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FRACTIONAL TRAPEZOID AND NEWTON TYPE INEQUALITIES FOR DIFFERENTIABLE S-CONVEX FUNCTIONS

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Abstract. In the present paper, we prove that our main inequality reduces to some trapezoid and Newton type inequalities for differentiable s-convex functions. These inequalities are established by using the well-known Riemann-Liouville fractional integrals. With the help of special cases of our main results, we also present some new and previously obtained trapezoid and Newton type inequalities.

1. Introduction

The inequality theory is a popular topic in many mathematical areas and remains an interesting research area with a great deal applications. One of the most famous inequalities for the case of convex functions is Hermite–Hadamard-type inequality because of its rich geometrical importance and applications. Therefore, Hermite–Hadamard-type inequalities and related these inequalities such as trapezoid, midpoint, and Simpson's inequality have been investigated by considerable number of mathematicians. In addition, the convex functions still employ a central role in the theory of inequalities since the convex functions develop a series of inequalities. Another significant result concerned with convex function is the Hermite–Hadamard inequality. Numerous mathematicians have studied to the Hermite–Hadamard inequality with great interest to generalise, refine, and extend it to the case of different classes of functions such as quasi-convex functions, log-convex, s-convex functions, etc.

Fractional calculus has been the focus of attraction for researchers in mathematical sciences owing to its fundamental definitions, properties, and applications in tackling real-life problems. Because of the importance of fractional

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calculus, mathematicians have investigated different fractional integral inequalities. It can be obtained the bounds of new formulas by using the Hermite-Hadamard and Simpson's type inequalities.

İşcan and Wu established Hermite-Hadamard type inequalities with the aid of harmonic convexity in [20]. By using the k-fractional integrals, the authors established some Ostrowski's type inequalities to the case of differentiable mappings in the paper [12]. In the paper [21], Khan et al. investigated some new estimates of Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals and conformable fractional integrals. With the aid of fractional integrals, Tunc [32] presented some new variants of Hermite-Hadamard inequalities for h-convex functions.

Sarikaya and Ertugral [30] established a new class of fractional integrals, which is known generalized fractional, and they used these integrals to prove the general version of Hermite-Hadamard type inequalities for convex functions. Zhao et al. proved Hermite-Hadamard type inequalities for interval-valued functions by using generalized fractional integrals in the paper [35]. It is referred the reader to [1, 2, 3, 34, 4, 21, 32] for a better understanding of fractional integral inequalities.

Simpson's inequalities are inequalities that are created from Simpson's rules. Simpson's first rule namely the rule of Simpson's quadrature formula are established by many mathematicians. For instance, some of Simpson's type inequalities with the aid of the Riemann-Liouville fractional integrals are established in the paper [28]. Moreover, it is investigated to several fractional Simpson type inequalities for functions whose second derivatives in absolute value are convex in the paper [15]. It can be referred to [7, 8, 33, 26] and the references therein for further significant information about Simpson type inequalities and some properties of Riemann–Liouville fractional integrals and various fractional integral operators.

Simpson's second rule comprises the rule of three-point Newton-Cotes quadrature, so evaluations based on three steps quadratic kernel are called Newton-type results. The inequalities obtained with this Newton-type result are known as Newton-type inequalities in the literature. A lot of research has been done on this inequality, which attracts the attention of mathematicians. For instance, some Newton's type inequalities for differentiable convex functions are proved by the aid of the Riemann-Liouville fractional integrals in the paper [31]. In addition, the authors also present some inequalities of Riemann-Liouville fractional Newton's type for functions of bounded variation. Some new inequalities of Newton's type based on convexity are given and some applications for special cases of real functions are also presented in the paper [13]. Moreover, some new integral inequalities of Newton's type for functions whose first derivative in absolute value at certain power are arithmetically-harmonically convex are obtained in the paper [11]. We refer the reader to [18, 19, 24, 25] for more pieces of significant information and unexplained subjects about Newton's type of inequalities including convex differentiable functions.

With the aid of the ongoing studies and mentioned papers above, we will establish trapezoid type and Newton's formula type inequalities for the case of differentiable s-convex functions by Riemann-Liouville fractional integrals. In Sect. 2, the fundamental definitions of fractional calculus and other relevant research in this discipline will be presented. In Sect. 3, we will prove an integral equality which is critical in establishing the primary results of the given paper. Moreover, it will be proved some new trapezoid and Newton type inequalities for differentiable s-convex functions with the aid of the Riemann-Liouville fractional integrals. In addition, we also give some new and previously obtained trapezoid and Newton type inequalities with the help of special cases of our main results. In Sect. 4, we will give some ideas for future work on this subject.

2. Preliminaries

In this section, we will present the basic definitions and facts about the fractional integral notations and different fractional integrals are also presented to recall various inequalities.

Definition 2.1. *A function $\mathscr{F}:[0,\infty)\to[0,\infty)$ will be called $s-$convex in the second sense if the following inequality*

$$\mathscr{F}\left(tx+(1-t)\,y\right)\leq t^{s}\mathscr{F}\left(x\right)+\left(1-t\right)^{s}\mathscr{F}\left(y\right)$$

is valid for all $x,y\in[0,\infty)$ with $t\in[0,1]$ and for some fixed $s\in(0,1]$.

The s-convex function was introduced in the paper [6] and some properties and connections with s-convexity in the first sense were established in paper [17]. Moreover, it can be easily seen that for $s=1$, s-convexity reduces to the ordinary convexity of functions defined on $[0,\infty)$.

In the paper [10], Dragomir and Fitzpatrick established a variant of Hermite-Hadamard inequality which is valid for the case of the s-convex functions.

Theorem 2.2. [10] *Let $\mathscr{F}:[0,\infty)\to[0,\infty)$ be a s-convex function in the second sense, where $s\in[0,1)$ and let $\sigma,\delta\in[0,\infty),\sigma<\delta$. If $\mathscr{F}\in L_1[0,1]$, then the following inequalities hold:*

$$2^{s-1}\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\leq\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathscr{F}\left(x\right)dx\leq\frac{\mathscr{F}\left(\sigma\right)+\mathscr{F}\left(\delta\right)}{s+1}. \tag{1}$$

For recent results and generalizations concerning s-convex functions see [5, 23, 10].

Remark 2.3. [27] *If we assign $s=1$ in inequalities (1), then it can be obtained the classical Hermite-Hadamard inequality:*

$$\mathscr{F}\left(\frac{\sigma+\delta}{2}\right)\leq\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathscr{F}\left(x\right)dx\leq\frac{\mathscr{F}\left(\sigma\right)+\mathscr{F}\left(\delta\right)}{2}.$$

From the fact of fractional calculus theory, mathematical preliminaries will be given as follows:

Definition 2.4. [22] *The well-known gamma function and incomplete beta function are defined*

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt$$

and

$$\beta(x, y, r) := \int_0^r t^{x-1} (1-t)^{y-1} dt,$$

respectively for $0 < x, y < \infty$ *and* $x, y \in \mathbb{R}$.

Definition 2.5. [14, 22] *Let us note that* $\mathscr{F}$ *belongs to* $L_1[\sigma, \delta]$. *The Riemann–Liouville integrals* $J^{\alpha}_{\sigma+}\mathscr{F}$ *and* $J^{\alpha}_{\delta-}\mathscr{F}$ *of order* $\alpha > 0$ *with* $\sigma \geq 0$ *equal to*

$$J^{\alpha}_{\sigma+}\mathscr{F}(x) = \frac{1}{\Gamma(\alpha)} \int_{\sigma}^{x} (x-t)^{\alpha-1} \mathscr{F}(t) dt, \quad x > \sigma$$

and

$$J^{\alpha}_{\delta-}\mathscr{F}(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\delta} (t-x)^{\alpha-1} \mathscr{F}(t) dt, \quad x < \delta,$$

respectively.

Remark 2.6. *If it is chosen* $\alpha = 1$ *in Definition 2.5, then the fractional integral reduces to the classical integral.*

3. Main results of parameterized inequalities

Lemma 3.1. *Let* $\mathscr{F} : [\sigma, \delta] \to \mathbb{R}$ *be an absolutely continuous function on* (σ, δ) *so that* $\mathscr{F}' \in L_1[\sigma, \delta]$, $\alpha > 0$, *and* $\lambda \geq 0$. *Then, the following equality holds:*

$$\begin{aligned} (2) \qquad & \lambda \mathscr{F}(\sigma) + \frac{1-2\lambda}{2} \left[\mathscr{F}\left(\frac{\sigma+2\delta}{3}\right) + \mathscr{F}\left(\frac{2\sigma+\delta}{3}\right) \right] + \lambda \mathscr{F}(\delta) \\ & - \frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^{\alpha}} \left[J^{\alpha}_{\sigma+}\mathscr{F}(\delta) + J^{\alpha}_{\delta-}\mathscr{F}(\sigma) \right] \\ & = \frac{\delta-\sigma}{2} \left[I_1 + I_2 + I_3 \right]. \end{aligned}$$

Here,

$$\begin{cases} I_1 = \int\limits_0^{\frac{1}{3}} (t^\alpha - \lambda) \left[\mathscr{F}'(t\delta + (1-t)\sigma) - \mathscr{F}'(t\sigma + (1-t)\delta)\right] dt, \\ I_2 = \int\limits_{\frac{1}{3}}^{\frac{2}{3}} \left(t^\alpha - \frac{1}{2}\right) \left[\mathscr{F}'(t\delta + (1-t)\sigma) - \mathscr{F}'(t\sigma + (1-t)\delta)\right] dt, \\ I_3 = \int\limits_{\frac{2}{3}}^{1} (t^\alpha - (1-\lambda)) \left[\mathscr{F}'(t\delta + (1-t)\sigma) - \mathscr{F}'(t\sigma + (1-t)\delta)\right] dt. \end{cases}$$

Proof. Using the facts of the fundamental rules of integration by parts, we derive

(3)

$$\begin{aligned} I_1 &= \int\limits_0^{\frac{1}{3}} (t^\alpha - \lambda) \left[\mathscr{F}'(t\delta + (1-t)\sigma) - \mathscr{F}'(t\sigma + (1-t)\delta)\right] dt \\ &= \frac{1}{\delta - \sigma} (t^\alpha - \lambda) \mathscr{F}(t\delta + (1-t)\sigma)\Big|_0^{\frac{1}{3}} - \frac{\alpha}{\delta - \sigma} \int\limits_0^{\frac{1}{3}} t^{\alpha-1} \mathscr{F}(t\delta + (1-t)\sigma) dt \\ &\quad + \frac{1}{\delta - \sigma} (t^\alpha - \lambda) \mathscr{F}(t\sigma + (1-t)\delta)\Big|_0^{\frac{1}{3}} - \frac{\alpha}{\delta - \sigma} \int\limits_0^{\frac{1}{3}} t^{\alpha-1} \mathscr{F}(t\sigma + (1-t)\delta) dt \\ &= \frac{1}{\delta - \sigma} \left[\lambda \mathscr{F}(\sigma) + \left(\left(\frac{1}{3}\right)^\alpha - \lambda\right) \left[\mathscr{F}\left(\frac{\sigma + 2\delta}{3}\right) + \mathscr{F}\left(\frac{2\sigma + \delta}{3}\right)\right] + \lambda \mathscr{F}(\delta)\right] \\ &\quad - \frac{\alpha}{\delta - \sigma} \int\limits_0^{\frac{1}{3}} t^{\alpha-1} \mathscr{F}(t\delta + (1-t)\sigma) dt - \frac{\alpha}{\delta - \sigma} \int\limits_0^{\frac{1}{3}} t^{\alpha-1} \mathscr{F}(t\sigma + (1-t)\delta) dt. \end{aligned}$$

Similarly, we obtain

(4)

$$\begin{aligned} I_2 &= \frac{1}{\delta - \sigma} \left(\left(\frac{2}{3}\right)^\alpha - \left(\frac{1}{3}\right)^\alpha\right) \left[\mathscr{F}\left(\frac{\sigma + 2\delta}{3}\right) + \mathscr{F}\left(\frac{2\sigma + \delta}{3}\right)\right] \\ &\quad - \frac{\alpha}{\delta - \sigma} \int\limits_{\frac{1}{3}}^{\frac{2}{3}} t^{\alpha-1} \mathscr{F}(t\delta + (1-t)\sigma) dt - \frac{\alpha}{\delta - \sigma} \int\limits_{\frac{1}{3}}^{\frac{2}{3}} t^{\alpha-1} \mathscr{F}(t\sigma + (1-t)\delta) dt \end{aligned}$$

and

(5)

$$
\begin{aligned}
I_3 = &\frac{1}{\delta-\sigma}\left[\lambda\mathscr{F}(\sigma)-\left(\left(\frac{2}{3}\right)^{\alpha}-(1-\lambda)\right)\right.\\
&\times\left.\left[\mathscr{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathscr{F}\left(\frac{2\sigma+\delta}{3}\right)\right]+\lambda\mathscr{F}(\delta)\right]\\
&-\frac{\alpha}{\delta-\sigma}\int_{\frac{2}{3}}^{1}t^{\alpha-1}\mathscr{F}(t\delta+(1-t)\sigma)\,dt-\frac{\alpha}{\delta-\sigma}\int_{\frac{2}{3}}^{1}t^{\alpha-1}\mathscr{F}(t\sigma+(1-t)\delta)\,dt.
\end{aligned}
$$

If we add equalities (3)–(5), then we get

(6)

$$
\begin{aligned}
&I_1+I_2+I_3\\
&=\frac{1}{\delta-\sigma}\left[2\lambda\mathscr{F}(\sigma)+(1-2\lambda)\left[\mathscr{F}\left(\frac{2\sigma+\delta}{3}\right)+\mathscr{F}\left(\frac{\sigma+2\delta}{3}\right)\right]+2\lambda\mathscr{F}(\delta)\right]\\
&\quad-\frac{\alpha}{\delta-\sigma}\left[\int_0^1 t^{\alpha-1}\mathscr{F}(t\delta+(1-t)\sigma)\,dt+\int_0^1 t^{\alpha-1}\mathscr{F}(t\sigma+(1-t)\delta)\,dt\right].
\end{aligned}
$$

By using the equality (6) and with the help of the change of the variable $x=t\delta+(1-t)\sigma$ and $x=t\sigma+(1-t)\delta$ for $t\in[0,1]$ respectively, it can be rewritten as follows

(7)

$$
\begin{aligned}
&I_1+I_2+I_3\\
&=\frac{1}{\delta-\sigma}\left[2\lambda\mathscr{F}(\sigma)+(1-2\lambda)\left[\mathscr{F}\left(\frac{2\sigma+\delta}{3}\right)+\mathscr{F}\left(\frac{\sigma+2\delta}{3}\right)\right]+2\lambda\mathscr{F}(\delta)\right]\\
&\quad-\frac{\Gamma(\alpha+1)}{(\delta-\sigma)^{\alpha+1}}\left[J_{\sigma+}^{\alpha}\mathscr{F}(\delta)+J_{\delta-}^{\alpha}\mathscr{F}(\sigma)\right].
\end{aligned}
$$

Multiplying the both sides of (7) by $\frac{\delta-\sigma}{2}$, the equality (2) is obtained. This is the end of proof of Lemma 3.1. □

Remark 3.2. *If we choose $\lambda=\frac{1}{2}$ in Lemma 3.1, then Lemma 3.1 equals to*

$$
\begin{aligned}
&\frac{\mathscr{F}(\sigma)+\mathscr{F}(\delta)}{2}-\frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^{\alpha}}\left[J_{\sigma+}^{\alpha}\mathscr{F}(\delta)+J_{\delta-}^{\alpha}\mathscr{F}(\sigma)\right]\\
&=\frac{\delta-\sigma}{2}\int_0^1\left(t^{\alpha}-\frac{1}{2}\right)\left[\mathscr{F}'(t\delta+(1-t)\sigma)-\mathscr{F}'(t\sigma+(1-t)\delta)\right]dt
\end{aligned}
$$

$$= \frac{\delta-\sigma}{2}\int_0^1 \left((1-t)^\alpha - t^\alpha\right)\mathscr{F}'\left(t\sigma+(1-t)\delta\right)dt,$$

which is established by [29, Lemma 2].

Remark 3.3. *Let us consider $\alpha = 1$ in Remark 3.2. Then, Remark 3.2 reduces to* [9, Lemma 2.1].

Remark 3.4. *If we take $\lambda = \frac{1}{8}$ in Lemma 3.1, then Lemma 3.1 is equal to*

$$\begin{aligned}
&\frac{1}{8}\left[\mathscr{F}(\sigma)+3\left[\mathscr{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathscr{F}\left(\frac{2\sigma+\delta}{3}\right)\right]+\mathscr{F}(\delta)\right] \\
&\quad -\frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^\alpha}\left[J_{\sigma+}^\alpha\mathscr{F}(\delta)+J_{\delta-}^\alpha\mathscr{F}(\sigma)\right] \\
&= \frac{\delta-\sigma}{2}\left[I_1+I_2+I_3\right],
\end{aligned}$$

which is given by [16, Lemma 1]. *Here,*

$$\begin{cases}
I_1 = \int_0^{\frac{1}{3}}\left(t^\alpha-\frac{1}{8}\right)\left[\mathscr{F}'(t\delta+(1-t)\sigma)-\mathscr{F}'(t\sigma+(1-t)\delta)\right]dt, \\
I_2 = \int_{\frac{1}{3}}^{\frac{2}{3}}\left(t^\alpha-\frac{1}{2}\right)\left[\mathscr{F}'(t\delta+(1-t)\sigma)-\mathscr{F}'(t\sigma+(1-t)\delta)\right]dt, \\
I_3 = \int_{\frac{2}{3}}^{1}\left(t^\alpha-\frac{7}{8}\right)\left[\mathscr{F}'(t\delta+(1-t)\sigma)-\mathscr{F}'(t\sigma+(1-t)\delta)\right]dt.
\end{cases}$$

Remark 3.5. *Let us consider $\alpha = 1$ in Remark 3.4. Then, Remark 3.4 reduces to Erden et al.* [11, Lemma 1].

Corollary 3.6. *For $\lambda = 0$, Lemma 3.1 equals to*

$$\begin{aligned}
&\frac{1}{2}\left[\mathscr{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathscr{F}\left(\frac{2\sigma+\delta}{3}\right)\right]-\frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^\alpha}\left[J_{\sigma+}^\alpha\mathscr{F}(\delta)+J_{\delta-}^\alpha\mathscr{F}(\sigma)\right] \\
&= \frac{(\delta-\sigma)}{2}\left[I_1+I_2+I_3\right],
\end{aligned}$$

where

$$\begin{cases}
I_1 = \int_0^{\frac{1}{3}} t^\alpha\left[\mathscr{F}'(t\delta+(1-t)\sigma)-\mathscr{F}'(t\sigma+(1-t)\delta)\right]dt, \\
I_2 = \int_{\frac{1}{3}}^{\frac{2}{3}}\left(t^\alpha-\frac{1}{2}\right)\left[\mathscr{F}'(t\delta+(1-t)\sigma)-\mathscr{F}'(t\sigma+(1-t)\delta)\right]dt, \\
I_3 = \int_{\frac{2}{3}}^{1}\left(t^\alpha-1\right)\left[\mathscr{F}'(t\delta+(1-t)\sigma)-\mathscr{F}'(t\sigma+(1-t)\delta)\right]dt.
\end{cases}$$

Theorem 3.7. *Assume that the assumptions of Lemma 3.1 hold. Assume also that the function $|\mathscr{F}'|$ is s-convex on $[\sigma,\delta]$. Then, the following inequality holds:*

$$\begin{aligned}&\left|\lambda\mathscr{F}(\sigma)+\frac{1-2\lambda}{2}\left[\mathscr{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathscr{F}\left(\frac{2\sigma+\delta}{3}\right)\right]+\lambda\mathscr{F}(\delta)\right.\\&\quad\left.-\frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^{\alpha}}\left[J_{\sigma+}^{\alpha}\mathscr{F}(\delta)+J_{\delta-}^{\alpha}\mathscr{F}(\sigma)\right]\right|\\&\leq\frac{\delta-\sigma}{2}\left(A_1^{\alpha}(s,\lambda)+A_2^{\alpha}(s,\lambda)+A_3^{\alpha}(s)+A_4^{\alpha}(s)+A_5^{\alpha}(s,\lambda)+A_6^{\alpha}(s,\lambda)\right)\\&\quad\times\left[|\mathscr{F}'(\sigma)|+|\mathscr{F}'(\delta)|\right].\end{aligned}$$

Here,

$$\begin{aligned}A_1^{\alpha}(s,\lambda)&=\int_0^{\frac{1}{3}}t^s\left|t^{\alpha}-\lambda\right|dt\\&=\begin{cases}\frac{1}{s+\alpha+1}\left[\frac{2\alpha\lambda^{\frac{s+\alpha+1}{\alpha}}}{s+1}+\left(\frac{1}{3}\right)^{s+\alpha+1}\right]-\frac{\lambda}{s+1}\left(\frac{1}{3}\right)^{s+1}, & 0\leq\lambda^{\frac{1}{\alpha}}<\frac{1}{3},\\ \frac{\lambda}{s+1}\left(\frac{1}{3}\right)^{s+1}-\frac{1}{s+\alpha+1}\left(\frac{1}{3}\right)^{s+\alpha+1}, & \lambda^{\frac{1}{\alpha}}\geq\frac{1}{3},\end{cases}\end{aligned}$$

$$\begin{aligned}A_2^{\alpha}(s,\lambda)&=\int_0^{\frac{1}{3}}(1-t)^s\left|t^{\alpha}-\lambda\right|dt\\&=\begin{cases}\frac{\lambda}{s+1}\left(1+\left(\frac{2}{3}\right)^{s+1}-2\left(1-\lambda^{\frac{1}{\alpha}}\right)^{\alpha+1}\right)\\ +\beta\left(\alpha+1,s+1,\frac{1}{3}\right)\\ -2\beta\left(\alpha+1,s+1,\lambda^{\frac{1}{\alpha}}\right), & 0\leq\lambda^{\frac{1}{\alpha}}<\frac{1}{3},\\ \frac{\lambda}{s+1}\left(1-\left(\frac{2}{3}\right)^{s+1}\right)-\beta\left(\alpha+1,s+1,\frac{1}{3}\right), & \lambda^{\frac{1}{\alpha}}\geq\frac{1}{3},\end{cases}\end{aligned}$$

$$A_3^{\alpha}(s)=\int_{\frac{1}{3}}^{\frac{2}{3}}t^s\left|t^{\alpha}-\frac{1}{2}\right|dt$$

$$= \begin{cases} \frac{1}{s+\alpha+1}\left(\left(\frac{2}{3}\right)^{s+\alpha+1} - \left(\frac{1}{3}\right)^{s+\alpha+1}\right) \\ -\frac{1}{2(s+1)}\left(\left(\frac{2}{3}\right)^{s+1} - \left(\frac{1}{3}\right)^{s+1}\right), & 0 \leq \alpha < \frac{\ln\frac{1}{2}}{\ln\frac{1}{3}}, \\ 2\left(\frac{1}{2(s+1)}\left(\frac{1}{2}\right)^{\frac{s+1}{\alpha}} - \frac{1}{s+\alpha+1}\left(\frac{1}{2}\right)^{\frac{s+\alpha+1}{\alpha}}\right) \\ -\frac{1}{2(s+1)}\left(\left(\frac{2}{3}\right)^{s+1} + \left(\frac{1}{3}\right)^{s+1}\right) & \frac{\ln\frac{1}{2}}{\ln\frac{1}{3}} \leq \alpha < \frac{\ln\frac{1}{2}}{\ln\frac{2}{3}}, \\ +\frac{1}{s+\alpha+1}\left(\left(\frac{2}{3}\right)^{s+\alpha+1} + \left(\frac{1}{3}\right)^{s+\alpha+1}\right), \\ \frac{1}{2(s+1)}\left(\left(\frac{2}{3}\right)^{s+1} - \left(\frac{1}{3}\right)^{s+1}\right) \\ -\frac{1}{s+\alpha+1}\left(\left(\frac{2}{3}\right)^{s+\alpha+1} - \left(\frac{1}{3}\right)^{s+\alpha+1}\right), & \alpha \geq \frac{\ln\frac{1}{2}}{\ln\frac{2}{3}}, \end{cases}$$

$$A_4^{\alpha}(s) = \int_{\frac{1}{3}}^{\frac{2}{3}} (1-t)^s \left|t^{\alpha} - \frac{1}{2}\right| dt$$

$$= \begin{cases} \beta\left(\alpha+1, s+1, \frac{2}{3}\right) - \beta\left(\alpha+1, s+1, \frac{1}{3}\right) \\ +\frac{1}{2(s+1)}\left(\left(\frac{1}{3}\right)^{s+1} - \left(\frac{2}{3}\right)^{s+1}\right), & 0 \leq \alpha < \frac{\ln\frac{1}{2}}{\ln\frac{1}{3}}, \\ \frac{1}{2(s+1)}\left[\left(\left(\frac{2}{3}\right)^{s+1} + \left(\frac{1}{3}\right)^{s+1}\right)\right. \\ \left.-2\left(1 - \left(\frac{1}{2}\right)^{\frac{1}{\alpha}}\right)^{s+1}\right] & \frac{\ln\frac{1}{2}}{\ln\frac{1}{3}} \leq \alpha < \frac{\ln\frac{1}{2}}{\ln\frac{2}{3}}, \\ +\beta\left(\alpha+1, s+1, \frac{1}{3}\right) - \beta\left(\alpha+1, s+1, \frac{2}{3}\right) \\ -2\beta\left(\alpha+1, s+1, \left(\frac{1}{2}\right)^{\frac{1}{\alpha}}\right) \\ \frac{1}{2(s+1)}\left(\left(\frac{2}{3}\right)^{s+1} - \left(\frac{1}{3}\right)^{s+1}\right) \\ -\beta\left(\alpha+1, s+1, \frac{2}{3}\right) + \beta\left(\alpha+1, s+1, \frac{1}{3}\right), & \alpha \geq \frac{\ln\frac{1}{2}}{\ln\frac{2}{3}}, \end{cases}$$

$$A_5^{\alpha}(s,\lambda) = \int_{\frac{2}{3}}^{1} t^s \left|t^{\alpha} - (1-\lambda)\right| dt$$

$$= \begin{cases} \frac{1}{s+\alpha+1}\left(1 - \left(\frac{2}{3}\right)^{s+\alpha+1}\right) \\ -\frac{1-\lambda}{s+1}\left(1 - \left(\frac{2}{3}\right)^{s+1}\right), & 0 \leq 1-\lambda < \left(\frac{2}{3}\right)^{\alpha}, \\ \frac{1}{s+\alpha+1}\left[\frac{2\alpha(1-\lambda)^{\frac{s+\alpha+1}{\alpha}}}{s+1}\right. \\ \left.+\left(1 + \left(\frac{2}{3}\right)^{s+\alpha+1}\right)\right] & \left(\frac{2}{3}\right)^{\alpha} \leq 1-\lambda < 1, \\ -\frac{1-\lambda}{s+1}\left(1 + \left(\frac{2}{3}\right)^{s+1}\right), \end{cases}$$

$$A_6^{\alpha}(s,\lambda)=\int_{\frac{2}{3}}^{1}(1-t)^s\left|t^{\alpha}-(1-\lambda)\right|dt$$

$$=\begin{cases}\begin{array}{l}\beta(\alpha+1,s+1,1)\\-\beta\left(\alpha+1,s+1,\frac{2}{3}\right)\\-\frac{1-\lambda}{s+1}\left(\frac{1}{3}\right)^{s+1},\end{array} & 0\leq 1-\lambda<\left(\frac{2}{3}\right)^{\alpha},\\ \\ \begin{array}{l}\frac{1-\lambda}{s+1}\left[\left(\frac{1}{3}\right)^{s+1}-2\left(1-(1-\lambda)^{\frac{1}{\alpha}}\right)^{s+1}\right]\\+\beta\left(\alpha+1,s+1,\frac{2}{3}\right)\\+\beta(\alpha+1,s+1,1)\\-2\beta\left(\alpha+1,s+1,(1-\lambda)^{\frac{1}{\alpha}}\right),\end{array} & \left(\frac{2}{3}\right)^{\alpha}\leq 1-\lambda<1.\end{cases}$$

Proof. With the aid of modulus, Lemma 3.1 becomes

$$\begin{aligned}(8)\qquad &\left|\lambda\mathscr{F}(\sigma)+\frac{1-2\lambda}{2}\left[\mathscr{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathscr{F}\left(\frac{2\sigma+\delta}{3}\right)\right]+\lambda\mathscr{F}(\delta)\right.\\ &\left.-\frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^{\alpha}}\left[J_{\sigma+}^{\alpha}\mathscr{F}(\delta)+J_{\delta-}^{\alpha}\mathscr{F}(\sigma)\right]\right|\\ &\leq\frac{\delta-\sigma}{2}\left[\int_0^{\frac{1}{3}}\left|t^{\alpha}-\lambda\right|\left|\mathscr{F}'(t\delta+(1-t)\sigma)-\mathscr{F}'(t\sigma+(1-t)\delta)\right|dt\right.\\ &+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|t^{\alpha}-\frac{1}{2}\right|\left|\mathscr{F}'(t\delta+(1-t)\sigma)-\mathscr{F}'(t\sigma+(1-t)\delta)\right|dt\\ &\left.+\int_{\frac{2}{3}}^{1}\left|t^{\alpha}-(1-\lambda)\right|\left|\mathscr{F}'(t\delta+(1-t)\sigma)-\mathscr{F}'(t\sigma+(1-t)\delta)\right|dt\right].\end{aligned}$$

From the fact that $|\mathscr{F}'|$ is s-convex, it follows

$$\begin{aligned}&\left|\lambda\mathscr{F}(\sigma)+\frac{1-2\lambda}{2}\left[\mathscr{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathscr{F}\left(\frac{2\sigma+\delta}{3}\right)\right]+\lambda\mathscr{F}(\delta)\right.\\ &\left.-\frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^{\alpha}}\left[J_{\sigma+}^{\alpha}\mathscr{F}(\delta)+J_{\delta-}^{\alpha}\mathscr{F}(\sigma)\right]\right|\\ &\leq\frac{\delta-\sigma}{2}\left[\int_0^{\frac{1}{3}}\left|t^{\alpha}-\lambda\right|\left[t^s\left|\mathscr{F}'(\delta)\right|+(1-t)^s\left|\mathscr{F}'(\sigma)\right|\right.\right.\\ &\left.+t^s\left|\mathscr{F}'(\sigma)\right|+(1-t)^s\left|\mathscr{F}'(\delta)\right|\right]dt\end{aligned}$$

$$+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|t^{\alpha}-\frac{1}{2}\right|\left[t^{s}\left|\mathscr{F}'(\delta)\right|+(1-t)^{s}\left|\mathscr{F}'(\sigma)\right|\right.$$
$$\left.+t^{s}\left|\mathscr{F}'(\sigma)\right|+(1-t)^{s}\left|\mathscr{F}'(\delta)\right|\right]dt$$
$$+\int_{\frac{2}{3}}^{1}\left|t^{\alpha}-(1-\lambda)\right|\left[t^{s}\left|\mathscr{F}'(\delta)\right|+(1-t)^{s}\left|\mathscr{F}'(\sigma)\right|\right.$$
$$\left.+t^{s}\left|\mathscr{F}'(\sigma)\right|+(1-t)^{s}\left|\mathscr{F}'(\delta)\right|\right]dt]$$
$$=\frac{\delta-\sigma}{2}\left(A_1^{\alpha}(s,\lambda)+A_2^{\alpha}(s,\lambda)+A_3^{\alpha}(s)+A_4^{\alpha}(s)+A_5^{\alpha}(s,\lambda)+A_6^{\alpha}(s,\lambda)\right)$$
$$\times\left[\left|\mathscr{F}'(\sigma)\right|+\left|\mathscr{F}'(\delta)\right|\right].$$

This finishes the proof of Theorem 3.7. □

Corollary 3.8. *Let us consider $\lambda=\frac{1}{2}$ and $s=1$ in Theorem 3.7. Then, the following trapezoid type inequality holds:*

$$\left|\frac{\mathscr{F}(\sigma)+\mathscr{F}(\delta)}{2}-\frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^{\alpha}}\left[J_{\sigma+}^{\alpha}\mathscr{F}(\delta)+J_{\delta-}^{\alpha}\mathscr{F}(\sigma)\right]\right| \tag{9}$$
$$\leq\frac{\delta-\sigma}{2}\left(A_1^{\alpha}\left(1,\frac{1}{2}\right)+A_2^{\alpha}\left(1,\frac{1}{2}\right)+A_3^{\alpha}(1)\right.$$
$$\left.+A_4^{\alpha}(1)+A_5^{\alpha}\left(1,\frac{1}{2}\right)+A_6^{\alpha}\left(1,\frac{1}{2}\right)\right)\left[\left|\mathscr{F}'(\sigma)\right|+\left|\mathscr{F}'(\delta)\right|\right]$$
$$=\frac{\delta-\sigma}{2}\int_{0}^{1}\left|t^{\alpha}-\frac{1}{2}\right|dt\left[\left|\mathscr{F}'(\sigma)\right|+\left|\mathscr{F}'(\delta)\right|\right]$$
$$=\frac{\delta-\sigma}{2}\left(\frac{\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}+\frac{1-\alpha}{2(\alpha+1)}\right)\left[\left|\mathscr{F}'(\sigma)\right|+\left|\mathscr{F}'(\delta)\right|\right].$$

Remark 3.9. *If we select $\alpha=1$ in Corollary 3.8, then Corollary 3.8 becomes*

$$\left|\frac{\mathscr{F}(\sigma)+\mathscr{F}(\delta)}{2}-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathscr{F}(x)dx\right|\leq\frac{\delta-\sigma}{8}\left[\left|\mathscr{F}'(\sigma)\right|+\left|\mathscr{F}'(\delta)\right|\right],$$

which is given by [9, Theorem 2.2].

Remark 3.10. *Let us note $\lambda=\frac{1}{8}$ and $s=1$ in Theorem 3.7. Then, Theorem 3.7 reduces to the following Newton type inequality*

$$\left|\frac{1}{8}\left[\mathscr{F}(\sigma)+3\left[\mathscr{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathscr{F}\left(\frac{2\sigma+\delta}{3}\right)\right]+\mathscr{F}(\delta)\right]\right.$$

$$
\begin{aligned}
& -\frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^{\alpha}}\left[J_{\sigma+}^{\alpha}\mathscr{F}(\delta)+J_{\delta-}^{\alpha}\mathscr{F}(\sigma)\right]\Bigg| \\
\leq & \frac{\delta-\sigma}{2}\left(A_1^{\alpha}\left(1,\frac{1}{8}\right)+A_2^{\alpha}\left(1,\frac{1}{8}\right)+A_3^{\alpha}(1)\right. \\
& \left.+A_4^{\alpha}(1)+A_5^{\alpha}\left(1,\frac{1}{8}\right)+A_6^{\alpha}\left(1,\frac{1}{8}\right)\right)\left[\left|\mathscr{F}'(\sigma)\right|+\left|\mathscr{F}'(\delta)\right|\right],
\end{aligned}
$$

which is proved by [16, Theorem 3].

Remark 3.11. *For $\alpha = 1$ in Remark 3.10, the following inequality holds:*

$$
\begin{aligned}
& \left|\frac{1}{8}\left[\mathscr{F}(\sigma)+3\left[\mathscr{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathscr{F}\left(\frac{2\sigma+\delta}{3}\right)\right]+\mathscr{F}(\delta)\right]-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathscr{F}(t)\,dt\right| \\
\leq & \frac{25(\delta-\sigma)^2}{576}\left[\left|\mathscr{F}'(\sigma)\right|+\left|\mathscr{F}'(\delta)\right|\right],
\end{aligned}
$$

which is given by [31, Remark 3].

Corollary 3.12. *Let us consider $\lambda = 0$ in Theorem 3.7. Then, Theorem 3.7 reduces to*

$$
\begin{aligned}
& \left|\frac{1}{2}\left[\mathscr{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathscr{F}\left(\frac{2\sigma+\delta}{3}\right)\right]-\frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^{\alpha}}\left[J_{\sigma+}^{\alpha}\mathscr{F}(\delta)+J_{\delta-}^{\alpha}\mathscr{F}(\sigma)\right]\right| \\
\leq & \frac{\delta-\sigma}{2}\left(A_1^{\alpha}(s,0)+A_2^{\alpha}(s,0)+A_3^{\alpha}(s)+A_4^{\alpha}(s)+A_5^{\alpha}(s,0)+A_6^{\alpha}(s,0)\right) \\
& \times\left[\left|\mathscr{F}'(\sigma)\right|+\left|\mathscr{F}'(\delta)\right|\right].
\end{aligned}
$$

Corollary 3.13. *If we assign $\alpha = s = 1$ in Corollary 3.12, then Corollary 3.12 reduces to*

$$
\begin{aligned}
& \left|\frac{1}{2}\left[\mathscr{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathscr{F}\left(\frac{2\sigma+\delta}{3}\right)\right]-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathscr{F}(t)\,dt\right| \\
\leq & \frac{5(\delta-\sigma)}{72}\left[\left|\mathscr{F}'(\sigma)\right|+\left|\mathscr{F}'(\delta)\right|\right].
\end{aligned}
$$

Theorem 3.14. *Suppose that the assumptions of Lemma 3.1 are valid. Suppose also that the function $|\mathscr{F}'|^q$, $q > 1$ is s-convex on $[\sigma, \delta]$. Then, the following inequality holds:*

$$
\begin{aligned}
& \left|\lambda\mathscr{F}(\sigma)+\frac{1-2\lambda}{2}\left[\mathscr{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathscr{F}\left(\frac{2\sigma+\delta}{3}\right)\right]+\lambda\mathscr{F}(\delta)\right. \\
& \left.-\frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^{\alpha}}\left[J_{\sigma+}^{\alpha}\mathscr{F}(\delta)+J_{\delta-}^{\alpha}\mathscr{F}(\sigma)\right]\right| \\
\leq & \frac{\delta-\sigma}{2}\left[\left(\varphi_1(\alpha,\lambda,p)+\varphi_3(\alpha,\lambda,p)\right)\right.
\end{aligned}
$$

$$\times\left[\left(\frac{1}{s+1}\left(\left(\frac{1}{3}\right)^{s+1}\left|\mathscr{F}'(\delta)\right|^q+\left(1-\left(\frac{2}{3}\right)^{s+1}\right)\left|\mathscr{F}'(\sigma)\right|^q\right)\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{1}{s+1}\left(\left(\frac{1}{3}\right)^{s+1}\left|\mathscr{F}'(\sigma)\right|^q+\left(1-\left(\frac{2}{3}\right)^{s+1}\right)\left|\mathscr{F}'(\delta)\right|^q\right)\right)^{\frac{1}{q}}\right]$$

$$\left.+2\varphi_2(\alpha,p)\left(\frac{1}{s+1}\left(\left(\frac{2}{3}\right)^{s+1}-\left(\frac{1}{3}\right)^{s+1}\right)\left(\left|\mathscr{F}'(\sigma)\right|^q+\left|\mathscr{F}'(\delta)\right|^q\right)\right)^{\frac{1}{q}}\right].$$

Here, $\frac{1}{p}+\frac{1}{q}=1$ *and*

$$\begin{cases}\varphi_1(\alpha,\lambda,p)=\left(\int_0^{\frac{1}{3}}\left|t^\alpha-\lambda\right|^p dt\right)^{\frac{1}{p}},\\ \varphi_2(\alpha,p)=\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|t^\alpha-\frac{1}{2}\right|^p dt\right)^{\frac{1}{p}},\\ \varphi_3(\alpha,\lambda,p)=\left(\int_{\frac{2}{3}}^{1}\left|t^\alpha-(1-\lambda)\right|^p dt\right)^{\frac{1}{p}}.\end{cases}$$

Proof. By applying Hölder inequality in inequality (8), it follows

$$\left|\lambda\mathscr{F}(\sigma)+\frac{1-2\lambda}{2}\left[\mathscr{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathscr{F}\left(\frac{2\sigma+\delta}{3}\right)\right]+\lambda\mathscr{F}(\delta)\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^\alpha}\left[J_{\sigma+}^\alpha\mathscr{F}(\delta)+J_{\delta-}^\alpha\mathscr{F}(\sigma)\right]\right|$$
$$\leq\frac{\delta-\sigma}{2}\left[\left(\int_0^{\frac{1}{3}}\left|t^\alpha-\lambda\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{3}}\left|\mathscr{F}'(t\delta+(1-t)\sigma)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_0^{\frac{1}{3}}\left|t^\alpha-\lambda\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{3}}\left|\mathscr{F}'(t\sigma+(1-t)\delta)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|t^\alpha-\frac{1}{2}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\mathscr{F}'(t\delta+(1-t)\sigma)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|t^\alpha-\frac{1}{2}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\mathscr{F}'(t\sigma+(1-t)\delta)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{2}{3}}^{1}\left|t^{\alpha}-(1-\lambda)\right|^{p}dt\right)^{\frac{1}{p}}\left(\int_{\frac{2}{3}}^{1}\left|\mathscr{F}'\left(t\delta+(1-t)\sigma\right)\right|^{q}dt\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_{\frac{2}{3}}^{1}\left|t^{\alpha}-(1-\lambda)\right|^{p}dt\right)^{\frac{1}{p}}\left(\int_{\frac{2}{3}}^{1}\left|\mathscr{F}'\left(t\sigma+(1-t)\delta\right)\right|^{q}dt\right)^{\frac{1}{q}}\right].$$

From the fact that $|\mathscr{F}'|^{q}$ is s-convex, we get

$$\begin{aligned}
&\left|\lambda\mathscr{F}(\sigma)+\frac{1-2\lambda}{2}\left[\mathscr{F}\left(\frac{\sigma+2\delta}{2}\right)+\mathscr{F}\left(\frac{2\sigma+\delta}{2}\right)\right]+\lambda\mathscr{F}(\delta)\right.\\
&\left.-\frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^{\alpha}}\left[J_{\sigma+}^{\alpha}\mathscr{F}(\delta)+J_{\delta-}^{\alpha}\mathscr{F}(\sigma)\right]\right|\\
&\leq\frac{\delta-\sigma}{2}\left[\left(\int_{0}^{\frac{1}{3}}\left|t^{\alpha}-\lambda\right|^{p}dt\right)^{\frac{1}{p}}\left[\left(\int_{0}^{\frac{1}{3}}t^{s}\left|\mathscr{F}'(\delta)\right|^{q}+(1-t)^{s}\left|\mathscr{F}'(\sigma)\right|^{q}dt\right)^{\frac{1}{q}}\right.\right.\\
&\left.+\left(\int_{0}^{\frac{1}{3}}t^{s}\left|\mathscr{F}'(\sigma)\right|^{q}+(1-t)^{s}\left|\mathscr{F}'(\delta)\right|^{q}dt\right)^{\frac{1}{q}}\right]+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|t^{\alpha}-\frac{1}{2}\right|^{p}dt\right)^{\frac{1}{p}}\\
&\times\left[\left(\int_{\frac{1}{3}}^{\frac{2}{3}}t^{s}\left|\mathscr{F}'(\delta)\right|^{q}+(1-t)^{s}\left|\mathscr{F}'(\sigma)\right|^{q}dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}t^{s}\left|\mathscr{F}'(\sigma)\right|^{q}+(1-t)^{s}\left|\mathscr{F}'(\delta)\right|^{q}dt\right)^{\frac{1}{q}}\right]\\
&+\left(\int_{\frac{2}{3}}^{1}\left|t^{\alpha}-(1-\lambda)\right|^{p}dt\right)^{\frac{1}{p}}\left[\left(\int_{\frac{2}{3}}^{1}t^{s}\left|\mathscr{F}'(\delta)\right|^{q}+(1-t)^{s}\left|\mathscr{F}'(\sigma)\right|^{q}dt\right)^{\frac{1}{q}}\right.\\
&\left.\left.+\left(\int_{\frac{2}{3}}^{1}t^{s}\left|\mathscr{F}'(\sigma)\right|^{q}+(1-t)^{s}\left|\mathscr{F}'(\delta)\right|^{q}dt\right)^{\frac{1}{q}}\right]\right]
\end{aligned}$$

$$= \frac{\delta-\sigma}{2}\left[\left(\left(\int_{0}^{\frac{1}{3}}\left|t^{\alpha}-\lambda\right|^{p}dt\right)^{\frac{1}{p}}+\left(\int_{\frac{2}{3}}^{1}\left|t^{\alpha}-(1-\lambda)\right|^{p}dt\right)^{\frac{1}{p}}\right)\right.$$

$$\times\left[\left(\frac{1}{s+1}\left(\left(\frac{1}{3}\right)^{s+1}\left|\mathscr{F}'(\delta)\right|^{q}+\left(1-\left(\frac{2}{3}\right)^{s+1}\right)\left|\mathscr{F}'(\sigma)\right|^{q}\right)\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{1}{s+1}\left(\left(\frac{1}{3}\right)^{s+1}\left|\mathscr{F}'(\sigma)\right|^{q}+\left(1-\left(\frac{2}{3}\right)^{s+1}\right)\left|\mathscr{F}'(\delta)\right|^{q}\right)\right)^{\frac{1}{q}}\right]$$

$$+2\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|t^{\alpha}-\frac{1}{2}\right|^{p}dt\right)^{\frac{1}{p}}\left[\frac{1}{s+1}\left(\left(\frac{2}{3}\right)^{s+1}-\left(\frac{1}{3}\right)^{s+1}\right)\right.$$

$$\left.\left.\times\left(\left|\mathscr{F}'(\sigma)\right|^{q}+\left|\mathscr{F}'(\delta)\right|^{q}\right)\right]^{\frac{1}{q}}\right].$$

This completes the proof of Theorem 3.14. □

Corollary 3.15. *Let us consider $\lambda=\frac{1}{2}$ and $s=1$ in Theorem 3.14. Then, the following trapezoid type inequality holds:*

$$\left|\frac{\mathscr{F}(\sigma)+\mathscr{F}(\delta)}{2}-\frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^{\alpha}}\left[J_{\sigma+}^{\alpha}\mathscr{F}(\delta)+J_{\delta-}^{\alpha}\mathscr{F}(\sigma)\right]\right|$$

$$\leq\frac{\delta-\sigma}{2}\left[\left(\varphi_{1}\left(\alpha,\frac{1}{2},p\right)+\varphi_{3}\left(\alpha,\frac{1}{2},p\right)\right)\right.$$

$$\times\left[\left(\frac{5\left|\mathscr{F}'(\sigma)\right|^{q}+\left|\mathscr{F}'(\delta)\right|^{q}}{18}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathscr{F}'(\sigma)\right|^{q}+5\left|\mathscr{F}'(\delta)\right|^{q}}{18}\right)^{\frac{1}{q}}\right]$$

$$\left.+2\varphi_{2}(\alpha,p)\left(\frac{\left|\mathscr{F}'(\sigma)\right|^{q}+\left|\mathscr{F}'(\delta)\right|^{q}}{6}\right)^{\frac{1}{q}}\right].$$

Corollary 3.16. *Consider $\alpha=1$ in Corollary 3.15. Then, we obtain*

$$\left|\frac{\mathscr{F}(\sigma)+\mathscr{F}(\delta)}{2}-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathscr{F}(t)\,dt\right|\leq\frac{\delta-\sigma}{9}\left[\left(\frac{3^{p+1}-1}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}\right.$$

$$\times\left[\left(\frac{5\left|\mathscr{F}'(\sigma)\right|^{q}+\left|\mathscr{F}'(\delta)\right|^{q}}{6}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathscr{F}'(\sigma)\right|^{q}+5\left|\mathscr{F}'(\delta)\right|^{q}}{6}\right)^{\frac{1}{q}}\right]$$

$$\left.+\left(\frac{1}{2^{p}(p+1)}\right)^{\frac{1}{p}}\left(\frac{\left|\mathscr{F}'(\sigma)\right|^{q}+\left|\mathscr{F}'(\delta)\right|^{q}}{2}\right)^{\frac{1}{q}}\right].$$

Remark 3.17. *If we choose* $\lambda = \frac{1}{8}$ *and* $s = 1$ *in Theorem 3.14, then the following Newton type inequality*

$$\begin{aligned}
&\left| \frac{1}{8}\left[\mathscr{F}(\sigma) + 3\left[\mathscr{F}\left(\frac{\sigma+2\delta}{3}\right) + \mathscr{F}\left(\frac{2\sigma+\delta}{3}\right)\right] + \mathscr{F}(\delta)\right] \right.\\
&\quad \left. - \frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^{\alpha}}\left[J^{\alpha}_{\sigma+}\mathscr{F}(\delta) + J^{\alpha}_{\delta-}\mathscr{F}(\sigma)\right] \right| \\
&\leq \frac{(\delta-\sigma)}{2}\left\{\left(\varphi_1\left(\alpha, \frac{1}{8}, p\right) + \varphi_3\left(\alpha, \frac{1}{8}, p\right)\right)\left[\left(\frac{5\left|\mathscr{F}'(\sigma)\right|^q + \left|\mathscr{F}'(\delta)\right|^q}{18}\right)^{\frac{1}{q}}\right.\right.\\
&\quad \left.\left. + \left(\frac{\left|\mathscr{F}'(\sigma)\right|^q + 5\left|\mathscr{F}'(\delta)\right|^q}{18}\right)^{\frac{1}{q}}\right] + 2\varphi_2(\alpha, p)\left(\frac{\left|\mathscr{F}'(\sigma)\right|^q + \left|\mathscr{F}'(\delta)\right|^q}{6}\right)^{\frac{1}{q}}\right\},
\end{aligned}$$

which is given by [16, Theorem 4].

Remark 3.18. *If we assign* $\alpha = 1$ *in Remark 3.17, then the following Newton's type inequality holds:*

$$\begin{aligned}
&\left| \frac{1}{8}\left[\mathscr{F}(\sigma) + 3\mathscr{F}\left(\frac{2\sigma+\delta}{3}\right) + 3\mathscr{F}\left(\frac{\sigma+2\delta}{3}\right) + \mathscr{F}(\delta)\right] - \frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathscr{F}(t)\,dt \right| \\
&\leq \frac{(\delta-\sigma)}{9}\left[\left(\frac{5^{p+1}+3^{p+1}}{8^{p+1}(p+1)}\right)^{\frac{1}{p}}\right.\\
&\quad \times \left[\left(\frac{5\left|\mathscr{F}'(\sigma)\right|^q + \left|\mathscr{F}'(\delta)\right|^q}{6}\right)^{\frac{1}{q}} + \left(\frac{\left|\mathscr{F}'(\sigma)\right|^q + 5\left|\mathscr{F}'(\delta)\right|^q}{6}\right)^{\frac{1}{q}}\right]\\
&\quad \left. + \left(\frac{1}{2^p(p+1)}\right)^{\frac{1}{p}}\left(\frac{\left|\mathscr{F}'(\sigma)\right|^q + \left|\mathscr{F}'(\delta)\right|^q}{2}\right)^{\frac{1}{q}}\right],
\end{aligned}$$

which is established by [31, Remark 5].

Corollary 3.19. *Let us note* $\lambda = 0$ *and* $s = 1$ *in Theorem 3.14. Then, we obtain*

$$\begin{aligned}
&\left| \frac{1}{2}\left[\mathscr{F}\left(\frac{\sigma+2\delta}{3}\right) + \mathscr{F}\left(\frac{2\sigma+\delta}{3}\right)\right] - \frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^{\alpha}}\left[J^{\alpha}_{\sigma+}\mathscr{F}(\delta) + J^{\alpha}_{\delta-}\mathscr{F}(\sigma)\right] \right| \\
&\leq \frac{(\delta-\sigma)}{2}\left[\left(\varphi_1(\alpha, 0, p) + \varphi_3(\alpha, 0, p)\right)\right.\\
&\quad \times \left[\left(\frac{5\left|\mathscr{F}'(\sigma)\right|^q + \left|\mathscr{F}'(\delta)\right|^q}{18}\right)^{\frac{1}{q}} + \left(\frac{\left|\mathscr{F}'(\sigma)\right|^q + 5\left|\mathscr{F}'(\delta)\right|^q}{18}\right)^{\frac{1}{q}}\right]\\
&\quad \left. + 2\varphi_2(\alpha, p)\left(\frac{\left|\mathscr{F}'(\sigma)\right|^q + \left|\mathscr{F}'(\delta)\right|^q}{6}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Corollary 3.20. *If it is chosen $\alpha = 1$ in Corollary 3.19, then the following inequality holds:*

$$\left| \frac{1}{2}\left[\mathscr{F}\left(\frac{\sigma+2\delta}{3}\right) + \mathscr{F}\left(\frac{2\sigma+\delta}{3}\right) \right] - \frac{1}{\delta-\sigma}\int_{\sigma}^{\delta} \mathscr{F}(t)\,dt \right|$$
$$\leq \frac{(\delta-\sigma)}{9}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left[\left(\frac{5\left|\mathscr{F}'(\sigma)\right|^{q} + \left|\mathscr{F}'(\delta)\right|^{q}}{6}\right)^{\frac{1}{q}} + \left(\frac{\left|\mathscr{F}'(\sigma)\right|^{q} + 5\left|\mathscr{F}'(\delta)\right|^{q}}{6}\right)^{\frac{1}{q}} \right.$$
$$\left. + \frac{1}{2}\left(\frac{\left|\mathscr{F}'(\sigma)\right|^{q} + \left|\mathscr{F}'(\delta)\right|^{q}}{2}\right)^{\frac{1}{q}} \right].$$

Theorem 3.21. *Let us consider that the assumptions of Lemma 3.1 are provided. Let us also consider that the function $|\mathscr{F}'|^{q}$, $q \geq 1$ is $s-$convex on $[\sigma, \delta]$. Then, the following inequality holds:*

$$\left| \lambda\mathscr{F}(\sigma) + \frac{1-2\lambda}{2}\left[\mathscr{F}\left(\frac{\sigma+2\delta}{3}\right) + \mathscr{F}\left(\frac{2\sigma+\delta}{3}\right) \right] + \lambda\mathscr{F}(\delta) \right.$$
$$\left. - \frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^{\alpha}}\left[J_{\sigma+}^{\alpha}\mathscr{F}(\delta) + J_{\delta-}^{\alpha}\mathscr{F}(\sigma)\right] \right|$$
$$\leq \frac{(\delta-\sigma)}{2}\left\{ \left(A_{1}^{\alpha}(0,\lambda)\right)^{1-\frac{1}{q}} \left[\left(A_{1}^{\alpha}(s,\lambda)\left|\mathscr{F}'(\delta)\right|^{q} + A_{2}^{\alpha}(s,\lambda)\left|\mathscr{F}'(\sigma)\right|^{q}\right)^{\frac{1}{q}} \right.\right.$$
$$\left. + \left(A_{1}^{\alpha}(s,\lambda)\left|\mathscr{F}'(\sigma)\right|^{q} + A_{2}^{\alpha}(s,\lambda)\left|\mathscr{F}'(\delta)\right|^{q}\right)^{\frac{1}{q}} \right]$$
$$+ \left(A_{3}^{\alpha}(0)\right)^{1-\frac{1}{q}} \left[\left(A_{3}^{\alpha}(s)\left|\mathscr{F}'(\delta)\right|^{q} + A_{4}^{\alpha}(s)\left|\mathscr{F}'(\sigma)\right|^{q}\right)^{\frac{1}{q}} \right.$$
$$\left. + \left(A_{3}^{\alpha}(s)\left|\mathscr{F}'(\sigma)\right|^{q} + A_{4}^{\alpha}(s)\left|\mathscr{F}'(\delta)\right|^{q}\right)^{\frac{1}{q}} \right]$$
$$+ \left(A_{5}^{\alpha}(0,\lambda)\right)^{1-\frac{1}{q}} \left[\left(A_{5}^{\alpha}(s,\lambda)\left|\mathscr{F}'(\delta)\right|^{q} + A_{6}^{\alpha}(s,\lambda)\left|\mathscr{F}'(\sigma)\right|^{q}\right)^{\frac{1}{q}} \right.$$
$$\left.\left. + \left(A_{5}^{\alpha}(s,\lambda)\left|\mathscr{F}'(\sigma)\right|^{q} + A_{6}^{\alpha}(s,\lambda)\left|\mathscr{F}'(\delta)\right|^{q}\right)^{\frac{1}{q}} \right] \right\}.$$

Proof. Using the facts of power-mean inequality in inequality (8), it follows

$$\left| \lambda\mathscr{F}(\sigma) + \frac{1-2\lambda}{2}\left[\mathscr{F}\left(\frac{\sigma+2\delta}{3}\right) + \mathscr{F}\left(\frac{2\sigma+\delta}{3}\right) \right] + \lambda\mathscr{F}(\delta) \right.$$
$$\left. - \frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^{\alpha}}\left[J_{\sigma+}^{\alpha}\mathscr{F}(\delta) + J_{\delta-}^{\alpha}\mathscr{F}(\sigma)\right] \right|$$
$$\leq \frac{(\delta-\sigma)}{2}\left[\left(\int_{0}^{\frac{1}{3}} \left|t^{\alpha}-\lambda\right| dt \right)^{1-\frac{1}{q}} \left(\int_{0}^{\frac{1}{3}} \left|t^{\alpha}-\lambda\right| \left|\mathscr{F}'(t\delta+(1-t)\sigma)\right|^{q} dt \right)^{\frac{1}{q}} \right.$$

$$+\left(\int_0^{\frac{1}{3}}|t^\alpha-\lambda|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{3}}|t^\alpha-\lambda|\,|\mathscr{F}'(t\sigma+(1-t)\delta)|^q\,dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|t^\alpha-\frac{1}{2}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|t^\alpha-\frac{1}{2}\right||\mathscr{F}'(t\delta+(1-t)\sigma)|^q\,dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|t^\alpha-\frac{1}{2}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|t^\alpha-\frac{1}{2}\right||\mathscr{F}'(t\sigma+(1-t)\delta)|^q\,dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{2}{3}}^{1}|t^\alpha-(1-\lambda)|\,dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{2}{3}}^{1}|t^\alpha-(1-\lambda)|\,|\mathscr{F}'(t\delta+(1-t)\sigma)|^q\,dt\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_{\frac{2}{3}}^{1}|t^\alpha-(1-\lambda)|\,dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{2}{3}}^{1}|t^\alpha-(1-\lambda)|\,|\mathscr{F}'(t\sigma+(1-t)\delta)|^q\,dt\right)^{\frac{1}{q}}\right].$$

From the facts that $|\mathscr{F}'|^q$ is s-convex, we obtain

$$\left|\lambda\mathscr{F}(\sigma)+\frac{1-2\lambda}{2}\left[\mathscr{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathscr{F}\left(\frac{2\sigma+\delta}{3}\right)\right]+\lambda\mathscr{F}(\delta)\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^\alpha}\left[J^\alpha_{\sigma+}\mathscr{F}(\delta)+J^\alpha_{\delta-}\mathscr{F}(\sigma)\right]\right|$$
$$\leq\frac{(\delta-\sigma)}{2}\left[\left(\int_0^{\frac{1}{3}}|t^\alpha-\lambda|\,dt\right)^{1-\frac{1}{q}}\right.$$
$$\times\left[\left(\int_0^{\frac{1}{3}}|t^\alpha-\lambda|\left[t^s|\mathscr{F}'(\delta)|^q+(1-t)^s|\mathscr{F}'(\sigma)|^q\right]dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^{\frac{1}{3}}|t^\alpha-\lambda|\left[t^s|\mathscr{F}'(\sigma)|^q+(1-t)^s|\mathscr{F}'(\delta)|^q\right]dt\right)^{\frac{1}{q}}\right]$$

$$
\begin{aligned}
&+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|t^{\alpha}-\frac{1}{2}\right|dt\right)^{1-\frac{1}{q}}\left[\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|t^{\alpha}-\frac{1}{2}\right|\left[t^{s}\left|\mathscr{F}'(\delta)\right|^{q}+(1-t)^{s}\left|\mathscr{F}'(\sigma)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|t^{\alpha}-\frac{1}{2}\right|\left[t^{s}\left|\mathscr{F}'(\sigma)\right|^{q}+(1-t)^{s}\left|\mathscr{F}'(\delta)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right]\\
&+\left(\int_{\frac{2}{3}}^{1}\left|t^{\alpha}-(1-\lambda)\right|dt\right)^{1-\frac{1}{q}}\\
&\times\left[\left(\int_{\frac{2}{3}}^{1}\left|t^{\alpha}-(1-\lambda)\right|\left[t^{s}\left|\mathscr{F}'(\delta)\right|^{q}+(1-t)^{s}\left|\mathscr{F}'(\sigma)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right.\\
&\left.\left.+\left(\int_{\frac{2}{3}}^{1}\left|t^{\alpha}-(1-\lambda)\right|\left[t^{s}\left|\mathscr{F}'(\sigma)\right|^{q}+(1-t)^{s}\left|\mathscr{F}'(\delta)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right]\right].\\
=&\frac{(\delta-\sigma)}{2}\left\{\left(A_{1}^{\alpha}(0,\lambda)\right)^{1-\frac{1}{q}}\left[\left(A_{1}^{\alpha}(s,\lambda)\left|\mathscr{F}'(\delta)\right|^{q}+A_{2}^{\alpha}(s,\lambda)\left|\mathscr{F}'(\sigma)\right|^{q}\right)^{\frac{1}{q}}\right.\right.\\
&\left.+\left(A_{1}^{\alpha}(s,\lambda)\left|\mathscr{F}'(\sigma)\right|^{q}+A_{2}^{\alpha}(s,\lambda)\left|\mathscr{F}'(\delta)\right|^{q}\right)^{\frac{1}{q}}\right]\\
&+\left(A_{3}^{\alpha}(0)\right)^{1-\frac{1}{q}}\left[\left(A_{3}^{\alpha}(s)\left|\mathscr{F}'(\delta)\right|^{q}+A_{4}^{\alpha}(s)\left|\mathscr{F}'(\sigma)\right|^{q}\right)^{\frac{1}{q}}\right.\\
&\left.+\left(A_{3}^{\alpha}(s)\left|\mathscr{F}'(\sigma)\right|^{q}+A_{4}^{\alpha}(s)\left|\mathscr{F}'(\delta)\right|^{q}\right)^{\frac{1}{q}}\right]\\
&+\left(A_{5}^{\alpha}(0)\right)^{1-\frac{1}{q}}\left[\left(A_{5}^{\alpha}(s)\left|\mathscr{F}'(\delta)\right|^{q}+A_{6}^{\alpha}(s)\left|\mathscr{F}'(\sigma)\right|^{q}\right)^{\frac{1}{q}}\right.\\
&\left.\left.+\left(A_{5}^{\alpha}(s)\left|\mathscr{F}'(\sigma)\right|^{q}+A_{6}^{\alpha}(s)\left|\mathscr{F}'(\delta)\right|^{q}\right)^{\frac{1}{q}}\right]\right\}.
\end{aligned}
$$

Consequently, we obtain the desired result of Theorem 3.21. □

Corollary 3.22. *Let us consider $\lambda=\frac{1}{2}$ in Theorem 3.21. Then, the following trapezoid type inequality holds:*

$$
\begin{aligned}
&\left|\frac{\mathscr{F}(\sigma)+\mathscr{F}(\delta)}{2}-\frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^{\alpha}}\left[J_{\sigma+}^{\alpha}\mathscr{F}(\delta)+J_{\delta-}^{\alpha}\mathscr{F}(\sigma)\right]\right|\\
\leq&\frac{(\delta-\sigma)}{2}\left\{\left(A_{1}^{\alpha}\left(0,\frac{1}{2}\right)\right)^{1-\frac{1}{q}}\left[\left(A_{1}^{\alpha}\left(s,\frac{1}{2}\right)\left|\mathscr{F}'(\delta)\right|^{q}+A_{2}^{\alpha}\left(s,\frac{1}{2}\right)\left|\mathscr{F}'(\sigma)\right|^{q}\right)^{\frac{1}{q}}\right.\right.
\end{aligned}
$$

$$+\left(A_1^\alpha\left(s,\frac{1}{2}\right)\left|\mathscr{F}'(\sigma)\right|^q+A_2^\alpha\left(s,\frac{1}{2}\right)\left|\mathscr{F}'(\delta)\right|^q\right)^{\frac{1}{q}}\Bigg]$$
$$+\left(A_3^\alpha(0)\right)^{1-\frac{1}{q}}\left[\left(A_3^\alpha(s)\left|\mathscr{F}'(\delta)\right|^q+A_4^\alpha(s)\left|\mathscr{F}'(\sigma)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(A_3^\alpha(s)\left|\mathscr{F}'(\sigma)\right|^q+A_4^\alpha(s)\left|\mathscr{F}'(\delta)\right|^q\right)^{\frac{1}{q}}\right]$$
$$+\left(A_5^\alpha\left(0,\frac{1}{2}\right)\right)^{1-\frac{1}{q}}\left[\left(A_5^\alpha\left(s,\frac{1}{2}\right)\left|\mathscr{F}'(\delta)\right|^q+A_6^\alpha\left(s,\frac{1}{2}\right)\left|\mathscr{F}'(\sigma)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.\left.+\left(A_5^\alpha\left(s,\frac{1}{2}\right)\left|\mathscr{F}'(\sigma)\right|^q+A_6^\alpha\left(s,\frac{1}{2}\right)\left|\mathscr{F}'(\delta)\right|^q\right)^{\frac{1}{q}}\right]\right\}.$$

Corollary 3.23. *If we select $\alpha = s = 1$ in Corollary 3.22, then we have the following trapezoid type inequality*

$$\left|\frac{\mathscr{F}(\sigma)+\mathscr{F}(\delta)}{2}-\frac{1}{\delta-\sigma}\int_\sigma^\delta \mathscr{F}(t)\,dt\right|$$
$$\leq\frac{(\delta-\sigma)}{2}\left[\frac{2}{9}\left[\left(\frac{5\left|\mathscr{F}'(\delta)\right|^q+31\left|\mathscr{F}'(\sigma)\right|^q}{36}\right)^{\frac{1}{q}}+\left(\frac{5\left|\mathscr{F}'(\sigma)\right|^q+31\left|\mathscr{F}'(\delta)\right|^q}{36}\right)^{\frac{1}{q}}\right]\right.$$
$$\left.+\frac{1}{18}\left(\frac{\left|\mathscr{F}'(\sigma)\right|^q+\left|\mathscr{F}'(\delta)\right|^q}{2}\right)^{\frac{1}{q}}\right].$$

Remark 3.24. *Consider $\lambda = \frac{1}{8}$ and $s = 1$ in Theorem 3.21. Then, the following Newton type inequality holds:*

$$\left|\frac{1}{8}\left[\mathscr{F}(\sigma)+3\left[\mathscr{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathscr{F}\left(\frac{2\sigma+\delta}{3}\right)\right]+\mathscr{F}(\delta)\right]\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^\alpha}\left[J_{\sigma+}^\alpha\mathscr{F}(\delta)+J_{\delta-}^\alpha\mathscr{F}(\sigma)\right]\right|$$
$$\leq\frac{(\delta-\sigma)}{2}\left\{\left(A_1^\alpha\left(0,\frac{1}{8}\right)\right)^{1-\frac{1}{q}}\left[\left(A_1^\alpha\left(1,\frac{1}{8}\right)\left|\mathscr{F}'(\delta)\right|^q+A_2^\alpha\left(1,\frac{1}{8}\right)\left|\mathscr{F}'(\sigma)\right|^q\right)^{\frac{1}{q}}\right.\right.$$
$$\left.+\left(A_1^\alpha\left(1,\frac{1}{8}\right)\left|\mathscr{F}'(\sigma)\right|^q+A_2^\alpha\left(1,\frac{1}{8}\right)\left|\mathscr{F}'(\delta)\right|^q\right)^{\frac{1}{q}}\right]$$
$$+\left(A_3^\alpha(0)\right)^{1-\frac{1}{q}}\left[\left(A_3^\alpha(1)\left|\mathscr{F}'(\delta)\right|^q+A_4^\alpha(1)\left|\mathscr{F}'(\sigma)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(A_3^\alpha(1)\left|\mathscr{F}'(\sigma)\right|^q+A_4^\alpha(1)\left|\mathscr{F}'(\delta)\right|^q\right)^{\frac{1}{q}}\right]$$
$$+\left(A_5^\alpha\left(0,\frac{1}{8}\right)\right)^{1-\frac{1}{q}}\left[\left(A_5^\alpha\left(1,\frac{1}{8}\right)\left|\mathscr{F}'(\delta)\right|^q+A_6^\alpha\left(1,\frac{1}{8}\right)\left|\mathscr{F}'(\sigma)\right|^q\right)^{\frac{1}{q}}\right.$$

$$+\left(A_5^{\alpha}\left(1,\frac{1}{8}\right)|\mathscr{F}'(\sigma)|^q+A_6^{\alpha}\left(1,\frac{1}{8}\right)|\mathscr{F}'(\delta)|^q\right)^{\frac{1}{q}}\Bigg]\Bigg\}.$$

which is established by [16, Theorem 5].

Remark 3.25. *If we select* $\alpha=1$ *in Remark 3.24, then we obtain the following Newton type inequality*

$$\left|\frac{1}{8}\left[\mathscr{F}(\sigma)+3\left[\mathscr{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathscr{F}\left(\frac{2\sigma+\delta}{3}\right)\right]+\mathscr{F}(\delta)\right]-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathscr{F}(t)\,dt\right|$$
$$\leq\frac{(\delta-\sigma)}{2}\left[\frac{17}{32\cdot 9}\left[\left(\frac{251|\mathscr{F}'(\delta)|^q+973|\mathscr{F}'(\sigma)|^q}{72}\right)^{\frac{1}{q}}\right.\right.$$
$$\left.+\left(\frac{251|\mathscr{F}'(\sigma)|^q+973|\mathscr{F}'(\delta)|^q}{72}\right)^{\frac{1}{q}}\right]$$
$$\left.+\frac{1}{18}\left(\frac{|\mathscr{F}'(\sigma)|^q+|\mathscr{F}'(\delta)|^q}{2}\right)^{\frac{1}{q}}\right],$$

which is presented by [31, Remark 4].

Corollary 3.26. *Let us consider* $\lambda=0$ *in Theorem 3.21. Then, the following inequality holds:*

$$\left|\frac{1}{2}\left[\mathscr{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathscr{F}\left(\frac{2\sigma+\delta}{3}\right)\right]-\frac{\Gamma(\alpha+1)}{2(\delta-\sigma)^{\alpha}}\left[J_{\sigma+}^{\alpha}\mathscr{F}(\delta)+J_{\delta-}^{\alpha}\mathscr{F}(\sigma)\right]\right|$$
$$\leq\frac{(\delta-\sigma)}{2}\Big\{\left(A_1^{\alpha}(0,0)\right)^{1-\frac{1}{q}}\Big[\left(A_1^{\alpha}(s,0)|\mathscr{F}'(\delta)|^q+A_2^{\alpha}(s,0)|\mathscr{F}'(\sigma)|^q\right)^{\frac{1}{q}}$$
$$+\left(A_1^{\alpha}(s,0)|\mathscr{F}'(\sigma)|^q+A_2^{\alpha}(s,0)|\mathscr{F}'(\delta)|^q\right)^{\frac{1}{q}}\Big]$$
$$+\left(A_3^{\alpha}(0)\right)^{1-\frac{1}{q}}\Big[\left(A_3^{\alpha}(s)|\mathscr{F}'(\delta)|^q+A_4^{\alpha}(s)|\mathscr{F}'(\sigma)|^q\right)^{\frac{1}{q}}$$
$$+\left(A_3^{\alpha}(s)|\mathscr{F}'(\sigma)|^q+A_4^{\alpha}(s)|\mathscr{F}'(\delta)|^q\right)^{\frac{1}{q}}\Big]$$
$$+\left(A_5^{\alpha}(0,0)\right)^{1-\frac{1}{q}}\Big[\left(A_5^{\alpha}(s,0)|\mathscr{F}'(\delta)|^q+A_6^{\alpha}(s,0)|\mathscr{F}'(\sigma)|^q\right)^{\frac{1}{q}}$$
$$+\left(A_5^{\alpha}(s,0)|\mathscr{F}'(\sigma)|^q+A_6^{\alpha}(s,0)|\mathscr{F}'(\delta)|^q\right)^{\frac{1}{q}}\Big]\Big\}.$$

Corollary 3.27. *If we select* $\alpha=s=1$ *in Corollary 3.26, then we have*

$$\left|\frac{1}{2}\left[\mathscr{F}\left(\frac{\sigma+2\delta}{3}\right)+\mathscr{F}\left(\frac{2\sigma+\delta}{3}\right)\right]-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathscr{F}(t)\,dt\right|$$
$$\leq\frac{(\delta-\sigma)}{18}\left[\left(\frac{2|\mathscr{F}'(\delta)|^q+7|\mathscr{F}'(\sigma)|^q}{9}\right)^{\frac{1}{q}}+\left(\frac{2|\mathscr{F}'(\sigma)|^q+7|\mathscr{F}'(\delta)|^q}{9}\right)^{\frac{1}{q}}\right.$$

$$+\frac{1}{2}\left(\frac{|\mathscr{F}'(\sigma)|^q+|\mathscr{F}'(\delta)|^q}{2}\right)^{\frac{1}{q}}\Bigg].$$

4. Concluding Remarks

Some trapezoid and Newton type inequalities are established for differentiable s-convex functions involving Riemann-Liouville fractional integrals. Moreover, we also give some new and previously obtained trapezoid and Newton type inequalities with the help of special cases of our main results.

In the forthcoming works, one can be generalized our results by using various versions of convex function classes. The ideas and strategies for our results about trapezoid and Newton type inequalities by Riemann-Liouville fractional integrals may open new avenues for further research in this field. In addition, one can obtain likewise inequalities of trapezoid and Newton type inequalities via Riemann-Liouville fractional integrals for the case of differentiable s-convex functions with the help of the quantum calculus. Furthermore, one can apply these resulting inequalities to different types of fractional integrals.

Author contributions

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