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RETARDED NONLINEAR INTEGRAL INEQUALITIES OF GRONWALL-BELLMAN-PACHPATTE TYPE AND THEIR APPLICATIONS

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Abstract. In this article, we state and prove several new retarded nonlinear integral and integro-differential inequalities of Gronwall-Bellman-Pachpatte type. These inequalities generalize some former famous inequalities and can be used in examining the existence, uniqueness, boundedness, stability, asymptotic behaviour, quantitative and qualitative properties of solutions of nonlinear differential and integral equations. Applications are provided to demonstrate the strength of our inequalities in estimating the boundedness and global existence of the solution to initial value problem for nonlinear integro-differential equation and Volterra type retarded nonlinear equation. This research work will ensure to open the new opportunities for studying of nonlinear dynamic inequalities on time scale structure of varying nature.

1. Introduction

Integral inequalities have significant applications to the questions of existence, stability, boundedness, uniqueness, asymptotic behaviour, quantitative and qualitative properties of solutions of nonlinear differential and integral equations (such as [1-3] and references therein). These inequalities play a very important role in the study of integro-differential equations. Throughout this article, the set of real numbers is denoted by $\mathbb{R}$, where as $\mathbb{R}_1 = [0, \infty)$ is the subset of $\mathbb{R}$ and derivative is presented through $'$. Moreover the sets of all continuous functions and continuously differentiable functions from $\mathbb{R}_1$ into $\mathbb{R}_1$ are denoted by $\mathbb{J}(\mathbb{R}_1, \mathbb{R}_1)$ and $\mathbb{J}'(\mathbb{R}_1, \mathbb{R}_1)$, respectively.

We recall by introducing the famous inequality that has huge number of applications in the area of differential, integral and integro-differential equations.

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Theorem 1.1 (Gronwall inequality [4]). Let $x : [a, a+h] \to \mathbb{R}$ be a continuous function and

$$0 \le x(r) \le \int_a^r (b\, x(\mu) + c)\, d\mu, \qquad \forall\, r \in [a, a+h],$$

where a, h, b, and c are nonnegative constants. Then

$$0 \le x(r) \le ch\, \mathtt{exp}(bh), \qquad \forall\, r \in [a, a+h].$$

A very important generalization of above inequality is established by Bellman which is stated below:

Theorem 1.2 (Gronwall-Bellman inequality [5]). Let x and g be nonnegative continuous functions defined on the interval $[0, h]$, and suppose x_0 and h are positive constants for which the inequality

$$x(r) \le x_0 + \int_0^r g(\mu)x(\mu)d\mu, \qquad \forall\, r \in [0, h],$$

holds, then

$$x(r) \le x_0\, \mathtt{exp}\left(\int_0^r g(\mu)d\mu\right), \quad \forall\, r \in [0, h].$$

A huge number of useful generalizations of above inequalities are given by many mathematicians and scientists (see [6-17]). So, another following important generalization of above inequalities is given by Pachpatte which has many applications.

Theorem 1.3 (Pachpatte inequality [3]). Let $x,\ g_1,\ g_2\ \in \mathbb{J}(\mathbb{R}_1, \mathbb{R}_1)$ be nonnegative functions and x_0 be a positive constant for which the inequality

$$x(r) \le x_0 + \int_0^r g_1(\lambda)\Big(x(\lambda) + \int_0^\lambda g_2(\mu)x(\mu)d\mu\Big)d\lambda, \qquad \forall\, r \in \mathbb{R}_1,$$

holds. Then

$$x(r) \le x_0\Big(1 + \int_0^r g_1(\lambda)\mathtt{exp}\Big(\int_0^\lambda \big(g_1(\mu) + g_2(\mu)\big)d\mu\Big)d\lambda\Big), \qquad \forall\, r \in \mathbb{R}_1.$$

In 2020, Tian and Fan [18] established nonlinear integral inequality with power and gave its application in delay integro-differential equations while El-Deeb and Rashid [19] studied new double dynamic inequalities associated with Leibniz integral rule on time scales in 2021. Recent articles published on delay nonlinear dynamic inequalities of Gronwall-Bellman-Pachpatte type by [20-22] in 2022. The objective of this article is to establish several new retarded nonlinear integral and integro-differential inequalities of Gronwall-Bellman-Pachpatte type which will extend certain former famous inequalities in [3, 5, 6] that can be used to examine the existence, stability, boundedness, uniqueness, asymptotic behaviour, quantitative and qualitative properties of solutions of nonlinear differential and integral equations. This research work will ensure to open the new opportunities for studying of nonlinear dynamic inequalities on time scale structure of varying nature.

The rest of this article is organized as follows: Firstly, we present some new retarded nonlinear integral and integro-differential inequalities of Gronwall-Bellman-Pachpatte type with a differentiable function instead of a constant function outside the integral sign. Secondly, we give applications to demonstrate the strength of our inequalities in estimating the boundedness and global existence of the solution to initial value problem for nonlinear integro-differential equation and Volterra type retarded nonlinear equation. This study will be concluded at the end of this article.

2. Main Results

This Section begins with a generalization of a nonlinear retarded integral inequality of Gronwall-Bellman type presented in [5, 6], which can be used in estimating the boundedness and global existence of the solution to initial value problem for nonlinear integro-differential equation.

Theorem 2.1. Let $x,\ g_1,\ g_2,\ g_3 \in \mathbb{J}(\mathbb{R}_1, \mathbb{R}_1)$ be nonnegative functions and $l,\ \alpha \in \mathbb{J}'(\mathbb{R}_1, \mathbb{R}_1)$ be nondecreasing with $l(r) \geq 1$, $\alpha(r) \leq r$ on $\mathbb{R}_1$. If the inequality

$$x(r) \leq l(r) + \int_0^{\alpha(r)} g_1(\lambda)x(\lambda)d\lambda + \int_0^{\alpha(r)} g_2(\lambda)\Big(x^p(\lambda) + \int_0^{\lambda} g_3(\mu)x^q(\mu)d\mu\Big)^{\frac{1}{p}} d\lambda, \qquad \forall\ r \in \mathbb{R}_1, \tag{1}$$

holds for $p > q \geq 0$, then

$$x(r) \leq \Big[\frac{(p-q)}{p}\int_0^{\alpha(r)} g_3(\lambda)\mathtt{exp}\Big((p-q)\int_\lambda^{\alpha(r)}\Big(l'\big(\alpha^{-1}(\sigma)\big) + g_1(\sigma) + g_2(\sigma)\Big)d\sigma\Big)d\lambda + l^{p-q}(0)\mathtt{exp}\Big((p-q)\int_0^{\alpha(r)}\Big(l'\big(\alpha^{-1}(\lambda)\big) + g_1(\lambda) + g_2(\lambda)\Big)d\lambda\Big)\Big]^{\frac{1}{p-q}}, \qquad \forall\ r \in \mathbb{R}_1. \tag{2}$$

Proof. Let $y(r)$ be the right hand side of (1), then we note that $y(r)$ is a positive and nondecreasing function, $x\big(\alpha(r)\big) \leq y\big(\alpha(r)\big) \leq y(r)$ and $y(0) = l(0)$.

Differentiating $y(r)$, we obtain

$$\begin{aligned} y'(r) &= l'(r) + \alpha'(r) g_1\big(\alpha(r)\big) x\big(\alpha(r)\big) + \alpha'(r) g_2\big(\alpha(r)\big) \\ &\quad \times \Big(x^p(\alpha(r)) + \int_0^{\alpha(r)} g_3(\lambda) x^q(\lambda) d\lambda \Big)^{\frac{1}{p}} \\ &\le l'(r) + \alpha'(r) g_1\big(\alpha(r)\big) y(r) + \alpha'(r) g_2\big(\alpha(r)\big) w(r), \quad \forall\, r \in \mathbb{R}_1, \end{aligned} \tag{3}$$

where

$$w(r) = \Big(y^p(r) + \int_0^{\alpha(r)} g_3(\lambda) y^q(\lambda) d\lambda \Big)^{\frac{1}{p}}, \qquad \forall\, r \in \mathbb{R}_1,$$

or equivalently

$$w^p(r) = y^p(r) + \int_0^{\alpha(r)} g_3(\lambda) y^q(\lambda) d\lambda, \qquad \forall\, r \in \mathbb{R}_1. \tag{4}$$

Thus we have $x(r) \le y(r) \le w(r)$, and $w(0) = l(0)$. Now differentiating the equality (4) and using (3), we obtain

$$\begin{aligned} p w^{p-1}(r) w'(r) &= p y^{p-1}(r) y'(r) + \alpha'(r) g_3\big(\alpha(r)\big) y^q\big(\alpha(r)\big) \\ &\le p w^{p-1}(r) \big(l'(r) + \alpha'(r) g_1(\alpha(r)) w(r) + \alpha'(r) g_2(\alpha(r)) w(r) \big) \\ &\quad + \alpha'(r) g_3(\alpha(r)) w^q(r), \end{aligned}$$

$\forall\, r \in \mathbb{R}_1$. Dividing both sides by $p w^{p-1}(r)$, we get

$$\begin{aligned} w'(r) &\le l'(r) + \alpha'(r) g_1(\alpha(r)) w(r) + \alpha'(r) g_2(\alpha(r)) w(r) \\ &\quad + \frac{1}{p} \alpha'(r) g_3(\alpha(r)) w^{q+1-p}(r), \quad \forall\, r \in \mathbb{R}_1. \end{aligned}$$

If we let $v(r) = w^{p-q}(r)$, $v(0) = l^{p-q}(0)$ and $w'(r) = \frac{1}{p-q} v'(r) w^{q-p+1}(r)$, then above inequality can be written as

$$\begin{aligned} \frac{1}{p-q} v'(r) w^{q-p+1}(r) &\le l'(r) + \alpha'(r) g_1(\alpha(r)) w(r) + \alpha'(r) g_2(\alpha(r)) w(r) \\ &\quad + \frac{1}{p} \alpha'(r) g_3(\alpha(r)) w^{q+1-p}(r), \end{aligned}$$

$\forall\, r \in \mathbb{R}_1$. As $l(r) \ge 1$, $w(r) \ge 1$ which implies that $\frac{l'(r)}{w(r)} \le l'(r)$, so dividing the above inequality by $w^{q-p+1}(r)$, we have

$$\begin{aligned} & v'(r) - (p-q) \Big[l'(r) + \alpha'(r) \Big(g_1\big(\alpha(r)\big) + g_2\big(\alpha(r)\big) \Big) \Big] v(r) \\ & \qquad \le \frac{(p-q)}{p} \alpha'(r) g_3\big(\alpha(r)\big), \qquad \forall\, r \in \mathbb{R}_1. \end{aligned} \tag{5}$$

Applying integration from 0 to r on the above inequality, implies an estimation for $v(r)$ as follows:

$$\begin{aligned} v(r) \leq\ & \frac{(p-q)}{p}\int_0^{\alpha(r)} g_3(\lambda)\mathtt{exp}\Big((p-q)\int_\lambda^{\alpha(r)}\Big(l'\big(\alpha^{-1}(\sigma)\big)+g_1(\sigma) \\ & +g_2(\sigma)\Big)d\sigma\Big)d\lambda + l^{p-q}(0)\mathtt{exp}\Big((p-q)\int_0^{\alpha(r)}\Big(l'\big(\alpha^{-1}(\lambda)\big) \\ & +g_1(\lambda)+g_2(\lambda)\Big)d\lambda\Big), \qquad \forall\ r\in\mathbb{R}_1. \end{aligned} \tag{6}$$

Using $x(r) \leq y(r) \leq w(r)$ and $v(r) = w^{p-q}(r)$ in (6), we obtain required inequality (2). This completes the proof. □

Remark 2.2. We deduce the following famous inequalities by changing the given assumptions in Theorem 2.1:

1. If we take $l(r) = x_0$ (a constant), $g_2(r) = 0$ and $\alpha(r) = r$, then Theorem 2.1 is converted into the well known Gronwall-Bellman inequality [5].
2. When we suppose $l(r) = x_0$ (a constant), $g_1(r) = 0$, and $\alpha(r) = r$, then Theorem 2.1 reduced to Theorem 2.1 [6].

Before proceeding to the next result, we first state a lemma which will be helpful in the proof of upcoming results.

Lemma 2.3 [7]. Suppose that $a \geq 0$, $m \geq n \geq 0$ and $m \neq 0$, then

$$a^{\frac{n}{m}} \leq \frac{n}{m}K^{\frac{n-m}{m}}a + \frac{m-n}{m}K^{\frac{n}{m}},$$

for any $K > 0$.

Now, we state and prove another new nonlinear retarded integral inequality which will generalize the results in [3, 5, 6].

Theorem 2.4. Let $x,\ g_1,\ g_2,\ g_3 \in \mathbb{J}(\mathbb{R}_1,\mathbb{R}_1)$ be nonnegative functions and $l,\ \alpha \in \mathbb{J}'(\mathbb{R}_1,\mathbb{R}_1)$ be nondecreasing with $l(r) \geq 1$, $\alpha(r) \leq r$ on $\mathbb{R}_1$. If the inequality

$$\begin{aligned} x^p(r) \leq\ & l(r) + \int_0^{\alpha(r)} g_1(\lambda)x(\lambda)d\lambda + \int_0^{\alpha(r)} g_2(\lambda)\Big(x^p(\lambda) \\ & + \int_0^\lambda g_3(\mu)x^q(\mu)d\mu\Big)^{\frac{1}{p}}d\lambda, \qquad \forall\ r\in\mathbb{R}_1, \end{aligned} \tag{7}$$

holds for $p \geq q \geq 1$, then

$$\begin{aligned} x^p(r) \leq & \int_0^{\alpha(r)} \Big(l'\big(\alpha^{-1}(\lambda)\big) + \frac{p-1}{p} K^{\frac{1}{p}} \big(g_1(\lambda) + g_2(\lambda)\big) + \frac{p-q}{p} K^{\frac{q}{p}} g_3(\lambda) \Big) \\ & \times \mathtt{exp}\Big(\frac{1}{p} \int_\lambda^{\alpha(r)} \Big(K^{\frac{1-p}{p}} \big(g_1(\sigma) + g_2(\sigma)\big) + q K^{\frac{q-p}{p}} g_3(\sigma) \Big) d\sigma \Big) d\lambda \\ & + l(0) \mathtt{exp}\Big(\frac{1}{p} \int_0^{\alpha(r)} \Big(K^{\frac{1-p}{p}} \big(g_1(\lambda) + g_2(\lambda)\big) + q K^{\frac{q-p}{p}} g_3(\lambda) \Big) d\lambda \Big), \end{aligned} \tag{8}$$

$\forall\, r \in \mathbb{R}_1$, for any $K > 0$.

Proof. Let $y(r)$ be the right hand side of (7), then we note that $y(r)$ is nondecreasing and $x^p(r) \leq y(r)$, $x\big(\alpha(r)\big) \leq y^{\frac{1}{p}}\big(\alpha(r)\big) \leq y^{\frac{1}{p}}(r)$, $y(0) = l(0)$. After differentiating $y(r)$, we obtain

$$\begin{aligned} y'(r) &= l'(r) + \alpha'(r) g_1\big(\alpha(r)\big) x\big(\alpha(r)\big) + \alpha'(r) g_2\big(\alpha(r)\big) \\ & \quad \times \Big(x^p(\alpha(r)) + \int_0^{\alpha(r)} g_3(\lambda) x^q(\lambda) d\lambda \Big)^{\frac{1}{p}} \\ &\leq l'(r) + \alpha'(r) g_1\big(\alpha(r)\big) y^{\frac{1}{p}}(r) + \alpha'(r) g_2\big(\alpha(r)\big) w^{\frac{1}{p}}(r), \quad \forall\, r \in \mathbb{R}_1, \end{aligned} \tag{9}$$

where

$$w(r) = y(r) + \int_0^{\alpha(r)} g_3(\lambda) y^{\frac{q}{p}}(\lambda) d\lambda, \qquad \forall\, r \in \mathbb{R}_1. \tag{10}$$

Thus we have $x^p(r) \leq y(r) \leq w(r)$, and $w(0) = l(0)$. Now differentiating the equality (10) and using (9), we obtain

$$\begin{aligned} w'(r) &= y'(r) + \alpha'(r) g_3\big(\alpha(r)\big) y^{\frac{q}{p}}\big(\alpha(r)\big) \\ &\leq l'(r) + \alpha'(r)\Big(g_1\big(\alpha(r)\big) + g_2\big(\alpha(r)\big) \Big) w^{\frac{1}{p}}(r) + \alpha'(r) g_3\big(\alpha(r)\big) w^{\frac{q}{p}}(r), \end{aligned} \tag{11}$$

$\forall\, r \in \mathbb{R}_1$. With the help of Lemma 2.3, the inequality (11) can be written as

$$\begin{aligned} w'(r) &\leq l'(r) + \alpha'(r)\Big(g_1\big(\alpha(r)\big) + g_2\big(\alpha(r)\big) \Big) \big(\frac{1}{p} K^{\frac{1-p}{p}} w(r) + \frac{p-1}{p} K^{\frac{1}{p}} \big) \\ & \quad + \alpha'(r) g_3\big(\alpha(r)\big) \big(\frac{q}{p} K^{\frac{q-p}{p}} w(r) + \frac{p-q}{p} K^{\frac{q}{p}} \big) \\ &\leq l'(r) + \frac{\alpha'(r)}{p} \Big(K^{\frac{1-p}{p}} \big(g_1\big(\alpha(r)\big) + g_2\big(\alpha(r)\big) \big) + q K^{\frac{q-p}{p}} g_3\big(\alpha(r)\big) \Big) w(r) \\ & \quad + \frac{\alpha'(r)}{p} \Big((p-1) K^{\frac{1}{p}} \big(g_1\big(\alpha(r)\big) + g_2\big(\alpha(r)\big) \big) + (p-q) K^{\frac{q}{p}} g_3\big(\alpha(r)\big) \Big), \end{aligned}$$

$\forall\, r \in \mathbb{R}_1$. Rearranging the above inequality and integrating from 0 to r, implies an estimation for $w(r)$ as follows:

$$\begin{aligned} w(r) \;\le\; & \int\limits_0^{\alpha(r)} \Bigg(l'\big(\alpha^{-1}(\lambda)\big) + \frac{p-1}{p} K^{\frac{1}{p}} \big(g_1(\lambda)+g_2(\lambda)\big) + \frac{p-q}{p} K^{\frac{q}{p}} g_3(\lambda) \Bigg) \\ & \times \mathtt{exp}\Bigg(\frac{1}{p} \int\limits_{\lambda}^{\alpha(r)} \Big(K^{\frac{1-p}{p}} \big(g_1(\sigma)+g_2(\sigma)\big) + qK^{\frac{q-p}{p}} g_3(\sigma)\Big) d\sigma \Bigg) d\lambda \\ & + l(0)\mathtt{exp}\Bigg(\frac{1}{p} \int\limits_0^{\alpha(r)} \Big(K^{\frac{1-p}{p}} \big(g_1(\lambda)+g_2(\lambda)\big) + qK^{\frac{q-p}{p}} g_3(\lambda)\Big) d\lambda \Bigg), \end{aligned} \tag{12}$$

$\forall\, r \in \mathbb{R}_1$. Using $x^p(r) \le y(r) \le w(r)$ in (12), we obtain required inequality (8). Proof is completed. □

Remark 2.5. We observe that Theorem 2.4 generalizes the inequalities [3, 5, 6] as follows:

1. If we put $l(r) = x_0$ (a constant), $g_1(r) = 0$, $\alpha(r) = r$ and $p = q = 1$, then Theorem 1.3 [3] is obtained.
2. If we take $l(r) = x_0$ (a constant), $g_2(r) = 0$, $\alpha(r) = r$ and $p = 1$, then Theorem 2.4 becomes the well known Gronwall-Bellman inequality [5].
3. When we consider $l(r) = x_0$ (a constant), $g_1(r) = 0$, and $\alpha(r) = r$, then Theorem 2.4 is reduced to Theorem 2.2 [6].

Here, we give another new nonlinear retarded integral inequality of Gronwall-Bellman-Pachpatte type, which can be used in analyzing the boundedness and global existence of the solution to Volterra type retarded nonlinear differential equations.

Theorem 2.6. Let $x,\ g_1,\ g_2,\ g_3 \in \mathbb{J}(\mathbb{R}_1, \mathbb{R}_1)$ be nonnegative functions and $l,\ \alpha \in \mathbb{J}'(\mathbb{R}_1, \mathbb{R}_1)$ be nondecreasing with $l(r) \ge 1$, $\alpha(r) \le r$ on $\mathbb{R}_1$. If the inequality

$$\begin{aligned} x^p(r) \;\le\; & l(r) + \int\limits_0^{\alpha(r)} g_1(\lambda) x(\lambda) d\lambda + \int\limits_0^{\alpha(r)} g_2(\lambda) \Big(x^p(\lambda) \\ & + \int\limits_0^{\lambda} g_3(\mu) x^q(\mu) d\mu \Big)^{\frac{1}{q}} d\lambda, \qquad \forall\, r \in \mathbb{R}_1, \end{aligned} \tag{13}$$

holds for $p \geq q \geq 1$, then

$$x^p(r) \leq \int_0^{\alpha(r)} \Big(l'\big(\alpha^{-1}(\lambda)\big) + \frac{p-1}{p}K^{\frac{1}{p}}g_1(\lambda) + \frac{q-1}{q}K^{\frac{1}{q}}g_2(\lambda) + \frac{p-q}{p}K^{\frac{q}{p}}g_3(\lambda)\Big)$$
$$\times \mathtt{exp}\left(\int_\lambda^{\alpha(r)} \Big(\frac{1}{p}K^{\frac{1-p}{p}}g_1(\sigma) + \frac{1}{q}K^{\frac{1-q}{q}}g_2(\sigma) + \frac{q}{p}K^{\frac{q-p}{p}}g_3(\sigma)\Big)d\sigma\right)d\lambda$$
$$+l(0)\mathtt{exp}\left(\int_0^{\alpha(r)} \Big(\frac{1}{p}K^{\frac{1-p}{p}}g_1(\lambda) + \frac{1}{q}K^{\frac{1-q}{q}}g_2(\lambda) + \frac{q}{p}K^{\frac{q-p}{p}}g_3(\lambda)\Big)d\lambda\right), \tag{14}$$

$\forall\ r \in \mathbb{R}_1$, for any $K > 0$.

Proof. Let $y(r)$ be the right hand side of (13), then we note that $y(r)$ is nondecreasing and $x^p(r) \leq y(r)$, $x\big(\alpha(r)\big) \leq y^{\frac{1}{p}}\big(\alpha(r)\big) \leq y^{\frac{1}{p}}(r)$, $y(0) = l(0)$. Differentiating $y(r)$, we obtain

$$y'(r) = l'(r) + \alpha'(r)g_1\big(\alpha(r)\big)x\big(\alpha(r)\big) + \alpha'(r)g_2\big(\alpha(r)\big)$$
$$\times\Big(x^p(\alpha(r)) + \int_0^{\alpha(r)} g_3(\lambda)x^q(\lambda)d\lambda\Big)^{\frac{1}{q}}$$
$$\leq l'(r) + \alpha'(r)g_1\big(\alpha(r)\big)y^{\frac{1}{p}}(r) + \alpha'(r)g_2\big(\alpha(r)\big)w^{\frac{1}{q}}(r), \quad \forall\ r \in \mathbb{R}_1, \tag{15}$$

where

$$w(r) = y(r) + \int_0^{\alpha(r)} g_3(\lambda)y^{\frac{q}{p}}(\lambda)d\lambda, \qquad \forall\ r \in \mathbb{R}_1. \tag{16}$$

Thus we have $x^p(r) \leq y(r) \leq w(r)$, and $w(0) = l(0)$. Now differentiating the equality (16) and using (15), we obtain

$$w'(r) = y'(r) + \alpha'(r)g_3\big(\alpha(r)\big)y^{\frac{q}{p}}\big(\alpha(r)\big)$$
$$\leq l'(r) + \alpha'(r)g_1\big(\alpha(r)\big)w^{\frac{1}{p}}(r) + \alpha'(r)g_2\big(\alpha(r)\big)w^{\frac{1}{q}}(r)$$
$$+\alpha'(r)g_3\big(\alpha(r)\big)w^{\frac{q}{p}}(r), \qquad \forall\ r \in \mathbb{R}_1.$$

With the help of Lemma 2.3, the above inequality can be written as

$$w'(r) \leq l'(r) + \alpha'(r)g_1\big(\alpha(r)\big)\Big(\frac{1}{p}K^{\frac{1-p}{p}}w(r) + \frac{p-1}{p}K^{\frac{1}{p}}\Big) + \alpha'(r)g_2\big(\alpha(r)\big)$$
$$\times\Big(\frac{1}{q}K^{\frac{1-q}{q}}w(r) + \frac{q-1}{q}K^{\frac{1}{q}}\Big) + \alpha'(r)g_3\big(\alpha(r)\big)\Big(\frac{q}{p}K^{\frac{q-p}{p}}w(r) + \frac{p-q}{p}K^{\frac{q}{p}}\Big)$$
$$\leq l'(r) + \alpha'(r)\Big(\frac{1}{p}K^{\frac{1-p}{p}}g_1\big(\alpha(r)\big) + \frac{1}{q}K^{\frac{1-q}{q}}g_2\big(\alpha(r)\big) + \frac{q}{p}K^{\frac{q-p}{p}}g_3\big(\alpha(r)\big)\Big)w(r)$$
$$+\alpha'(r)\Big(\frac{p-1}{p}K^{\frac{1}{p}}g_1\big(\alpha(r)\big) + \frac{q-1}{q}K^{\frac{1}{q}}g_2\big(\alpha(r)\big) + \frac{p-q}{p}K^{\frac{q}{p}}g_3\big(\alpha(r)\big)\Big),$$

$\forall\ r \in \mathbb{R}_1$. Rearranging the above inequality and integrating from 0 to r gives an estimation for $w(r)$ as follows:

$$\begin{aligned} w(r) \leq & \int_0^{\alpha(r)} \Big(l'\big(\alpha^{-1}(\lambda)\big) + \frac{p-1}{p} K^{\frac{1}{p}} g_1(\lambda) + \frac{q-1}{q} K^{\frac{1}{q}} g_2(\lambda) + \frac{p-q}{p} K^{\frac{q}{p}} g_3(\lambda) \Big) \\ & \times \texttt{exp}\Bigg(\int_\lambda^{\alpha(r)} \Big(\frac{1}{p} K^{\frac{1-p}{p}} g_1(\sigma) + \frac{1}{q} K^{\frac{1-q}{q}} g_2(\sigma) + \frac{q}{p} K^{\frac{q-p}{p}} g_3(\sigma) \Big) d\sigma \Bigg) d\lambda \\ & + l(0)\texttt{exp}\Bigg(\int_0^{\alpha(r)} \Big(\frac{1}{p} K^{\frac{1-p}{p}} g_1(\lambda) + \frac{1}{q} K^{\frac{1-q}{q}} g_2(\lambda) + \frac{q}{p} K^{\frac{q-p}{p}} g_3(\lambda) \Big) d\lambda \Bigg), \end{aligned} \tag{17}$$

$\forall\ r \in \mathbb{R}_1$. Employing $x^p(r) \leq y(r) \leq w(r)$ in (17), we obtain desired inequality (14). This completes the proof. □

Remark 2.7. We deduce the following famous inequalities by changing the given assumptions in Theorem 2.6:

1. If we put $l(r) = x_0$ (a constant), $g_1(r) = 0$, $\alpha(r) = r$ and $p = q = 1$, then Theorem 2.6 yields Theorem 1.3 [3].
2. Considering $l(r) = x_0$ (a constant), $g_2(r) = 0$, $\alpha(r) = r$ and $p = 1$, show that Theorem 2.6 is reduced to Gronwall-Bellman inequality [5].
3. When we replace $l(r) = x_0$ (a constant), $g_1(r) = 0$, and $\alpha(r) = r$, then Theorem 2.6 implies Theorem 2.3 [6].

Now, we present new nonlinear retarded integro-differential inequality, which can be used in analyzing the boundedness and global existence of the solution to Volterra type retarded nonlinear integro-differential equation.

Theorem 2.8. Let $x,\ x'\ g_1,\ g_2,\ g_3 \in \mathbb{J}(\mathbb{R}_1, \mathbb{R}_1)$ be nonnegative functions and $l,\ \alpha \in \mathbb{J}'(\mathbb{R}_1, \mathbb{R}_1)$ be nondecreasing with $l(r) \geq 1$, $\alpha(r) \leq r$ on $\mathbb{R}_1$ and $x(0) = 0$. If the inequality

$$\begin{aligned} x'(r) \leq & \ l(r) + \int_0^{\alpha(r)} g_1(\lambda) x(\lambda) d\lambda + \int_0^{\alpha(r)} g_2(\lambda) \Big(x^p(\lambda) \\ & + \int_0^\lambda g_3(\mu) x^q(\mu) d\mu \Big)^{\frac{1}{p}} d\lambda, \qquad \forall\ r \in \mathbb{R}_1, \end{aligned} \tag{18}$$

holds for $p > q \geq 0$, then

$$\begin{aligned} x(r) \leq & \Bigg[\frac{p-q}{p} \int_0^{\alpha(r)} g_3(\lambda) \texttt{exp}\Bigg((p-q) \int_\lambda^{\alpha(r)} \Big(\alpha^{-1}(\sigma) \Big(l'\big(\alpha^{-1}(\sigma)\big) + g_1(\sigma) \\ & + g_2(\sigma) \Big) + \frac{1}{\alpha^{-1}(\sigma)} \Big) d\sigma \Bigg) d\lambda \Bigg]^{\frac{1}{p-q}}, \qquad \forall\ r \in \mathbb{R}_1. \end{aligned} \tag{19}$$

Proof. Let $y(r)$ be the right hand side of (18), then we note that $y(r)$ is nondecreasing and $x(r) \le ry(r)$, $x'(r) \le y(r)$, $y(0) = l(0)$. After differentiating $y(r)$, we obtain

$$\begin{aligned} y'(r) &= l'(r) + \alpha'(r) g_1\big(\alpha(r)\big) x\big(\alpha(r)\big) + \alpha'(r) g_2\big(\alpha(r)\big) \\ &\quad \times \Big(x^p(\alpha(r)) + \int_0^{\alpha(r)} g_3(\lambda) x^q(\lambda) d\lambda \Big)^{\frac{1}{p}} \\ &\le l'(r) + \alpha'(r) g_1\big(\alpha(r)\big) r y(r) + \alpha'(r) g_2\big(\alpha(r)\big) w(r), \qquad \forall\, r \in \mathbb{R}_1, \end{aligned} \tag{20}$$

where

$$w(r) = \Big(r^p y^p(r) + \int_0^{\alpha(r)} g_3(\lambda) \lambda^q y^q(\lambda) d\lambda \Big)^{\frac{1}{p}}, \qquad \forall\, r \in \mathbb{R}_1,$$

or equivalently

$$w^p(r) = r^p y^p(r) + \int_0^{\alpha(r)} g_3(\lambda) \lambda^q y^q(\lambda) d\lambda, \qquad \forall\, r \in \mathbb{R}_1. \tag{21}$$

Thus we have $x(r) \le ry(r) \le w(r)$, and $w(0) = 0$. Now differentiating the equality (21) and using (20), we obtain

$$\begin{aligned} p w^{p-1}(r) w'(r) &= p r^p y^{p-1}(r) y'(r) + p r^{p-1} y^p(r) + \alpha'(r) g_3\big(\alpha(r)\big) \big(\alpha(r)\big)^q y^q\big(\alpha(r)\big) \\ &\le p r w^{p-1}(r) \Big(l'(r) + \alpha'(r) g_1(\alpha(r)) w(r) + \alpha'(r) g_2(\alpha(r)) w(r) \Big) \\ &\quad + p r^{-1} w^p(r) + \alpha'(r) g_3(\alpha(r)) w^q(r), \qquad \forall\, r \in \mathbb{R}_1. \end{aligned}$$

Dividing both sides by $pw^{p-1}(r)$, we get

$$\begin{aligned} w'(r) &\le r l'(r) + \Big(\alpha'(r) r g_1(\alpha(r)) + \alpha'(r) r g_2(\alpha(r)) + \frac{1}{r} \Big) w(r) \\ &\quad + \frac{1}{p} \alpha'(r) g_3(\alpha(r)) w^{q+1-p}(r), \qquad \forall\, r \in \mathbb{R}_1. \end{aligned}$$

If we let $v(r) = w^{p-q}(r)$, $v(0) = 0$ and $w'(r) = \frac{1}{p-q} v'(r) w^{q-p+1}(r)$, then above inequality can be written as

$$\begin{aligned} \frac{1}{p-q} v'(r) w^{q-p+1}(r) &\le r l'(r) + \Big(\alpha'(r) r g_1(\alpha(r)) + \alpha'(r) r g_2(\alpha(r)) + \frac{1}{r} \Big) w(r) \\ &\quad + \frac{1}{p} \alpha'(r) g_3(\alpha(r)) w^{q+1-p}(r), \qquad \forall\, r \in \mathbb{R}_1. \end{aligned}$$

As $l(r) \geq 1$, $w(r) \geq 1$ which implies that $\frac{l'(r)}{w(r)} \leq l'(r)$, so dividing the above inequality by $w^{q-p+1}(r)$, we have

$$\begin{aligned} v'(r) \quad &\leq \quad (p-q)\Big(rl'(r) + \alpha'(r)\Big(rg_1(\alpha(r)) + rg_2(\alpha(r))\Big) + \frac{1}{r}\Big)v(r) \\ &\quad + \frac{p-q}{p}\alpha'(r)g_3\big(\alpha(r)\big), \qquad \forall\, r \in \mathbb{R}_1. \end{aligned}$$

Rearranging the above inequality and integrating from 0 to r implies an estimation for $v(r)$ as follows:

$$\begin{aligned} v(r) \quad &\leq \quad \frac{p-q}{p}\int_0^{\alpha(r)} g_3(\lambda)\texttt{exp}\Big((p-q)\int_\lambda^{\alpha(r)} \Big(\alpha^{-1}(\sigma)\Big(l'\big(\alpha^{-1}(\sigma)\big) + g_1(\sigma) \\ &\quad + g_2(\sigma)\Big) + \frac{1}{\alpha^{-1}(\sigma)}\Big)d\sigma\Big)d\lambda, \qquad \forall\, r \in \mathbb{R}_1. \end{aligned} \tag{22}$$

Employing $x(r) \leq ry(r) \leq w(r)$ and $v(r) = w^{p-q}(r)$ in (22), we obtain required inequality (19). This completes the proof. □

At the end of this Section, we present following new nonlinear retarded integro-differential inequality.

Theorem 2.9. Let $x,\ x'\ g_1,\ g_2,\ g_3 \in \mathbb{J}(\mathbb{R}_1, \mathbb{R}_1)$ be nonnegative functions and $l,\ \alpha \in \mathbb{J}'(\mathbb{R}_1, \mathbb{R}_1)$ be nondecreasing with $l(r) \geq 1$, $\alpha(r) \leq r$ on $\mathbb{R}_1$ and $x(0) = 0$. If the inequality

$$\begin{aligned} x'(r) \quad &\leq \quad l(r) + \int_0^{\alpha(r)} g_1(\lambda)x(\lambda)d\lambda + \int_0^{\alpha(r)} g_2(\lambda)\Big(x'(\lambda) \\ &\quad + \int_0^{\lambda} g_3(\mu)x(\mu)d\mu\Big)^{\frac{1}{p}}d\lambda, \qquad \forall\, r \in \mathbb{R}_1, \end{aligned} \tag{23}$$

holds for $p \geq 1$, then

$$\begin{aligned} x'(r) \quad &\leq \quad \int_0^{\alpha(r)} \Big(l'\big(\alpha^{-1}(\lambda)\big) + \frac{p-1}{p}K^{\frac{1}{p}}g_2(\lambda)\Big)\texttt{exp}\Big(\int_\lambda^{\alpha(r)} \Big(\alpha^{-1}(\sigma)\Big(g_1(\sigma) \\ &\quad + g_3(\sigma)\Big) + \frac{1}{p}K^{\frac{1-p}{p}}g_2(\sigma)\Big)d\sigma\Big)d\lambda + l(0)\texttt{exp}\Big(\int_0^{\alpha(r)} \Big(\alpha^{-1}(\lambda) \\ &\quad \times\Big(g_1(\lambda) + g_3(\lambda)\Big) + \frac{1}{p}K^{\frac{1-p}{p}}g_2(\lambda)\Big)d\lambda\Big), \qquad \forall\, r \in \mathbb{R}_1, \end{aligned} \tag{24}$$

for any $K > 0$.

Proof. Let $y(r)$ be the right hand side of (23), then we note that $y(r)$ is nondecreasing and $x(r) \leq ry(r)$, $x'(r) \leq y(r)$, $y(0) = l(0)$. After differentiating $y(r)$, we obtain

$$\begin{aligned} y'(r) &= l'(r) + \alpha'(r)g_1\big(\alpha(r)\big)x\big(\alpha(r)\big) + \alpha'(r)g_2\big(\alpha(r)\big) \\ &\quad \times\Big(x'(\alpha(r)) + \int_0^{\alpha(r)} g_3(\lambda)x(\lambda)d\lambda\Big)^{\frac{1}{p}} \\ &\leq l'(r) + \alpha'(r)g_1\big(\alpha(r)\big)ry(r) + \alpha'(r)g_2\big(\alpha(r)\big)w^{\frac{1}{p}}(r), \qquad \forall\, r \in \mathbb{R}_1, \end{aligned} \tag{25}$$

where

$$w(r) = y(r) + \int_0^{\alpha(r)} g_3(\lambda)\lambda y(\lambda)d\lambda, \qquad \forall\, r \in \mathbb{R}_1. \tag{26}$$

Thus we have $x'(r) \leq y(r) \leq w(r)$, and $w(0) = y(0) = l(0)$. Now differentiating the equality (26) and using (25), we obtain

$$\begin{aligned} w'(r) &= y'(r) + \alpha'(r)g_3\big(\alpha(r)\big)\alpha(r)y\big(\alpha(r)\big) \\ &\leq l'(r) + \alpha'(r)g_1\big(\alpha(r)\big)rw(r) + \alpha'(r)g_2\big(\alpha(r)\big)w^{\frac{1}{p}}(r) \\ &\quad +\alpha'(r)g_3\big(\alpha(r)\big)rw(r), \qquad \forall\, r \in \mathbb{R}_1. \end{aligned}$$

With the help of Lemma 2.3, we have

$$\begin{aligned} w'(r) &\leq l'(r) + \alpha'(r)g_1\big(\alpha(r)\big)rw(r) + \alpha'(r)g_2\big(\alpha(r)\big)\Big(\frac{1}{p}K^{\frac{1-p}{p}}w(r) \\ &\quad +\frac{p-1}{p}K^{\frac{1}{p}}\Big) + \alpha'(r)g_3\big(\alpha(r)\big)rw(r) \\ &\leq l'(r) + \alpha'(r)\Big(rg_1\big(\alpha(r)\big) + \frac{1}{p}K^{\frac{1-p}{p}}g_2\big(\alpha(r)\big) + rg_3\big(\alpha(r)\big)\Big)w(r) \\ &\quad +\alpha'(r)\Big(\frac{p-1}{p}K^{\frac{1}{p}}g_2\big(\alpha(r)\big)\Big), \qquad \forall\, r \in \mathbb{R}_1. \end{aligned}$$

Rearranging the above inequality and integrating from 0 to r gives an estimation for $w(r)$ as follows:

$$\begin{aligned} w(r) &\leq \int_0^{\alpha(r)} \Big(l'\big(\alpha^{-1}(\lambda)\big) + \frac{p-1}{p}K^{\frac{1}{p}}g_2(\lambda)\Big)\mathtt{exp}\Big(\int_\lambda^{\alpha(r)} \Big(\alpha^{-1}(\sigma)\Big(g_1(\sigma) \\ &\quad +g_3(\sigma)\Big) + \frac{1}{p}K^{\frac{1-p}{p}}g_2(\sigma)\Big)d\sigma\Big)d\lambda + l(0)\mathtt{exp}\Big(\int_0^{\alpha(r)} \Big(\alpha^{-1}(\lambda) \\ &\quad \times\Big(g_1(\lambda) + g_3(\lambda)\Big) + \frac{1}{p}K^{\frac{1-p}{p}}g_2(\lambda)\Big)d\lambda\Big), \qquad \forall\, r \in \mathbb{R}_1. \end{aligned} \tag{27}$$

Using the inequality $x'(r) \leq y(r) \leq w(r)$ in (27), we obtain required inequality (24). This completes the proof. □

3. Applications

In some situations, the bounds and existence of solution to the initial value problem of nonlinear integro-differential equations given by the other inequalities are not directly applicable. Also, it is not possible to examine the stability and asymptotic behavior of solutions of classes of more general retarded nonlinear differential and integral equations. However, the integral inequalities established in this article allow us to study the global existence, uniqueness, stability, boundedness and asymptotic behavior and other properties of solutions of classes of more general retarded nonlinear differential and integral equations. In this section, we shall discuss the existence and boundedness behavior of solution of certain nonlinear differential and integral equations.

Consider the following initial value problem for nonlinear differential equation:

$$\begin{cases} x'(r) = l'(r) + H_1(r,x) + M\Big(r, x(r), H_2(r,x)\Big), & \forall\, r \in \mathbb{R}_1, \\ x(0) = l(0), \end{cases} \tag{28}$$

where $l(0) \neq 0$ is a constant, $M \in \mathbb{J}(\mathbb{R}_1^3, \mathbb{R})$, $H_i \in \mathbb{J}(\mathbb{R}_1^2, \mathbb{R})$, for $i = 1, 2$ satisfy the following conditions

$$|H_1(r,x)| \leq g_1(r)|x(r)|, \tag{29}$$

$$|H_2(r,x)| \leq g_3(r)|x(r)|^q, \tag{30}$$

$$|M(r,x,H_2)| \leq g_2(r)\Big(|x(r)|^p + \int_0^r |H_2(\lambda, x(\lambda))|d\lambda\Big)^{\frac{1}{p}}, \tag{31}$$

where g_1, g_2 and g_3 are nonnegative continuous functions on $\mathbb{R}_1$, and $p > q \geq 0$.

Corollary 3.1. Consider the initial value problem for nonlinear differential equation (28) and suppose that H_1, H_2 and M satisfy the conditions (29), (30) and (31) respectively. Then all solutions of (28) exist and are bounded on $\mathbb{R}_1$.

Proof. Integrating (28) from 0 to r, we obtain

$$x(r) = l(r) + \int_0^r H_1(\lambda, x)d\lambda + \int_0^r M\Big(\lambda, x(\lambda), H_2(\lambda, x)\Big)d\lambda, \forall r \in \mathbb{R}_1. \tag{32}$$

Using (29), (30) and (31) in (32), we get

$$\begin{aligned}|x(r)| &\leq |l(r)| + \int_0^r g_1(\lambda)|x(\lambda)|d\lambda + \int_0^r g_2(\lambda)\Big(|x(\lambda)|^p + \int_0^\lambda g_3(\mu)|x(\mu)|^q d\mu\Big)^{\frac{1}{p}} d\lambda \\ &\leq |l(r)| + \int_0^{\alpha(r)} \frac{g_1(\alpha^{-1}(\lambda))}{\alpha'(\alpha^{-1}(\lambda))}|x(\lambda)|d\lambda + \int_0^{\alpha(r)} \frac{g_2(\alpha^{-1}(\lambda))}{\alpha'(\alpha^{-1}(\lambda))}\Big(|x(\lambda)|^p \\ &\quad + \int_0^\lambda g_3(\mu)|x(\mu)|^q d\mu\Big)^{\frac{1}{p}} d\lambda, \qquad \forall\, r \in \mathbb{R}_1.\end{aligned}$$

If above inequality holds for $p > q \geq 0$, then as an application of Theorem 2.1, we obtain

$$\begin{aligned}|x(r)| &\leq \Bigg[\frac{(p-q)}{p}\int_0^{\alpha(r)} \frac{1}{\alpha'(\alpha^{-1}(\lambda))} g_3(\lambda)\mathtt{exp}\Big(\int_\lambda^{\alpha(r)} \frac{(p-q)}{\alpha'(\alpha^{-1}(\lambda))}\Big(l'\big(\alpha^{-1}(\sigma)\big) \\ &\quad + g_1(\sigma) + g_2(\sigma)\Big)d\sigma\Big)d\lambda + |l^{p-q}(0)|\mathtt{exp}\Big((p-q)\int_0^{\alpha(r)} \frac{1}{\alpha'(\alpha^{-1}(\lambda))} \\ &\quad \times\Big(l'\big(\alpha^{-1}(\lambda)\big) + g_1(\lambda) + g_2(\lambda)\Big)d\lambda\Big)\Bigg]^{\frac{1}{p-q}}, \qquad \forall\, r \in \mathbb{R}_1.\end{aligned}$$

This shows that the solution of the system (28) exists and is bounded on $\mathbb{R}_1$. Proof is completed. □

Now we consider the following Volterra type retarded nonlinear equation which arises very often in various problems such as describing physical processes with after effects:

$$\begin{aligned}x^5(r) &= l(r) + \int_0^{\alpha(r)} H_1(\lambda, x(\lambda))d\lambda + \int_0^{\alpha(r)} M\Big(\lambda, \big(x^5(\lambda), \\ &\quad \int_0^\lambda H_2(\sigma, x^4(\sigma))d\sigma\big)^{\frac{1}{4}}\Big)d\lambda, \qquad \forall\, r \in \mathbb{R}_1, \qquad (33)\end{aligned}$$

where $l(r) = r^2$ is nondecreasing on $\mathbb{R}_1$, $M \in \mathbb{J}(\mathbb{R}_1^3, \mathbb{R})$, $H_i \in \mathbb{J}(\mathbb{R}_1^2, \mathbb{R})$, for $i = 1, 2$ and satisfy the following conditions

$$H_1(r, x) \leq e^r x(r), \qquad \forall\, r \in \mathbb{R}_1, \qquad (34)$$

$$H_2(r, x^4) \leq \sin r x^4(r), \qquad \forall\, r \in \mathbb{R}_1, \qquad (35)$$

$$M\Big(r, \big(x^5, \int_0^r H_2 d\lambda\big)^{\frac{1}{4}}\Big) \leq \cos r\Big(x^5(r) + \int_0^r H_2 d\lambda\Big)^{\frac{1}{4}}, \quad \forall\, r \in \mathbb{R}_1. \tag{36}$$

Corollary 3.2. Consider the Volterra type retarded nonlinear equation (33) and suppose that H_1, H_2 and M satisfy the conditions (34), (35) and (36) respectively. Then we examine the boundedness and existence on $\mathbb{R}_1$ of the solution of (33).

Proof. By using (34), (35) and (36) in (33), we obtain

$$x^5(r) \leq r^2 + \int_0^{\alpha(r)} e^{\lambda} x(\lambda) d\lambda + \int_0^{\alpha(r)} \cos\lambda\Big(x^5(\lambda) + \int_0^{\lambda} \sin\sigma x^4(\sigma) d\sigma\Big)^{\frac{1}{4}} d\lambda, \tag{37}$$

$\forall\, r \in \mathbb{R}_1$. The inequality (37) is the particular form of (13), and also inequality (37) satisfy all the conditions of Theorem 2.6. So, as an application of Theorem 2.6, we obtain

$$\begin{aligned} x^5(r) \quad &\leq \int_0^{\alpha(r)} \Big(2\alpha^{-1}(\lambda) + \frac{4}{5}K^{\frac{1}{5}} e^{\lambda} + \frac{3}{4}K^{\frac{1}{4}} \cos\lambda + \frac{1}{5}K^{\frac{4}{5}} \sin\lambda\Big) \\ &\times \mathtt{exp}\Bigg(\int_{\lambda}^{\alpha(r)} \Big(\frac{1}{5}K^{\frac{-4}{5}} e^{\sigma} + \frac{1}{4}K^{\frac{-3}{4}} \cos\sigma + \frac{4}{5}K^{\frac{-1}{5}} \sin\sigma\Big) d\sigma\Bigg) d\lambda, \end{aligned} \tag{38}$$

$\forall\, r \in \mathbb{R}_1$, for any $K > 0$.

In particularly, if $K = 1$, then from (38), we get

$$\begin{aligned} x^5(r) \quad &\leq \int_0^{\alpha(r)} \Big(2\alpha^{-1}(\lambda) + \frac{4}{5}e^{\lambda} + \frac{3}{4}\cos\lambda + \frac{1}{5}\sin\lambda\Big) \\ &\times \mathtt{exp}\Bigg(\int_{\lambda}^{\alpha(r)} \Big(\frac{1}{5}e^{\sigma} + \frac{1}{4}\cos\sigma + \frac{4}{5}\sin\sigma\Big) d\sigma\Bigg) d\lambda, \ \forall\, r \in \mathbb{R}_1, \end{aligned} \tag{39}$$

which gives boundedness and global existence for x on $\mathbb{R}_1$. This completes the proof. □

4. Conclusion

Retarded nonlinear integral and integro-differential inequalities of Gronwall-Bellman-Pachpatte type are studied in this research work, and many fresh and existing famous inequalities might be achieved by taking appropriate selection of parameters. Moreover, the integral inequalities established in this article permit us to analyze the existence, uniqueness, stability, boundedness and asymptotic behavior and other properties of solutions of classes of more general retarded nonlinear differential and integral equations. Many renowned and existing important special cases can be explored on the basis of different choices of parameters (see remarks 2.2, 2.5 and 2.7) from our integral inequalities of this article. So, these inequalities can handle the problems of nonlinear partial differential equations in applied.

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