328+\frac{9663676416}{5}\sqrt{\frac{6}{59}}.\end{aligned}$$

Thus

$$|\Delta_1|\leq\frac{893770979}{4529848320}+\frac{1}{25}\sqrt{\frac{2}{177}}.$$

□

Lemma 2.8. *Let the function* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be in the class* $\mathcal{S}_L^*$. *Then*

$$\begin{aligned}|\Delta_2| =& |a_2(a_5a_7 - a_6^2) - a_3(a_4a_7 - a_5a_6) + a_4(a_4a_6 - a_5^2)| \\ \leq& \frac{33207941}{188743680} + \frac{1}{48\sqrt{59}} \approx 0.178654.\end{aligned}$$

Proof. On substituting the values of a_i $(i = 2, 3, 4, 5, 6, 7)$ from (5)–(10) in the expression of Δ_2, we get

$$2899102924800\Delta_2 = \boldsymbol{\chi}_4(p_i),$$

where

$$\begin{aligned}\boldsymbol{\chi}_4(p_i) =& 715273p_1^{11} - 7709336p_1^9p_2 + 13674816p_1^8p_3 + 384p_1^7(57911p_2^2 - 37200p_4) \\ &- 6144p_1^6(12839p_2p_3 + 14128p_5) - 1024p_1^5(2219p_2^3 - 237840p_2p_4 + 24(5900p_6 \\ &- 5613p_3^2)) - 24576p_1^4(5759p_2^2p_3 - 27112p_2p_5 + 2400p_3p_4) + 4096p_1^3(1933p_2^4 \\ &- 245760p_2^2p_4 + 120p_2(1360p_6 - 959p_3^2) + 144(125p_4^2 - 1228p_3p_5)) \\ &+ 131072p_1^2(7905p_2^3p_3 - 9696p_2^2p_5 + 8130p_2p_3p_4 + 4910p_3^3 - 9600p_3p_6 \\ &+ 9000p_4p_5) - 98304p_1(2045p_2^5 - 10240p_2^3p_4 + 40p_2^2(333p_3^2 - 40p_6) \\ &+ 240p_2(95p_4^2 - 92p_3p_5) + 96(25p_4(p_3^2 - 8p_6) + 192p_5^2)) - 1310720(15p_2^4p_3 \\ &+ 432p_2^3p_5 - 840p_2^2p_3p_4 + 8p_2(35p_3^3 + 120p_3p_6 - 108p_4p_5) \\ &+ 48p_3(15p_4^2 - 16p_3p_5)).\end{aligned}$$

After rearranging the terms and applying triangle inequality, above expression can be written as

$$\begin{aligned}|\boldsymbol{\chi}_4(p_i)| \leq& 7709336|p_1|^9\left|\frac{715273}{7709336}p_1^2 - p_2\right| + 14284800|p_1^7|\left|\frac{23741}{24800}p_1p_3 - p_4\right| \\ &+ 78882816|p_1|^6|p_2|\left|\frac{57911}{205424}p_1p_2 - p_3\right| + 141533184|p_1|^4|p_2|^2|p_3| \\ &+ 666304572|p_1|^4|p_5|\left|-\frac{883}{6778}p_1^2 + p_2\right| + 566231040|p_2|^3|p_5| \\ &+ 1065615360|p_1|^2|p_3||p_4|\left|-\frac{15}{271}p_1^2 + p_2\right| + 1258291200|p_2||p_3||p_6 \\ &+ 1006632960|p_2|^3|p_1|\left|-\frac{409}{2048}p_2^2 + p_4\right| + 73728000|p_4|^2\left|\frac{64}{5}p_3 - p_1^3\right| \\ &+ 1811939328|p_1||p_5|\left|\frac{115}{96}p_2p_3 - p_5\right| + 1006632960|p_3|^2\left|-\frac{35}{96}p_2p_3 + p5\right| \\ &+ 1132462080|p_1||p_4|\left|-\frac{95}{48}p_1p_4 + p_5\right| + 1101004800|p_2|^2|p_3|\left|-\frac{1}{56}p_2^2 + p_4\right|\end{aligned}$$

$$+ 1887436800|p_1||p_4|\left|-\frac{1}{8}p_3^2 + p_6\right| + 153286400|p_1||p_2|^2\left|-\frac{333}{40}p_3^2 + p_6\right|$$
$$+ 1270874112|p_1|^2|p_3|\left|\frac{491}{960}p_3^2 - p_6\right| + 1258291200|p_1|^2|p_3|\left|\frac{491}{960}p_3^2 - p_6\right|$$
$$+ 1179648000|p_1|^2|p_5|\left|-\frac{307}{500}p_1p_3 + p_4\right| + 668467200|p_1|^3|p_2|\left|\frac{959}{1360}p_3^2 - p_6\right|$$
$$+ 1006632960|p_1|^3|p_2|^2\left|\frac{1933}{245760}p_2^2 - p_4\right| + 144998400|p_1|^5\left|\frac{5613}{5900}p_3^2 - p_6\right|$$
$$+ 243548160|p_1|^5|p_2|\left|\frac{2219}{237840}p_2^2 - p_4\right|.$$

Proceeding similarly to Lemma 2.5, we obtain

$$|\boldsymbol{\chi}_4(p_i)| \leq 510073973760 + \frac{60397977600}{\sqrt{59}}.$$

Therefore

$$|\Delta_2| \leq \frac{33207941}{188743680} + \frac{1}{48\sqrt{59}},$$

which is the desired estimate. □

Lemma 2.9. *Let the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{S}_L^*$. Then*

$$|\Delta_3| = |a_3(a_5a_7 - a_6^2) - a_4(a_4a_7 - a_5a_6) + a_5(a_4a_6 - a_5^2)| \leq \frac{9329005817}{30198988800} \approx 0.308918.$$

Proof. On putting the values of a_i $(i = 2, 3, 4, 5, 6, 7)$ from (5)–(10) in expression of Δ_3, we get

$$69578470195200\Delta_3 = \boldsymbol{\chi}_5(p_i),$$

where

$$\begin{aligned}\boldsymbol{\chi}_5(p_i) =& - 667801p_1^{12} + 8788448p_1^{10}p_2 + 2434208p_1^9p_3 - 960p_1^8(54925p_2^2 - 108522p_4)\\ &- 3072p_1^7(68293p_2p_3 - 85112p_5) + 2048p_1^6(132250p_2^3 - 660210p_2p_4 + 6573p_3^2\\ &+ 123600p_6) + 12288p_1^5(147737p_2^2p_3 - 194576p_2p_5 + 73980p_3p_4)\\ &- 4096p_1^4(264943p_2^4 - 1423380p_2^2p_4 + 48p_2(8213p_3^2 + 8900p_6) - 24(12536p_3p5\\ &+ 2025p_4^2)) - 65536p_1^3(70843p_2^3p_3 - 115368p_2^2p_5 + 117780p_2p_3p_4 - 20(571p_3^3\\ &- 300p_3p_6 + 108p_4p_5)) + 393216p_1^2(5461p_2^5 - 25310p_2^3p_4 + 5p_2^2(4187p_3^2\\ &+ 2480p_6) + 4p_2(675p_4^2 - 5476p_3p_5) + 288(40p_3^2p_4 - 75p_4p_6 + 72p_5^2))\end{aligned}$$

$$+ 131072p_1(16845p_2^4p_3 - 63072p_2^3p_5 + 96360p_2^2p_3p_4 - 40p_2(1355p_3^3 - 2760p_3p_6 + 2376p_4p_5) + 240p_3(405p_4^2 - 448p_3p_5)) - 262144(4725p_2^6 - 22950p_2^4p_4 + 600p_2^3(29p_3^2 + 72p_6) - 1080p_2^2(96p_3p_5 + 5p_4^2) + 48p_2(25p_4(91p_3^2 - 72p_6) + 1728p_5^2) - 160(260p_3^4 - 480p_3^2p_6 + 864p_3p_4p_5 - 405p_4^3)).$$

After rearrangement of terms and on applying triangle inequality, above expression can be written as

$$\begin{aligned}
|\boldsymbol{\chi}_5(p_i)| \leq{} & 8788448|p_1|^{10}\left|\frac{667801}{8788448}p_1^2 - p_2\right| + 261464064|p_1|^7|p_5| + 600(2)|p_2|^2|p_3|^2 \\
&+ 138240|p_3||p_4|\,|92160p_1p_4 - p_5| + 14092861440|p_1||p_3|^2\left|\frac{1355}{2688}p_2p_3 - p_5\right| \\
&+ 76800|p_3|^2\left|\frac{13}{24}p_3^2 - p_6\right| + 1061683200|p_1|^2|p_2||p_4|\left|\frac{25310}{150}p_2^2 - p_4\right| \\
&+ 141557760|p_1|^3|p_4|\left|\frac{1963}{36}p_2p_3 - p_5\right| + 393216000|p_1|^3|p_3|\left|\frac{571}{300}p_3^2 - p_5\right| \\
&+ 8613003264|p_1|^2|p_2||p_3|\left|\frac{20935}{21904}p_2p_3 - p_5\right| + 8493465600|p_1|^2|p_4|\left|\frac{8}{5}p_3^2 - p_6\right| \\
&+ 209796096|p_1|^7|p_3|\left|\frac{76069}{6556128}p_1^2 - p_2\right| + 393216(5461)|p_1|^2|p_2|^5 \\
&+ 104181120|p_1|^8\left|\frac{54925}{108522}p_2^2 - p_4\right| + 393216(5)(2480)|p_1|^2|p_2|^2|p_6| \\
&+ 2390949888|p_1|^5|p_2|\left|\frac{147737}{194576}p_2p_3 - p_5\right| + 393216(288)(72)|p_1|^2|p_5|^2 \\
&+ 1085206528|p_1|^4|p_2|^3\left|\frac{66125}{264943}p_1^2 - p_2\right| + 600(72)|p_2|^3|p_1| \\
&+ 253132800|p_1|^6\left|\frac{22007}{4120}p_2p_4 - p_6\right| + 131072(40)(2760)|p_1||p_2||p_3||p_6| \\
&+ 1749811200|p_1|^4|p_2|\left|\frac{23723}{7120}p_2p_6 - p_6\right| + 64800|p_4|^2\left|\frac{1}{12}p_2^2 - p_4\right| \\
&+ 1232338944|p_1|^4|p_3|\left|\frac{8213}{6268}p_2p_3 - p_5\right| + 4096(24)(2025)|p_1|^4|p_4|^2 \\
&+ 8266973184|p_1||p_2|^3\left|\frac{16845}{63072}p_2p_3 - p_5\right| + 2048(6573)|p_1|^6|p_3|^2 \\
&+ 12457082880|p_2||p_1||p_4|\left|\frac{73}{72}p_2p_3 - p_5\right| + 909066240|p_1|^5|p_3||p_4|
\end{aligned}$$

$$+ 6016204800|p_2|^4 \left|\frac{7}{34}p_2^2 - p_4\right| + 82944|p_2||p_5| \left|\frac{5}{4}p_2p_3 - p_5\right|$$
$$+ 86400|p_2||p_4| \left|\frac{91}{72}p_3^2 - p_6\right| + 7560757248|p_1|^3|p_2|^2 \left|\frac{63713}{10488}p_2p_3 - p_5\right|.$$

As similar to Lemma 2.5, we get $|\boldsymbol{\chi}_5(p_i)| \leq 21494029402368$. Thus we conclude

$$|\Delta_3| \leq \frac{9329005817}{30198988800}.$$

□

Lemma 2.10. *Let the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{S}_L^*$. Then*

$$|\Delta_4| = |a_3(a_5a_8 - a_6a_7) - a_4(a_4a_8 - a_6^2) + a_5(a_4a_7 - a_5a_6)| \approx 0.421776.$$

Proof. We denote $\Theta_1 = a_5a_8 - a_6a_7$, $\Theta_2 = a_4a_8 - a_6^2$ and $\Theta_3 = a_4a_7 - a_5a_6$ so that $\Delta_4 = a_3\Theta_1 - a_4\Theta_2 + a_5\Theta_3$. To determine the estimate on Δ_4, we first compute the estimates on Θ_1, Θ_2 and Θ_3. Therefore, on substituting the values of a_i $(i = 5, 6, 7, 8)$ from (8)–(11) in the expression of Θ_1, we get

$$101468602368000\Theta_1 = \boldsymbol{\chi}_6(p_i),$$

where

$$\begin{aligned}\boldsymbol{\chi}_6(p_i) =&5(-49p_1^4 + 272p_1^2p_2 - 352p_1p_3 - 192(p_2^2 - 2p_4))(607537p_1^7 - 5077864p_1^5p_2\\ &+ 6101312p_1^4p_3 + 64p_1^3(198751p_2^2 - 116940p_4) + 512p_1^2(18168p_5 - 46901p_2p_3)\\ &- 512p_1(16371p_2^3 - 39780p_2p_4 + 1840(12p_6 - 11p_3^2)) + 24576(445p_2^2p_3\\ &- 516p_2p_5 - 530p_3p_4 + 480p_7)) - 7(543p_1^5 - 3568p_1^3p_2 + 4608p_1^2p_3\\ &+ 64p_1(77p_2^2 - 90p_4) - 6400p_2p_3 + 6144p_5)(-32303p_1^6 + 241688p_1^4p_2\\ &- 301888p_1^3p_3 + 64p_1^2(5940p_4 - 7457p_2^2) + 9728p_1(85p_2p_3 - 48p_5)\\ &+ 2560(57p_2^3 - 204p_2p_4 - 104p_3^2 + 192p_6)).\end{aligned}$$

On rearrangement of terms, above expression becomes

$$\begin{aligned}\boldsymbol{\chi}_6(p_i) =&344871184p_1^9\left(-\frac{1861633}{24633656}p_1^2 + p_2\right) + 3415957504p_1^5p_2^2\left(-\frac{3204693}{6671792}p_1^2 + p_2\right)\\ &+ 3013017600p_1p_2^4\left(-\frac{427621}{367800}p_1^2 + p_2\right) + 3693189120p_1^5p_4\left(\frac{65793}{961768}p_1^2 - p_2\right)\\ &+ 11747328000p_1p_2^2p_4\left(\frac{56587}{47800}p_1^2 - p_2\right) + 3643269120p_1^4p_3\left(-\frac{256114}{148245}p_2^2 + p_4\right)\\ &+ 10105651200p_2^2p_3\left(-\frac{403}{1028}p_2^2 + p_4\right) + 22649241600p_7\left(-\frac{11}{12}p_1p_3 + p_4\right)\\ &+ 16043212800p_1^2p_7\left(-\frac{49}{272}p_1^2 + p_2\right) + 25008537600p_3p_4\left(\frac{68}{53}p_1p_3 - p_4\right)\end{aligned}$$

$$
\begin{aligned}
&+ 34162606080p_1p_3p_5\left(\frac{1237}{3620}p_1^2 - p_2\right) + 10456793088p_1^2p_2p_5\left(-\frac{128159}{212744}p_1^2 + p_2\right) \\
&+ 9650831360p_1^2p_3^2\left(\frac{29025}{29452}p_1p_2 - p_3\right) + 3192098304p_1^6p_3\left(-\frac{3902549}{33251024}p_1^2 + p_2\right) \\
&+ 11450449920p_3^2\left(-\frac{5}{48}p_2p_3 + p_5\right) + 901447680p_1^5\left(-\frac{58769}{20960}p_3^2 + p_6\right) \\
&+ 17032151040p_1p_2^2p_3\left(-\frac{3933}{34652}p_1p_2 + p_3\right) + 885147648.p_1^6p_5 \\
&+ 17317232640p_1^2p_4\left(\frac{325}{5872}p_1p_4 - p_5\right) + 1887436800p_2p_4\left(\frac{153}{16}p_1p_4 - p_5\right) \\
&+ 11324620800p_2^2\left(\frac{25}{48}p_2p_5 - p_7\right) + 21139292160p_5\left(\frac{19}{20}p_1p_5 - p_6\right) \\
&+ 1887436800p_1p_6\left(-\frac{97}{30}p_2^2 + p_4\right) + 22020096000p_2p_6\left(-\frac{197}{1400}p_1^3 + p_3\right) \\
&+ 4042260480p_1^2p_3\left(-\frac{16505}{2056}p_2p_4 + p_6\right).
\end{aligned}
$$

As similar to Lemma 2.5, we have

$$|\boldsymbol{\chi}_6(p_i)| \leq 8391385300992 + 587202560000\sqrt{\frac{42}{401}}.$$

Therefore, we obtain

$$|\Theta_1| \leq \frac{170723171}{206438400} + \frac{25}{72}\sqrt{\frac{7}{2406}} \approx 0.845722. \tag{17}$$

Next, on substituting the values of a_4, a_6 and a_8 from (7), (9) and (11) respectively in the expression of Θ_2, we get

$$126835729600\Theta_2 = \boldsymbol{\chi}_7(p_i),$$

where

$$
\begin{aligned}
\boldsymbol{\chi}_7(p_i) =& 14722589p_1^{10} - 174684288p_1^8p_2 + 729681792p_1^6p_2^2 - 1152483328p_1^4p_2^3 \\
&+ 306991104p_1^2p_2^4 + 170636928p_1^7p_3 + 3165388800p_1^2p_2p_6 - 1548169728p_1^5p_2p_3 \\
&+ 3853713408p_1^3p_2^2p_3 + 466771200p_1^3p_7 - 445808640p_1p_2^3p_3 - 3303014400p_1p_2p_7 \\
&+ 842379264p_1^4p_3^2 - 5631344640p_1^2p_2p_3^2 + 58982400p_2^2p_3^2 + 3316121600p_1p_3^3 \\
&- 33239040p_1^4p_2p_4 - 934133760p_1^2p_2^2p_4 + 1217949040p_1^3p_3p_4 + 3971481600p_1p_2p_3p_4
\end{aligned}
$$

$$
\begin{aligned}
&- 4168089600p_3^2p_4 - 2786918400p_1^2p_4^2 + 44150784p_1^5p_5 + 254017536p_1^3p_2p_5 \\
&- 1779695616p_1^2p_3p_5 + 2548039680p_2p_3p_5 + 5945425920p_1p_4p_5 - 3170893824p_5^2 \\
&- 3617587200p_1p_3p_6 - 734822400p_1^4p_6 - 1535901696p_1p_2^2p_5 + 38979840p_1^6p_4 \\
&+ 3774873600p_3p_7.
\end{aligned}
$$

As similar to previous part, we get $|\boldsymbol{\chi}_7(p_i)| \leq 1037443956735$ which implies

$$|\Theta_2| \leq \frac{31660277}{38707200} \approx 0.817943.$$

Using (7), (8), (9) and (10), we have the following expression for Θ_3 as

$$9059696640\Theta_3 = \boldsymbol{\chi}_8(p_i),$$

where

$$
\begin{aligned}
\boldsymbol{\chi}_8(p_i) = &- 100655p_1^9 + 1080576p_1^7p_2 - 3944064p_1^5p_2^2 + 4317184p_1^3p_2^3 \\
&- 988800p_1^6p_3 + 9248256p_1^4p_2p_3 - 24526848p_1^2p_2^2p_3 - 5406720p_2^3p_3 \\
&+ 40796160p_1p_2p_3^2 + 6389760p_1^3p_6 - 17039360p_3^3 - 27525120p_1p_2p_6 \\
&- 946944p_1^5p_4 + 7163904p_1^3p_2p_4 + 31457280p_3p_6 - 6733824p_1p_2^2p_4 \\
&- 3932160p_2p_3p_4 + 26542080p_1p_4^2 - 2457600p_1^4p_5 + 6094848p_1^2p_2p_5 \\
&+ 14155776p_2^2p_5 - 3932160p_1p_3p_5 - 28311552p_4p_5 - 3317760p_1^3p_3^2 \\
&- 21233664p_1^2p_3p_4 + 3182592p_1p_2^4.
\end{aligned}
$$

As similar to calculation done for Θ_1, we get

$$|\Theta_3| \leq \frac{10915}{18432} \approx 0.592177.$$

Therefore using (12),(13) and the estimates on Θ_1, Θ_2, Θ_3, we have

$$
\begin{aligned}
|\Delta_4| &\leq |a_3||\Theta_1| + |a_4|\Theta_2| + |a_5||\Theta_3| \\
&\leq \frac{1}{4}(0.845722) + \frac{1}{6}(0.817943) + \frac{1}{8}(0.592177) \approx 0.421776.
\end{aligned}
$$

□

Lemma 2.11. *Let the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{S}_L^*$. Then*

$$|\Delta_5| = |a_3(a_6a_8 - a_7^2) - a_4(a_5a_8 - a_6a_7) + a_5(a_5a_7 - a_6^2)| \leq 0.693354.$$

Proof. We denote $\Theta_4 = a_6a_8 - a_7^2$ and $\Theta_5 = a_5a_7 - a_6^2$ so that $\Delta_5 = a_3\Theta_4 - a_4\Theta_1 + a_5\Theta_5$. To determine the estimate on Δ_5, we first compute the estimates on Θ_4 and Θ_5. Therefore, on substituting the values of a_i $(i = 5, 6, 7, 8)$ from (8)–(11) in the expressions of Θ_4 and Θ_5, we get

$$
\begin{aligned}
974098582732800\Theta_4 &= \boldsymbol{\chi}_9(p_i), \\
362387865600\Theta_5 &= \boldsymbol{\chi}_{10}(p_i),
\end{aligned}
$$

where

$$\begin{aligned}\boldsymbol{\chi}_9(p_i) =&24(543p_1^5-3568p_1^3p_2+4608p_1^2p_3+64p_1(77p_2^2-90p_4)-6400p_2p_3\\&+6144p_5)(607537p_1^7-5077864p_1^5p_2+6101312p_1^4p_3+64p_1^3(198751p_2^2\\&-116940p_4)+512p_1^2(18168p_5-46901p_2p_3)-512p_1(16371p_2^3-39780p_2p_4\\&+1840(12p_6-11p_3^2))+24576(445p_2^2p_3-516p_2p_5-530p_3p_4+480p_7))\\&-7(-32303p_1^6+241688p_1^4p_2-301888p_1^3p_3+64p_1^2(5940p_4-7457p_2^2)\\&+9728p_1(85p_2p_3-48p_5)+2560(57p_2^3-204p_2p_4-104p_3^2+192p_6))^2\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{\chi}_{10}(p_i) =&5(-49p_1^4+272p_1^2p_2-352p_1p_3-192(p_2^2-2p_4))(-32303p_1^6+241688p_1^4p_2\\&-301888p_1^3p_3+64p_1^2(5940p_4-7457p_2^2)+9728p_1(85p_2p_3-48p_5)\\&+2560(57p_2^3-204p_2p_4-104p_3^2+192p_6))-24(543p_1^5-3568p_1^3p_2\\&+4608p_1^2p_3+64p_1(77p_2^2-90p_4)-6400p_2p_3+6144p_5)^2.\end{aligned}$$

As similar to the proof of Lemma 2.10, we get

$$|\Theta_4| \leq 1.93683 \quad \text{and} \quad |\Theta_5| \leq 0.405893. \tag{18}$$

Therefore, in view of (12), (13) and the estimates on $|\Theta_1|$, $|\Theta_4|$ and $|\Theta_5|$ from (17) and (18) respectively, we have

$$\begin{aligned}|\Delta_5| &\leq |a_3||\Theta_4| + |a_4||\Theta_1| + |a_5||\Theta_5|\\&\leq \frac{1.93683}{4}+\frac{0.845722}{6}+\frac{0.405893}{8} \approx 0.675898.\end{aligned}$$

□

Lemma 2.12. *Let the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{S}_L^*$. Then*

$$|\Delta_6| = |a_4(a_6a_8 - a_7^2) - a_5(a_5a_8 - a_6a_7) + a_6(a_5a_7 - a_6^2)| \leq 0.476074.$$

Proof. Using triangle inequality, we have

$$\begin{aligned}|\Delta_6| &\leq |a_4||a_6a_8 - a_7^2| + |a_5||a_5a_8 - a_6a_7| + |a_6||a_5a_7 - a_6^2|\\&\leq |a_4||\Theta_4| + |a_5||\Theta_1| + |a_6||\Theta_5|.\end{aligned}$$

As similar to previous lemma, we get the desired estimate on $|\Delta_6|$. □

Proof of Theorem 2.1. From the definition (2), the Hankel determinant of fourth order $H_4^{(2)}(f)$ is given by

$$H_4^{(2)}(f) = a_8H_3^{(2)} - a_7\Delta_1 + a_6\Delta_2 - a_5\Delta_3$$

so that

$$|H_4^{(2)}(f)| \leq |a_8|\,|H_3^{(2)}(f)| + |a_7|\,|\Delta_1| + |a_6|\,|\Delta_2| + |a_5|\,|\Delta_3|.$$

In view of relevant estimates from (13), (14) and Lemmas 2.5, 2.7, 2.8, 2.9, we have

$$|H_4^{(2)}(f)| \leq \left(\frac{2-\sqrt{2}}{7}\right)(0.596806) + \left(\frac{2-\sqrt{2}}{6}\right)(0.201559) + \left(\frac{2-\sqrt{2}}{5}\right)(0.178654) + \frac{1}{8}(0.308918) \approx 0.129167.$$

□

Proof of Theorem 2.2. From the definition (2), the Hankel determinant of fourth order $H_4^{(3)}(f)$ is given by

$$H_4^{(3)}(f) = a_9 H_3^{(3)} - a_8\Delta_4 + a_7\Delta_5 - a_6\Delta_6.$$

So

$$|H_4^{(3)}(f)| \leq |a_9|\,|H_3^{(3)}(f)| + |a_8|\,|\Delta_4| + |a_7|\,|\Delta_5| + |a_6|\,|\Delta_6|.$$

In view of relevant estimates from (14) and Lemmas 2.6, 2.10, 2.11, 2.12, we have

$$|H_4^{(3)}(f)| \leq \left(\frac{2-\sqrt{2}}{8}\right)(0.093842) + \left(\frac{2-\sqrt{2}}{7}\right)(0.421776) + \left(\frac{2-\sqrt{2}}{6}\right)(0.675898) + \frac{2-\sqrt{2}}{5}(0.476078) \approx 0.163932.$$

□

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