Honam Mathematical J. **45** (2023), No. 1, pp. 1–24
https://doi.org/10.5831/HMJ.2023.45.1.1

GENERALISED COMMON FIXED POINT THEOREM FOR WEAKLY COMPATIBLE MAPPINGS VIA IMPLICIT CONTRACTIVE RELATION IN QUASI-PARTIAL S_b-METRIC SPACE WITH SOME APPLICATIONS

Lucas Wangwe and Santosh Kumar*

Abstract. In the present paper, we prove common fixed point theorems for a pair of weakly compatible mappings under implicit contractive relation in quasi-partial S_b-metric spaces. We also provide an illustrative example to support our results. Furthermore, we will use the results obtained for application to two boundary value problems for the second-order differential equation. Also, we prove a common solution for the nonlinear fractional differential equation.

1. Introduction

In 1906, Fréchet [20] introduced the study of sets of elements in abstract spaces. In 1922, Banach [10] proved a fixed point theorem using the concept of abstract spaces. This theorem gave an iterative procedure to find the fixed point and is famously known as the Banach contraction principle. The Banach contraction principle has several applications in nonlinear analysis and pure and applied mathematics. Researchers have generalised these results by refining the contraction conditions and replacing metric spaces with a more generalised abstract space.

1976, Jungck [33] initiated the concept of commuting mappings and proved fixed points results in metric space. Jungck [34] extended the concept of commuting mapping to compatible mappings and proved common fixed points results on metric spaces. Sessa [58] proved the results on a weak commutativity condition of mappings in fixed point considerations. Kaneko and Sessa [35] extended the concept of compatible mappings due to Jungck [33] to include multi-valued mappings as well as single-valued mappings. Moreover,

Received August 2, 2021. Accepted September 27, 2022.
2020 Mathematics Subject Classification. 47H10, 54H25.
Key words and phrases. common fixed point, quasi-partial S_b-metric space, implicit relation, weakly compatible mapping, an integral equation, fractional differential equation.
*Corresponding author

they proved coincidence and fixed point theorems for hybrid pairs of compatible mappings. For more literature, we refer the reader to [1, 25, 47, 35, 61] and the reference cited therein.

Similarly, Czerwik [17] established b-metric spaces by weakening the triangle inequality coefficient and generalising Banach's contraction principle to these spaces. Since then, several papers have been published on the fixed point theory of various classes of the single and multi-valued map in b-metric space. Aydi *et al.* [8] proved a common fixed points via implicit contractions on b-metric-like spaces. Aydi *et al.* [7] proved a fixed point theorem for set-valued quasi-contractions in b-metric spaces. For more details, one can see in [16, 38, 39, 41, 56] and the references therein.

Likewise, Matthew [42] introduced non-zero self-distance, which is applied in computer networking, data structure, and computer programming languages. The non-self distance generalises the metric to partial metric axioms, accommodating both metric and topological properties of abstract spaces. Some of these properties are complete spaces, Cauchy sequences and contraction fixed point theorem, which generalises the Banach contraction principle.

On the other hand, Popa [53] introduced the concept of implicit functions and proved the results for contractive mapping, whose strength lies in producing many contractions. Several researchers are working in this area. For more details, we refer the readers to [3, 4, 6, 8, 11, 12, 19, 30, 31, 47, 50, 51, 52, 54, 55, 62] and the references cited therein.

Moreover, Sedghi *et al.* [57] gave a generalisation of D-metric space and G-metric space to S-metric space. Since then, several researchers have been working on generalising the results using different contraction conditions in S-metric space. For more detail, one can see [5, 15, 40, 46, 59, 60] and the references therein. Nizar and Nabil [45] proved a fixed point theorem in S_b-metric spaces. Nizar [44] proved the results on a fixed point in partial S_b-metric spaces. Later, Mlaiki *et al.* [43] proved fixed point theorem for α-ψ-contractive mapping in S_b-metric spaces.

Motivated by Matthew [42], Karapinar [37] initiated the concept of quasi-partial metric space and discussed the existence of fixed points of self-mapping for this Space. Gupta and Gautam [26, 28] further generalised the quasi-partial metric Space to the class of quasi-partial b-metric spaces. Recently, Gautam and Verma [21] discussed fixed point results via implicit mapping in quasi-partial b-metric space. Gautam *et al.* [24] proved an interpolative Chatterjea and cyclic Chatterjea contraction on quasi-partial b-metric space. Gupta and Gautam [27] proved the topological structure of quasi-partial b-metric spaces. Gautam *et al.* [23] gave proof of common fixed point results on generalised weak compatible mapping in quasi-partial b-metric space. Aydi *et al.* [7] proved a fixed point theorem for set-valued quasi-contractions in b-metric spaces. Gautam *et al.* [22] proved fixed point of interpolative Rus-Reich-Ćirić contraction mapping on rectangular quasi-partial b-metric space.

This paper is motivated by the results of Gautam and Verma [21], and Nizar [44]. We prove common fixed point theorems for weakly compatible mappings satisfying an implicit relation in the quasi-partial S_b-metric space setting and obtain coincidence and a unique common fixed point of such mappings. Some examples are provided to verify the validity of our results. Finally, a solution to the second-order differential equation's two boundary value problem and the existence of a common solution of the Caputo-type fractional differential equation will be discussed.

We describe some definitions and theorems, which will help to develop our main results.

The property of quasi-partial b-metric space introduced in [26] is as follows:

Definition 1.1. [26] A quasi-partial b-metric space on a non empty set X is a mapping $qp_b : X \times X \to \mathbb{R}^+$ such that for some real number $s \geq 1$ and all $u, v, z \in X$:

(QPb1): $qp_b(u,u) = qp_b(u,v) = qp_b(v,v) \Rightarrow u = v$;
(QPb2): $qp_b(u,u) \leq qp_b(u,v)$;
(QPb3): $qp_b(u,u) \leq qp_b(v,u)$; and
(QPb4): $qp_b(u,v) \leq s[qp_b(u,z) + qp_b(v,z)] - qp_b(z,z)$.

A quasi-partial b-metric space is a pair (X, qp_b) such that X is a non-empty set and (X, qp_b) is a quasi partial b-metric on X. The number s is called the coefficient of (X, qp_b).

For a quasi-partial b-metric space (X, qp_b), the function $d_{qpb} : X \times X \to R^+$ defined by $d_{qp_b}(u,v) = qp_b(u,v) + qp_b(v,u) - qp_b(u,u) - qp_b(v,v)$ is a b-metric on X.

Lemma 1.2. [26] Every quasi-partial metric pace is a quasi-partial b-metric space, but the converse need not be true.

Lemma 1.3. [26] Let (X, qp_b) be a quasi-partial b-metric space and (X, d_{qp_b}) be the corresponding b-metric space. Then (X, d_{qp_b}) is complete if (X, qp_b) is complete.

Examples of quasi-partial b-metric space are given in [26], and [21].

In 2012, Sedghi *et al.* [57] gave a generalisation of D-metric Space and G-metric Space to S-metric space by formulating its properties as follows:

Definition 1.4. [57] Let X be a non-empty set. A S-metric on X is a function $S : X \times X \times X \to [0, \infty)$ that satisfies the following conditions for all $u, v, z, a \in X$.

(S1): $S(u,v,z) \geq 0$;
(S2): $S(u,v,z) = 0$ if and only if $u = v = z$; and
(S3): $S(u,v,z) \leq S(u,u,a) + S(v,v,a) + S(z,z,a)$.

The pair (X, S) is called an S-metric space.

Motivated by the results of Czerwik [17] and sedghi *et al.* [57], Nizar and Nabil [45] introduced the notion of S_b-metric space.

Definition 1.5. [45] Let X be a non-empty set and let $s \geq 1$ be a given number. A function $S_b : X \times X \times X \to [0, \infty)$ is said to be S_b-metric if and only if for all $u, v, z, t \in X$ the following conditions hold:

(S1): $S_b(u, v, z) = 0$, if and only if $u = v = z$;
(S2): $S_b(u, u, v) = S_b(v, v, u)$ for all $u, v \in X$; and
(S3): $S_b(u, v, z) \leq s[S_b(u, u, t) + S_b(v, v, t) + S_b(z, z, t)]$.

The pair (X, S_b) is called an S_b-metric space.

Inspired by Nizar [44], Nizar and Nabil [45], and Gautam and Verma [21], we introduce the concept of quasi partial S_b-metric space as follows:

Definition 1.6. *A quasi-partial S_b-metric space on a non empty set X is a mapping $S_{qp_b} : X \times X \times X \to \mathbb{R}^+$ such that for some real number $s \geq 1$ and all $u, v, z \in X$:*

(QPSb1): $S_{qp_b}(u, u, u) = S_{qp_b}(u, v, z) = S_{qp_b}(v, v, y) \Rightarrow u = v = z;$
(QPSb2): $S_{qp_b}(u, u, v) = Sqp_b(v, v, u);$
(QPSb2): $S_{qp_b}(u, u, u) \leq S_{qp_b}(u, u, v)$*; and*
(QPSb4): $S_{qp_b}(u, v, z) \leq s[S_{qp_b}(u, u, t) + S_{qp_b}(v, v, t) + S_{qp_b}(z, z, t)] - S_{qp_b}(t, t, t).$

A quasi-partial S_b-metric space is a pair (X, S_{qp_b}) such that X is a non-empty set and (X, S_{qp_b}) is a quasi partial S_b-metric on X. The number s is called the coefficient of (X, S_{qp_b}).

For a quasi-partial S_b-metric space (X, S_{qp_b}), the function $d_{S_{qp_b}} : X \times X \times X \to \mathbb{R}^+$ defined by $d_{Sqp_b}(u, u, v) = S_{qp_b}(u, u, v) + S_{qp_b}(v, v, u) - S_{qp_b}(u, u, u) - S_{qp_b}(v, v, v,)$ is a S_{qp_b}-metric on X.

The following are fundamental convergence properties of quasi- partial S_b-metric spaces.

Definition 1.7. *Let (X, S_{qp_b}) be a quasi-partial S_b-metric space, then:*

(i): *a sequence $\{u_n\} \subset X$ converges to a point $u \in X$ if and only if*

$$S_{qp_b}(u, u, u) \quad = \quad \lim_{n \to \infty} S_{qp_b}(u_n, u_n, u) = \lim_{n \to \infty} S_{qp_b}(u, u, u_n),$$

(ii): *a sequence $\{u_n\}$ of elements of X is called a Cauchy sequence if and only if*

$$\lim_{n,m \to \infty} S_{qp_b}(u_n, u_n, u_m) \text{ and } \lim_{n,m \to \infty} S_{qp_b}(u_m, u_m, u_n)$$

exists and is finite,

(iii): *the quasi-partial S_b-metric space (X, S_{qp_b}) is said to be complete if every Cauchy sequence $\{u_n\} \subset X$ converges to a point $u \in X$ such that*

$$\lim_{n,m \to \infty} S_{qp_b}(u_n, u_n, u_m) = \lim_{n,m \to \infty} S_{qp_b}(u_m, u_m, u_n) = S_{qp_b}(u, u, u).$$

Lemma 1.8. *Let (X, S_{qp_b}) be a quasi-partial b-metric space. Then the following holds:*

(i): *If $S_{qp_b}(u, u, u) = 0$, then $u = v$.*
(ii): *If $u \neq v$, then $S_{qp_b}(u, u, v) > 0$ and $S_{qp_b}(v, v, u) > 0$.*

From, Sedghi *et al.* [57], we proved the following lemma to satisfy quasi-partial S_b-metric space.

Lemma 1.9. In a S_{qp_b}-metric space, we have

$$S_{qp_b}(u, u, v) = S_{qp_b}(v, v, u).$$

Proof. By condition $(QPSb4)$ of Definition 1.6 and $x = t$ we get

$$\begin{aligned} S_{qp_b}(u, u, v) &\leq s[S_{qp_b}(u, u, t) + S_{qp_b}(u, u, t) + S_{qpb}(v, v, t)] - S_{qp_b}(t, t, t) \\ &\leq s[0 + 0 + S_{qp_b}(v, v, t)] - 0 \\ &= sS_{qpb}(v, v, t). \qquad (1) \end{aligned}$$

Similarly,

$$\begin{aligned} S_{qp_b}(v, v, u) &\leq s[S_{qp_b}(v, v, t) + S_{qp_b}(v, v, t) + S_{qpb}(u, u, t)] - S_{qp_b}(t, t, t) \\ &\leq s[0 + 0 + S_{qp_b}(xu, u, t)] - 0 \\ &= sS_{qp_b}(u, u, t). \qquad (2) \end{aligned}$$

Consequently, by (1) and (2) as a results

$$S_{qp_b}(u, u, t) = S_{qp_b}(v, v, t).$$

□

Example 1.10. Let $X = [0, 1]$. Define $S_{qp_b} : X \times X \times X \to \mathbb{R}^+$ as $S_{qp_b}(u, v, z) = (u - v)^2 + (v - z)^2 + u + v$. It is easy to show that (X, S_{qp_b}) is a quasi-partial S_b-metric space.

By $(QPSb1)$, for $u = v = z$ we have $S_{qp_b}(u, u, u) = S_{qp_b}(v, v, v) = S_{qp_b}(z, z, z)$

$$\begin{aligned} S_{qp_b}(u, u, u) &\leq (u - v)^2 + (v - z)^2 + u + v, \\ &= (u - u)^2 + (u - u)^2 + u + u, \\ &= 2u. \end{aligned}$$

By $(QPSb2)$, for all $u, v \in X$ we have

$$\begin{aligned} S_{qp_b}(u, u, v) &\leq (u - u)^2 + (u - v)^2 + u + u, \\ &= (u - v)^2 + u + u, \\ &= u^2 - 2uv + v^2 + 2u, \end{aligned}$$

and

$$\begin{aligned} S_{qp_b}(v, v, u) &\leq (v - v)^2 + (v - u)^2 + v + v, \\ &= (v - u)^2 + v + v, \\ &= v^2 - 2uv + u^2 + 2v, \end{aligned}$$

hence, $v^2 - 2uv + v^2 + 2u = v^2 - 2uv + x^2 + 2v$.

Similar, $(QPSb3)$ follows from $(QPSb2)$ and $(QPSb1)$

$$2u \leq u^2 - 2uv + v^2 + 2u.$$

Consequently, by $(QPSb4)$, we get

$$\begin{aligned} S_{qp_b}(u,u,t) &= (u-t)^2 + 2u, \\ S_{qpb}(v,v,t) &= (v-t)^2 + 2v, \\ S_{qp_b}(z,z,t) &= (z-t)^2 + 2z. \end{aligned}$$

Combining all the above equalities using $(QPSb4)$, we obtain

$$(u-v)^2 + (v-z)^2 + u + v \leq s[(u-t)^2 + 2u + (v-t)^2 + 2v] - ((z-t)^2 + 2z),$$

thus, all axioms are satisfied. Hence (X, S_{qp_b}) is complete.

Furthermore, Abbas and Jungck [1] and Pathak [48] gave the following definition for a unique common fixed point notion.

Definition 1.11. [1, 48]

(i): Let $\mathcal{S}$ and $\mathcal{A}$ be self maps of a set X. If $u^* = \mathcal{S}u = \mathcal{A}u$ for some u in X, then u is called a coincidence point of $\mathcal{S}$ and $\mathcal{A}$, and u^* is called a point of coincidence of $\mathcal{S}$ and $\mathcal{A}$.

(ii): Let $\mathcal{S}$ and $\mathcal{A}$ be weakly compatible self maps of a set X, we have $\mathcal{S}u^* = \mathcal{S}\mathcal{A}u = \mathcal{A}\mathcal{S}u = \mathcal{A}u^*$. If $\mathcal{S}$ and $\mathcal{A}$ have a unique point of coincidence $u^* = \mathcal{S}u = \mathcal{A}u$, then u^* is the unique common fixed point of $\mathcal{S}$ and $\mathcal{A}$.

2. Implicit mapping and related notion

In 2021, Gautam and Verma [21] proved the results for fixed point theorems of mappings satisfying implicit contractive relation in quasi-partial b-metric Space. They considered the family F_Q of all lower semi-continuous real functions $F : \mathbb{R}^5_+ \to \mathbb{R}^+$ and the following conditions:

(F1): F is non-increasing in the t_1 and t_5 variable;

(F2): for all $q, r \geq 0$, there exist $h \in [0,1)$ such that $F(q,r,r,q,s(q+r)) \leq 0$ implies $q \leq hr$;

(F3): $F(t,t,0,0,t) > 0$ for all $t > 0$.

We give some examples of functions that satisfy the above implicit relation conditions.

Example 2.1. The function of $F \in F_Q$ satisfies the properties (F1) - (F3) (see, [21]).

(1): $F(t_1,t_2,t_3,t_4,t_5) = t_1 - \alpha \max\{t_2,t_3,t_4,t_5\}$, where $\alpha \in [0, \frac{1}{2s})$;

(2): $F(t_1,t_2,t_3,t_4,t_5) = t_1 - a_1t_1 - a_2t_2 - a_3t_3 - a_4t_4 - a_5t_5$, where $a_i \geq 0$, i = 1,2,3,4, also $0 < a_1 + a_2 + a_3 + 2sa_4 < 1$ and $0 < a_1 + a_4 < 1$.

Gautam and Verma [21] proved the following theorem satisfying implicit mappings.

Theorem 2.2. [21] Let (X, qp_b) be a complete quasi-partial b-metric space and $T : X \to X$ is continuous self map for all $u \in X$. Suppose that

$$F\Big[qp_b(Tu,Tv), qp_b(x,y), qp_b(u,Tv), qp_b(v,Tv), \quad [qp_b(u,Tv) + qp_b(v,Tu)]\Big] \leq 0. \tag{3}$$

For some $F \in F_Q$ and if F satisfies $F(q, 0, r, r, 2sq) \leq 0$ for all $q, r \geq 0$, there exists $\beta \in [0, \frac{1}{s}]$ such that $q < \beta r$, then z is a unique fixed point of T. i.e, $Tz = z$ with $qp_b(z, z) = 0$.

We introduce a definition of a common fixed point via implicit mappings in quasi-partial S_b-metric space.

Motivated by the concept given by Gautam and Verma [21] above. We introduce the following definition.

Definition 2.3. *Consider $s \geq 1$. Let F_Q be the set of all functions $F_S(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}^5 \to \mathbb{R}$ such that*

(FS1): *F_S is non-increasing in the t_1 and t_5 variable;*
(FS2): *for all $q, r \geq 0$, there exist $\vartheta \in [0, \frac{1}{s}]$, such that $F_S(q, r, q, r, s(2q + r)) \leq 0$ implies $q \leq \vartheta r$;*
(FS3): *$F_S(t, t, 0, 0, t) > 0$ for all $t > 0$.*

Example 2.4. The function of $F_S \in F_Q$ satisfies the properties (FS1) - (FS3).

(1): $F_S(t_1, t_2, t_3, t_4, t_5) = t_1 - t_5$, where $\gamma \in [0, \frac{1}{2s})$;
(2): $F_S(t_1, t_2, t_3, t_4, t_5) = t_1 - \max\{t_2, t_3, t_5\}$, where $\alpha, \gamma \in [0, \frac{1}{2s})$;
(3): $F_S(t_1, t_2, t_3, t_4, t_5) = t_1 - \max\{t_2, t_3, t_4, t_5\}$, where $\alpha, \beta, \gamma \in [0, \frac{1}{s})$.

Proof. (1), Let $F_S : \mathbb{R}^5 \to \mathbb{R}^+$. Define $F_S(t_1, t_2, t_3, t_4, t_5) = t_1 - t_5$, where $\gamma \in [0, \frac{1}{s})$. Then F_S satisfies an implicit relation.

(FS1): F_S is non-increasing in the t_1 and t_5 variable;
(FS2): for all $q, r \geq 0$, we have

$$\begin{aligned} F_S(q, r, r, , q, s(2q + r)) &= t_1 - \gamma t_5 \leq 0, \\ q - \gamma s(2q + r) &\leq 0, \\ (1 - 2s\gamma)q &\leq s\gamma r, \\ q &\leq \frac{s\gamma r}{(1 - 2s\gamma)}. \end{aligned} \tag{4}$$

Thus $q \leq \vartheta r$, with $\vartheta = \frac{s\gamma}{(1-2s\gamma)} < 1$.

(FS3): $F_S(t,t,0,0,t) > 0$ for all $t > 0$.

$$\begin{aligned} F_S(t,t,0,0,t) &= t_1 - t_5 \leq 0, \\ u - s(2u+v) &\leq 0, \\ t - s(2t+t)) &\leq 0, \\ (1-3s)t &\leq 0, \\ t &\leq 0, \end{aligned}$$

which is a contradiction. Hence $F_S \in F_Q$ satisfies an implicit relation with $\gamma \in [0, \frac{1}{s})$. □

The example $(2,3)$ can be proved similarly by following the above steps to satisfy the implicit relation conditions imposed in Definition 2.3.

3. Main Results

We prove the following theorem, an extension of Theorem 2.2, from quasi-partial b-metric Space to quasi-partial S_b-metric space setting. By using a pair of self-mapping.

Theorem 3.1. *Let (X, S_{qp_b}, s) be a complete quasi-partial S_b-metric space with $s \geq 1$, and let $\mathcal{A}, \mathcal{S} : X \to X$ be a pair of self-mappings. Assume that there exists $F_S \in F_Q$, satisfies $(FS1 - FS4)$ such that the following conditions hold:*

(a): *there exists $\mathcal{A}X \subseteq \mathcal{S}X$ such that (X, qp_b) is complete,*
(b): *there exists $u_0 \in X$ such that $\mathcal{S}u_n = \mathcal{A}u_{n-1}$,*
(c): *$\mathcal{A}$ and $\mathcal{S}$ have a coincidence point in X,*
(d): *$(\mathcal{A}, \mathcal{S})$ is non-decreasing and weakly compatible for some point u^* in X,*
(e): *there exists an implicit function $F_S \in F_Q$ with*

$$F_S \left\{ \begin{array}{c} S_{qp_b}(\mathcal{A}u, \mathcal{A}u, \mathcal{A}v), S_{qp_b}(\mathcal{S}u, \mathcal{S}u, \mathcal{S}v), \\ S_{qp_b}(\mathcal{S}u, \mathcal{S}u, \mathcal{A}u), S_{qp_b}(\mathcal{S}v, \mathcal{S}v, \mathcal{A}v), \\ [S_{qp_b}(\mathcal{S}u, \mathcal{S}u, \mathcal{A}v) + S_{qp_b}(\mathcal{S}v, \mathcal{S}v, \mathcal{A}u)] \end{array} \right\} \leq 0, \tag{5}$$

$\forall u, v \in X$. Then $\mathcal{A}$ and $\mathcal{S}$ have a unique common fixed point.

Proof. Assume that $\mathcal{S}X \subseteq \mathcal{A}X$ and (X, S_{qp_b}) is a complete quasi-partial S_b-metric space, for u_0 with $(\mathcal{S}u_0, \mathcal{S}u_0, \mathcal{A}u_0,) \in X$, we construct a $\mathcal{S}$-$\mathcal{A}$-sequence $\{\mathcal{A}u_n\}$ with initial point u_0 satisfying

$$(\mathcal{S}u_0, \mathcal{S}u_0, \mathcal{A}u_0), (\mathcal{S}u_1, \mathcal{S}u_1, \mathcal{A}u_1), (\mathcal{S}u_2, \mathcal{S}u_2, \mathcal{A}x_2), \ldots, (\mathcal{S}u_{n+1}, \mathcal{S}u_{n+1}, \mathcal{A}u_{n+1})$$

$\forall n \in \mathbb{N}_{\mathbb{O}} = (\mathbb{N} \cup \{0\})$, thus, $\{\mathcal{A}u_n\}$, $\{\mathcal{S}u_n\} \in \mathcal{A}(X)$.

From assumption (b), let u_0 be an arbitrary element of X. If $\mathcal{S}u_0 = \mathcal{A}u_0$, then u_0 is a common fixed point of $\mathcal{A}$ and $\mathcal{S}$ and our proof completed. Otherwise, if $\mathcal{S}u_0 \neq \mathcal{A}u_0$, then $\mathcal{S}X \subseteq \mathcal{A}X$, now we choose $u_1 \in X$ such that

$\mathcal{S}u_1 = \mathcal{A}u_0$. Again we can choose $u_2 \in X$ such that $\mathcal{S}u_2 = \mathcal{A}u_1$. Repeating this process the same way, we construct a sequence $\{\mathcal{S}u_n\} \subset X$, such that

$$\mathcal{S}u_{n+1} = \mathcal{A}u_n, \forall n \in \mathbb{N}_0.$$

If $\mathcal{S}u_{n-1} = \mathcal{S}u_n = \mathcal{A}u_{n-1}$, for all $n \geq 1$, then u_{n-1} is a coincidence point of $\mathcal{A}$ and $\mathcal{S}$ in X. Suppose that $\mathcal{S}u_{n-1} \neq \mathcal{S}u_n$ $\forall n \geq 1$. Then $S_{qp_b}(\mathcal{S}u_{n+1}, \mathcal{S}u_{n+1}, \mathcal{S}u_n) = S_{qp_b}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{S}u_{n-1})$.

By taking $u = u_{n-1}$ and $v = u_n$ in (5), we have

$$F_S \left\{ \begin{matrix} S_{qp_b}(\mathcal{A}u_{n-1}, \mathcal{A}u_{n-1}, \mathcal{A}u_n), S_{qp_b}(\mathcal{S}u_{n-1}, \mathcal{S}u_{n-1}, \mathcal{S}u_n), \\ S_{qp_b}(\mathcal{S}u_{n-1}, \mathcal{S}u_{n-1}, \mathcal{A}u_{n-1}), S_{qp_b}(\mathcal{S}u_n, \mathcal{S}u_n, \mathcal{A}u_n), \\ [S_{qp_b}(\mathcal{S}u_{n-1}, \mathcal{S}u_{n-1}, \mathcal{A}u_n) + S_{qp_b}(\mathcal{S}u_n, \mathcal{S}u_n, \mathcal{A}u_{n-1})] \end{matrix} \right\} \leq 0.$$

It follows that

$$F_S \left\{ \begin{matrix} S_{qp_b}(\mathcal{S}u_n, \mathcal{S}u_n, \mathcal{S}u_{n+1}), S_{qp_b}(\mathcal{S}u_{n-1}, \mathcal{S}u_{n-1}, \mathcal{S}u_n), \\ S_{qp_b}(\mathcal{S}u_{n-1}, \mathcal{S}u_{n-1}, \mathcal{S}u_n), S_{qp_b}(\mathcal{S}u_n, \mathcal{S}u_n, \mathcal{S}u_{n+1}), \\ [S_{qp_b}(\mathcal{S}u_{n-1}, \mathcal{S}u_{n-1}, \mathcal{S}u_{n+1}) + S_{qp_b}(\mathcal{S}u_n, \mathcal{S}u_n, \mathcal{S}u_n)] \end{matrix} \right\} \leq 0. \tag{6}$$

By $(QPSb4)$ we have

$$\begin{aligned} S_{qp_b}(\mathcal{S}u_{n+1}, \mathcal{S}u_{n+1}, \mathcal{S}u_{n-1}) \quad \leq \quad & s[2S_{qp_b}(\mathcal{S}u_{n+1}, \mathcal{S}u_{n+1}, \mathcal{S}u_n) + \\ & S_{qp_b}(\mathcal{S}u_n, \mathcal{S}u_n, \mathcal{S}u_{n-1})] \\ & -S_{qp_b}(\mathcal{S}u_n, \mathcal{S}u_n, \mathcal{S}u_n). \end{aligned} \tag{7}$$

Using (7) in (5) we get

$$F_S \left\{ \begin{matrix} S_{qp_b}(\mathcal{S}u_n, \mathcal{S}u_n, \mathcal{S}u_{n+1}), S_{qp_b}(\mathcal{S}u_{n-1}, \mathcal{S}u_{n-1}, \mathcal{S}u_n), \\ S_{qp_b}(\mathcal{S}u_{n-1}, \mathcal{S}u_{n-1}, \mathcal{S}u_n), S_{qp_b}(\mathcal{S}u_n, \mathcal{S}u_n, \mathcal{S}u_{n+1}), \\ [s[2S_{qp_b}(\mathcal{S}u_{n+1}, \mathcal{S}u_{n+1}, \mathcal{S}u_n) + S_{qp_b}(\mathcal{S}u_n, \mathcal{S}u_n, \mathcal{S}u_{n-1})] \\ -S_{qp_b}(\mathcal{S}u_n, \mathcal{S}u_n, \mathcal{S}u_n) + S_{qp_b}(\mathcal{S}u_n, \mathcal{S}u_n, \mathcal{S}u_n)] \end{matrix} \right\} \leq 0.$$

Consequently,

$$F_S \left\{ \begin{matrix} S_{qp_b}(\mathcal{S}u_n, \mathcal{S}u_n, \mathcal{S}u_{n+1}), S_{qp_b}(\mathcal{S}u_{n-1}, \mathcal{S}u_{n-1}, \mathcal{S}u_n), \\ S_{qp_b}(\mathcal{S}u_{n-1}, \mathcal{S}u_{n-1}, \mathcal{S}u_n), S_{qp_b}(\mathcal{S}u_n, \mathcal{S}un, \mathcal{S}u_{n+1}), \\ s[2S_{qp_b}(\mathcal{S}u_{n+1}, \mathcal{S}u_{n+1}, \mathcal{S}u_n) + S_{qp_b}(\mathcal{S}u_n, \mathcal{S}u_n, \mathcal{S}u_{n-1})] \end{matrix} \right\} \leq 0.$$

By denoting $q = S_{qp_b}(\mathcal{S}u_{n+1}, \mathcal{S}u_{n+1}, \mathcal{S}u_n)$ and $r = S_{qp_b}(\mathcal{S}u_n, \mathcal{S}u_n, \mathcal{S}u_{n-1})$ in (8) we get

$$F_S \left\{ q, r, r, q, s(2q + r) \right\} \leq 0. \tag{8}$$

By (8), in view of condition $(FS2)$ there exists $\vartheta \in [0, \frac{1}{s})$ and q is nonincreasing in the first variable, such that $uq \leq \vartheta r$, this implies that

$$S_{qp_b}(\mathcal{S}u_{n+1}, \mathcal{S}u_{n+1}, \mathcal{S}u_n) \quad \leq \quad \vartheta S_{qp_b}(\mathcal{S}u_n, \mathcal{S}u_n, \mathcal{S}u_{n-1}); \tag{9}$$

$\forall n \in \mathbb{N}$.

By induction in (9), we get

$$\begin{aligned} S_{qp_b}(\mathcal{S}u_{n+1}, \mathcal{S}u_{n+1}, \mathcal{S}u_n) &\leq \vartheta S_{qp_b}(\mathcal{S}u_n, \mathcal{S}u_n, \mathcal{S}u_{n-1}), \\ &\leq \vartheta^2 S_{qp_b}(\mathcal{S}u_{n-1}, \mathcal{S}u_{n-1}, \mathcal{S}x_{n-2}), \\ &\leq \ldots \\ &\leq \vartheta^n S_{qp_b}(\mathcal{S}u_0, \mathcal{S}u_0, \mathcal{S}u_1). \qquad (10) \end{aligned}$$

Therefore, $\lim_{n\to\infty} S_{qp_b}(\mathcal{S}u_{n+1}, \mathcal{S}u_{n+1}, \mathcal{S}u_n) = 0$.

Now, we prove that $S_{qp_b}(\mathcal{S}u_{n+1}, \mathcal{S}u_{n+1}, \mathcal{S}u_n)$ is a Cauchy sequence. Let $n, m \in \mathbb{N}$, for any positive integers such that $n > m$, using $(QPSb4)$ we have

$$\begin{aligned} S_{qp_b}(\mathcal{S}u_n, \mathcal{S}u_n, \mathcal{S}u_m) &\leq s[2S_{qp_b}(\mathcal{S}u_n, \mathcal{S}u_n, \mathcal{S}u_{n-1}) + S_{qp_b}(\mathcal{S}u_{n-1}, \mathcal{S}u_{n-1}, \mathcal{S}u_m)] \\ &\quad -S_{qp_b}(\mathcal{S}u_{n-1}, \mathcal{S}u_{n-1}, \mathcal{S}u_{n-1}), \\ &= 2sS_{qp_b}(\mathcal{S}u_n, \mathcal{S}u_n, \mathcal{S}u_{n-1}) + 2s^2 S_{qp_b}(\mathcal{S}u_{n-1}, \mathcal{S}u_{n-1}, \mathcal{S}u_{n-2}) \\ &\quad +s^2 S_{qp_b}(\mathcal{S}u_{n-2}, \mathcal{S}u_{n-2}, \mathcal{S}u_m) + \\ &\quad \cdots + s^{m-n-1} S_{qp_b}(\mathcal{S}u_{m+1}, \mathcal{S}u_{m+1}, \mathcal{S}u_m) \\ &\leq 2[s\vartheta^{n-1} + s^2\vartheta^{n-2} + s^3\vartheta^{n-3} + \\ &\quad \cdots + s^{m-n+1}\vartheta^m] S_{qp_b}(\mathcal{S}u_0, \mathcal{S}u_0, \mathcal{S}u_1), \\ &\leq 2s\vartheta^{n-1}[1 + s\vartheta + s^2\vartheta^2 + \\ &\quad \cdots + s^{m-1}\vartheta^{m-n+1}] S_{qp_b}(\mathcal{S}u_0, \mathcal{S}u_0, \mathcal{S}u_1), \\ &\leq \frac{2s\vartheta^{n-1}}{1-s\vartheta} S_{qp_b}(\mathcal{S}u_0, \mathcal{S}u_0, \mathcal{S}u_1). \qquad (11) \end{aligned}$$

Since $\vartheta \in [0, \frac{1}{s})$, we conclude that $\frac{2s\vartheta^{n-1}}{1-s\vartheta} S_{qp_b}(\mathcal{S}u_0, \mathcal{S}u_0, \mathcal{S}u_1) \to 0$ as $n \to \infty$. Therefore, $\{\mathcal{S}u_n\}$ is a Cauchy sequence in $\mathcal{S}(X)$. Thus $S_{qp_b}(\mathcal{S}u_n, \mathcal{S}u_n, \mathcal{S}u_m) \to 0$ as $n, m \to \infty$.

Similarly, suppose that $\mathcal{A}X \subseteq \mathcal{S}X$. For every $u_0 \in X$ we consider the sequence $\{\mathcal{A}u_n\} \in X$ defined by

$$\begin{aligned} \mathcal{S}u_n &= \mathcal{A}u_{n-1}, \\ \mathcal{S}u_{n+1} &= \mathcal{A}u_n. \end{aligned}$$

If $\mathcal{S}u_{n+1} = \mathcal{A}u_n$, then u_n is a fixed point of $\mathcal{S}$ and $\mathcal{A}$ and the proof completed. On contrary, assume that $\mathcal{S}u_{n+1} \neq \mathcal{A}u_n$ and $u_{n+1} \neq u_n$. Then, $u = u_n$ and $v = u_{n+1}$ in (5) we have

$$(12)\ F_S \left\{ \begin{array}{c} S_{qp_b}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{A}u_{n+1}), S_{qp_b}(\mathcal{S}u_n, \mathcal{S}u_n, \mathcal{S}u_{n+1}), \\ S_{qp_b}(\mathcal{S}u_n, \mathcal{S}u_n, \mathcal{A}u_n), S_{qp_b}(\mathcal{S}u_{n+1}, \mathcal{S}u_{n+1}, \mathcal{A}u_{n+1}), \\ [S_{qp_b}(\mathcal{S}u_n, \mathcal{S}u_n, \mathcal{A}u_{n+1}) + S_{qp_b}(\mathcal{S}u_{n+1}, \mathcal{S}u_{n+1}, \mathcal{A}u_n)] \end{array} \right\} \leq 0.$$

By substituting $\mathcal{S}u_n = \mathcal{A}u_{n-1}$ and $\mathcal{S}u_{n+1} = \mathcal{A}u_n$ in (12), we get

$$(13) F_S \left\{ \begin{array}{c} S_{qp_b}(\mathcal{A}u_n, \mathcal{A}u_n, \zeta x_{n+1}), S_{qp_b}(\mathcal{A}u_{n-1}, \mathcal{A}u_{n-1}, \mathcal{A}u_n), \\ S_{qp_b}(\mathcal{A}u_{n-1}, \mathcal{A}u_{n-1}, \mathcal{A}u_n), S_{qp_b}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{A}u_{n+1}), \\ [S_{qp_b}(\mathcal{A}u_{n-1}, \mathcal{A}u_{n-1}, \mathcal{A}u_{n+1}) + S_{qp_b}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{A}u_n)] \end{array} \right\} \leq 0.$$

By $(QPSb4)$, we have

$$S_{qp_b}(\mathcal{A}u_{n-1},\mathcal{A}u_{n-1},\mathcal{A}u_{n+1}) \leq s[2S_{qp_b}(\mathcal{A}u_{n-1},\mathcal{A}u_{n-1},\mathcal{A}u_n)+ S_{qp_b}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{A}u_{n+1})] -S_{qp_b}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{A}u_n). \tag{14}$$

Using (14) in (13), we get

$$F_S\left\{\begin{array}{c} S_{qp_b}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{A}u_{n+1}), S_{qp_b}(\mathcal{A}u_{n-1},\mathcal{A}u_{n-1},\mathcal{A}u_n),\\ S_{qp_b}(\mathcal{A}u_{n-1},\mathcal{A}u_{n-1},\mathcal{A}u_n), S_{qp_b}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{A}u_{n+1}),\\ [s[2S_{qp_b}(\mathcal{A}u_{n-1},\mathcal{A}u_{n-1},\mathcal{A}u_n)+S_{qp_b}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{A}u_{n+1})]\\ -S_{qp_b}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{A}u_n)+S_{qp_b}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{A}u_n)] \end{array}\right\}\leq 0. \tag{15}$$

Since quasi-partial S_b is not symmetrical, by $(FS2)$, we reach similar results from the right-hand side of Cauchy convergence.

Using $(QPSb4)$ and $(FS1)$, since is a non-decreasing in the fifth variable and satisfy

$$q \leq \vartheta r,$$

where $\vartheta \in [0, \frac{1}{s})$.

Which implies that

$$\begin{aligned} S_{qp_b}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{A}u_{n+1}) &\leq \vartheta S_{qp_b}(\mathcal{A}u_{n-1},\mathcal{A}u_{n-1},\mathcal{A}u_n),+\\ &\cdots+\\ &\leq \vartheta^n S_{qp_b}(\mathcal{A}u_0,\mathcal{A}u_0,\mathcal{A}u_1). \end{aligned} \tag{16}$$

For $n\to\infty$ in (16), leads to $S_{qp_b}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{A}u_{n+1})\to 0$.

Using $(QPSb4)$, for all $n,m\in\mathbb{N}_0$ with $m>n$, we obtain

$$\begin{aligned} S_{qp_b}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{A}u_m) &\leq s[2S_{qp_b}(\mathcal{S}u_n,\mathcal{A}u_n,\mathcal{A}u_{n+1})+\\ &\quad S_{qp_b}(\mathcal{S}u_{n+1},\mathcal{A}u_{n+1},\mathcal{A}u_m)]\\ &\quad -S_{qp_b}(\mathcal{A}u_{n+1},\mathcal{A}u_{n+1},\mathcal{A}u_{n+1}),\\ &= 2sS_{qp_b}(\mathcal{A}u_n,\mathcal{A}u_n,\mathcal{A}u_{n+1})+\\ &\quad 2s^2S_{qp_b}(\mathcal{A}u_{n+1},\mathcal{A}u_{n+1},\mathcal{A}u_{n+2})\\ &\quad +s^2S_{qp_b}(\mathcal{A}u_{n+2},\mathcal{A}u_{n+2},\mathcal{A}u_m)+\\ &\quad \cdots+s^{m-n-1}S_{qp_b}(\mathcal{A}u_{m-1},\mathcal{A}u_{m-1},\mathcal{A}u_m)\\ &\leq 2[s\vartheta^n+s^2\vartheta^{n+1}+s^2\vartheta^{n+2}+\cdots+\\ &\quad s^{m-n-1}\vartheta^{m-1}]S_{qp_b}(\mathcal{A}u_0,\mathcal{A}u_0,\mathcal{A}u_1),\\ &= 2s\vartheta^n[1+s\vartheta+s^2\vartheta^2+\\ &\quad \cdots+s^{m-n-2}\vartheta^{m-n-1}]S_{qp_b}(\mathcal{A}u_0,\mathcal{A}u_0,\mathcal{A}u_1),\\ &\leq \frac{2s\vartheta^n}{1-s\vartheta}S_{qp_b}(\mathcal{A}u_0,\mathcal{A}u_0,\mathcal{A}u_1). \end{aligned} \tag{17}$$

Since $\vartheta \in [0, \frac{1}{s})$, we conclude that $\frac{2s\vartheta^n}{1-s\vartheta} S_{qp_b}(\mathcal{A}u_0, \mathcal{A}u_0, \mathcal{A}u_1) \to 0$ as $n \to \infty$. Therefore, $\{\mathcal{A}u_n\}$ is a Cauchy sequence in $\mathcal{A}(X)$. Thus $S_{qp_b}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{A}u_m) \to 0$ as $n, m \to \infty$.

Now we show that u^* is a fixed point of $\mathcal{A}u$ such that $u^* = \mathcal{A}u^*$ and $\lim_{n,m\to\infty} S_{qp_b}(u_n, u_n, \mathcal{A}u^*) = S_{qp_b}(u^*, u^*, \mathcal{A}u^*) = 0$. Let $u = u_n$ and $v = u^*$, using $(QPSb4)$ we obtain

$$\begin{aligned} S_{qp_b}(u^*, u^*, \mathcal{A}u^*) &\leq s[S_{qp_b}(u^*, u^*, u_{n+1}) + S_{qp_b}(u^*, u^*, u_{n+1}) + \\ &\quad S_{qp_b}(\mathcal{A}u^*, \mathcal{A}u^*, u_{n+1})] - \\ &\quad S_{qp_b}(u_{n+1}, u_{n+1}, u_{n+1}), \\ &= s[2S_{qp_b}(u^*, u^*, u_{n+1}) + S_{qp_b}(\mathcal{A}u^*, \mathcal{A}u^*, u_{n+1})] - \\ &\quad S_{qp_b}(u_{n+1}, u_{n+1}, u_{n+1}). \end{aligned} \tag{18}$$

Taking the limit $n \to \infty$ in (18), we get

$$\begin{aligned} S_{qp_b}(u^*, u^*, \mathcal{A}u^*) &\leq s[2S_{qp_b}(u^*, u^*, u^*) + S_{qp_b}(\mathcal{A}u^*, \mathcal{A}u^*, u^*)] - \\ &\quad S_{qp_b}(u^*, u^*, u^*), \\ &= s[0 + S_{qp_b}(\mathcal{A}u^*, \mathcal{A}u^*, u^*)] - 0, \\ &\leq sS_{qp_b}(\mathcal{A}u^*, \mathcal{A}u^*, u^*), \end{aligned}$$

which is a contradiction. Hence, $u^* = \mathcal{A}u^*$. Thus u^* is a fixed point of $\mathcal{A}$.

From Definition 1.11, we show that u^* is a coincidence point of $\mathcal{A}$ and $\mathcal{S}$. Since $\mathcal{A}X$ is complete there exists $u^*, v^* \in X$ such that $u^* = \mathcal{S}v^*$. Which implies that

$$\lim_{n\to\infty} \mathcal{A}u_n = \lim_{n\to\infty} \mathcal{S}u_n = \mathcal{S}v^* = u^*. \tag{19}$$

By taking $u = u_n$ and $v = v^*$ in (5), we obtain

$$F_S \left\{ \begin{array}{c} S_{qp_b}(\mathcal{A}u_n, \mathcal{A}u_n, \mathcal{A}v^*), S_{qp_b}(\mathcal{S}u_n, \mathcal{S}u_n, \mathcal{S}v^*), \\ S_{qp_b}(\mathcal{S}u_n, \mathcal{S}u_n, \mathcal{A}u_n), S_{qp_b}(\mathcal{S}v^*, \mathcal{S}v^*, \mathcal{A}v^*), \\ [S_{qp_b}(\mathcal{S}u_n, \mathcal{S}u_n, \mathcal{A}v^*) + S_{qp_b}(\mathcal{S}v^*, \mathcal{S}v^*, \mathcal{A}u_n)] \end{array} \right\} \leq 0. \tag{20}$$

Letting $n \to \infty$ in (20), we get

$$F_S \left\{ \begin{array}{c} S_{qp_b}(\mathcal{S}v^*, \mathcal{S}v^*, \mathcal{A}v^*), S_{qp_b}(\mathcal{S}v^*, \mathcal{S}v^*, \mathcal{S}v^*), \\ S_{qp_b}(\mathcal{S}v^*, \mathcal{S}v^*, \mathcal{A}v^*), S_{qp_b}(\mathcal{S}v^*, \mathcal{S}v^*, \mathcal{A}v^*), \\ [S_{qp_b}(\mathcal{S}v^*, \mathcal{S}v^*, \mathcal{A}v^*) + S_{qp_b}(\mathcal{S}v^*, \mathcal{S}v^*, \mathcal{A}v^*)] \end{array} \right\} \leq 0, \tag{21}$$

by assumption $(FS2)$ and continuity of F_S, we obtain $S_{qp_b}(\mathcal{A}v^*, \mathcal{A}v^*, \mathcal{S}v^*) \leq 0$.

Consequently,

$$\mathcal{A}v^* = \mathcal{S}v^* = u^*.$$

Thus, u^* is a coincidence point of $\mathcal{S}$ and $\mathcal{A}$.

Now, we assume that $\mathcal{S}$ and $\mathcal{A}$ are either $\mathcal{S}$ or $\mathcal{A}$-weakly compatible. Let

$$\begin{aligned}\lim_{n\to\infty} u_n &= u^*,\\ \lim_{n\to\infty} \mathcal{S}u_n &= \mathcal{S}u^*,\\ \lim_{n\to\infty} \mathcal{A}u_n &= \mathcal{A}u^*,\\ \mathcal{S}\mathcal{S}u^* &= \mathcal{S}\mathcal{A}u^*,\\ \mathcal{S}\mathcal{A}u^* &= \mathcal{A}\mathcal{S}u^*.\end{aligned}$$

Suppose $u = \mathcal{S}u^*$ and $v = v^*$, using (5) and definition 1.11, we get

$$F_S\left\{\begin{array}{c} S_{qp_b}(\mathcal{A}\mathcal{S}u^*, \mathcal{A}\mathcal{S}u^*, \mathcal{A}u^*), S_{qp_b}(\mathcal{S}\mathcal{S}u^*, \mathcal{S}\mathcal{S}u^*, \mathcal{S}u^*),\\ S_{qp_b}(\mathcal{S}\mathcal{S}u^*, \mathcal{S}\mathcal{S}u^*, \mathcal{A}\mathcal{S}u^*), S_{qp_b}(\mathcal{S}u^*, \mathcal{S}u^*, \mathcal{A}u^*),\\ [S_{qp_b}(\mathcal{S}\mathcal{S}u^*, \mathcal{S}\mathcal{S}u^*, \mathcal{A}u^*) + S_{qp_b}(\mathcal{S}u^*, \mathcal{S}u^*, \mathcal{A}\mathcal{S}u^*)]\end{array}\right\} \leq 0, \tag{22}$$

yields to,

$$F_S\left\{\begin{array}{c} S_{qp_b}(\mathcal{A}\mathcal{S}u^*, \mathcal{A}\mathcal{S}u^*, \mathcal{A}u^*), S_{qp_b}(\mathcal{S}\mathcal{A}u^*, \mathcal{S}\mathcal{A}u^*, \mathcal{S}u^*),\\ S_{qp_b}(\mathcal{S}\mathcal{A}u^*, \mathcal{S}\mathcal{A}u^*, \mathcal{A}\mathcal{S}u^*), S_{qp_b}(\mathcal{S}u^*, \mathcal{S}u^*, \mathcal{A}u^*),\\ [S_{qp_b}(\mathcal{S}\mathcal{A}u^*, \mathcal{S}\mathcal{A}u^*, \mathcal{A}u^*) + S_{qp_b}(\mathcal{S}u^*, \mathcal{S}u^*, \mathcal{A}\mathcal{S}u^*)]\end{array}\right\} \leq 0. \tag{23}$$

Which implies that

$$S_{qp_b}(\mathcal{A}u^*, \mathcal{A}u^*, \mathcal{S}\mathcal{A}u^*) \leq 0.$$

We have $\mathcal{S}u^* = \mathcal{S}\mathcal{A}u = \mathcal{A}\mathcal{S}u = \mathcal{A}u^*$. Thus, $\mathcal{S}$ and $\mathcal{A}$ are weakly compatible self-maps of a set X. Therefore, $\mathcal{S}$ and $\mathcal{A}$ have a unique point of coincidence $u^* = \mathcal{S}u = \mathcal{A}u$, then u^* is the unique common fixed point of $\mathcal{S}$ and $\mathcal{A}$. □

Pathak*et al.* [49], in their work, considered an example in which weakly compatible mapping is not compatible. In this work, we use one more example of this type, which satisfies quasi partial S_b-metric Space and uses it to formulate an implicit function that satisfies all conditions imposed in Definition 1.11 and Theorem 3.1.

Example 3.2. *Consider $X = [0, \infty]$ endowed with complete quasi-partial S_b-metric space, defined by metric $S_{qp_b}(u, u, v) = 2(u - v)^2$ on X. Define a pair of mappings $\mathcal{A}, \mathcal{S} : X \to X$ by*

$$\mathcal{S}u = \begin{cases} \cos u & \text{if } u \neq 1\\ 0 & \text{if } u = 1,\end{cases}$$

and

$$\mathcal{A}u = \begin{cases} e^u & \text{if } u \neq 1\\ 0 & \text{if } u = 1,\end{cases}$$

by Definition 1.11, it obvious that at $u = 0$, we have $u^* = \mathcal{S}u = \mathcal{S}\mathcal{S}u = \mathcal{A}\mathcal{S}u = \mathcal{A}u$, then $u^* = 0$ is the unique common fixed point of $\mathcal{S}$ and $\mathcal{A}$. Therefore, the mappings $\mathcal{S}$ and $\mathcal{A}$ are weakly compatible. Define continuous function $F : \mathbb{R}^5_+ \to \mathbb{R}$ by

$$F(t_1, t_2, t_3, t_4, t_5) = t_1 - \gamma t_5.$$

i.e.,

$$F_S(u,v,v,u,s(2u+v)) \quad = \quad t_1 - \gamma t_5.$$

With a view to verify assumptions (a) and (d) of Theorem 3.1. Consider $\mathcal{A}u, \mathcal{S}u \in X$ so that

$$\begin{aligned} t_1 &\leq \gamma t_5. \\ S_{qp_b}(\mathcal{A}u,\mathcal{A}u,\mathcal{S}v) &\leq \gamma[s[2S_{qp_b}(\mathcal{A}u,\mathcal{A}u,\mathcal{S}v) + S_{qp_b}(\mathcal{A}v,\mathcal{A}v,\mathcal{S}u)]]. \quad (24) \end{aligned}$$

Recall the quasi partial S_b-metric as,

$$\begin{aligned} S_{qp_b}(\mathcal{A}u,\mathcal{A}u,\mathcal{S}v) &= 2(\mathcal{A}u - \mathcal{S}v)^2, \\ &= 2(e^u - \cos v)^2. \quad (25) \end{aligned}$$

Similarly,

$$\begin{aligned} S_{qp_b}(\mathcal{A}v,\mathcal{A}v,\mathcal{S}u) &= 2(\mathcal{A}v - \mathcal{S}u)^2 \\ &= 2(e^v - \cos u)^2. \quad (26) \end{aligned}$$

Using (25) and (26) in (24), we get

$$\begin{aligned} 2(e^u - \cos v)^2 &\leq \gamma s[2(e^u - \cos v)^2 + 2(e^v - \cos u)^2], \\ 2(e^u - \cos v)^2(1-2\gamma s) &\leq \gamma s[4(e^v - \cos u)^2], \\ 2(e^u - \cos v)^2 &\leq \frac{\gamma s}{(1-2\gamma s)}[2(e^v - \cos u)^2], \quad (27) \end{aligned}$$

which means

$$q \quad \leq \quad \vartheta r.$$

Hence, F_S satisfies $FS1, FS2$ and $FS3$ for $\vartheta \in [0, \frac{1}{s}]$. Also, all assumptions of Theorem 3.1 and Definition 1.11 are satisfied. It is observed that the pair $(\mathcal{S}, \mathcal{A})$ has a common fixed point. Thus, they admit a coincidence fixed point.

4. Some Applications

This section has two applications. The first application covers the existence of the solution for two boundary value second-order differential equations. In the second application, we prove the existence solution for Caputo-type non-linear fractional differential equations. Finally, we use the two applications to utilise the results obtained in Theorem 3.1 where a common solution is applied in quasi partial S_b-metric space setting.

4.1. Existence of the two boundary value second order differential equation

In this subsection, we discuss the existence of a solution to the boundary value problem by considering space to be quasi-partial S_b-metric space. We now consider the second-order differential equation's two-point boundary value problem. The following example is motivated by [13, 18, 29, 47, 63]

$$\begin{cases} u^{''}(t) = f(t, u(t), u'(t)), \ 0 \leq t \leq T, \\ u(0) = \alpha, \\ u(T) = \beta, \end{cases} \tag{28}$$

where $T > 0$ and $f : [0, T] \times X \times X \longrightarrow X$ is a continuous function.

This boundary value problem is equivalent to the integral equation

$$u(t) \quad = \quad \alpha + \frac{\beta - \alpha}{T} t + \int_0^T G(t, s) f(s, u(s), u'(s)) ds, \forall\ t, s \in [0, T]. \tag{29}$$

where the Green's function associated with the above integral equation is given by

$$G(t, s) = \begin{cases} \dfrac{s(T - t)}{T}, & 0 \leq s \leq t \leq T, \\ \\ \dfrac{t(T - s)}{T}, & 0 \leq t \leq s \leq T, \end{cases}$$

and $\alpha, \beta > 0$.

We prove our results by establishing a common fixed point for a pair of weakly compatible self-mappings in quasi-partial S_b-metric space.

Theorem 4.1. *Let $\mathcal{A}, \mathcal{S} : C([0, T]) \longrightarrow C([0, T])$ be self maps of a quasi-partial S_b-metric space (X, S_{qp_b}) such that the following condition holds:*

(i) *there exists $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a continuous function and η-weakly increasing in the first and fifth variables with $\gamma \in [0, \frac{1}{s}]$ such that*

$$|f_1(t, u(t), u'(t))| - |f_2(t, v(t), V(t))| \quad \leq \quad \gamma \sqrt{\frac{\ln[(u - v)^2 + 1]}{u - v}},$$

where $|u(s) - v(s)| = \gamma \sqrt{\frac{\ln[(u-v)^2+1]}{u-v}}$ and for increasing of u and v, we have $u, v \in C^1([0, T], X)$,

(ii) *the Green's function is given by*

$$\int_0^T G(t, s) \leq \frac{1}{8}.$$

Then, the integral equation (29) has a common solution in $C^1([0, T], X)$.

Proof: Let $C^1([0,T],X) = f : [0,T] \to \mathbb{R}$ is a continuous function. Now, we define the function $S_{qp_b} : C[0,T] \times C[0,T] \times C[0,T] \to [0,\infty)$ with the quasi-partial S_b-metric

$$S_{qp_b}(u,u,v) = 2\left(\sup_{t\in[0,T]} |u(t)-v(t)|\right)^2 + 2\left(\sup_{t\in[0,T]} |u'(t)-v'(t)|\right)^2.$$

Then, (X, S_{qp_b}) is a complete quasi-partial S_b-metric space.

Let $\mathcal{A},\mathcal{S} : X \longrightarrow X$ be two $\mathcal{S}$-weakly compatible operator defined by

$$\mathcal{A}u(t) = \alpha + \frac{(\beta-\alpha)t}{T} + \int_0^T G(t,s)f_1(t,s,u(s),u'(s))ds, \forall\ t,s \in [0,T].$$

and

$$\mathcal{S}v(t) = \alpha + \frac{(\beta-\alpha)t}{T} + \int_0^T G(t,s)f_2(t,s,v(s),v'(s))ds, \forall\ t,s \in [0,T],$$

where f_1, f_2 and α, β are continuous functions.

Now, u^* is a solution of (29) if and only if u^* is a common fixed point of $\mathcal{A}$ and $\mathcal{S}$. Since $\mathcal{A}$ and $\mathcal{S}$ are increasing in the first and fifth variables, other assertions of Theorem 4.1 are satisfied. We shows that $\mathcal{A}$ and $\mathcal{S}$ are contraction in X.

For each $t \in [0,1]$, by (ii), we have

$$\int_a^b G(t,s)ds = \frac{1}{2}t(t-1).$$

and sup-norm of $t(1-t) = \frac{1}{4}$, therefore

$$\sup_{t\in[a,b]} \int_a^b G(t,s)ds = \frac{1}{8}.$$

By using condition (i) of Theorem 4.1, we discuss the following cases:

Case I.

$$\begin{aligned} |\mathcal{A}u(t) - \mathcal{S}v(t)| &= \int_0^T |f_1(s,u(s),u'(s)) - f_2(s,v(s),v'(s))|ds, \\ &\le 2\left(\int_0^T |G(t,s)|ds|u(s)-v(s)|\right)^2, \\ &\le 2\left(\frac{\gamma}{8}\sqrt{\frac{\ln[(|u-v|)^2+1]}{|u-v|}}\right)^2. \qquad (30) \end{aligned}$$

Case II.

$$\begin{aligned} |\mathcal{A}u'(t)-\mathcal{S}v'(t)| &= \int_0^T |f_1(s,u(s),u'(s))-f_2(s,v(s),v'(s))|ds, \\ &\leq 2\left(\int_0^T |G(t,s)|ds|u'(s)-v'(s)|\right)^2, \\ &\leq 2\left(\frac{\gamma'}{8}\sqrt{\frac{\ln[(|u'-v'|)^2+1]}{|u'-v'|}}\right)^2. \end{aligned} \tag{31}$$

By combining (30) and (31) , we obtain

$$\begin{aligned} |\mathcal{A}u(t)-\mathcal{S}v(t)|+|\mathcal{A}u'(t)-\mathcal{S}v'(t)| &\leq 2\left(\frac{\gamma}{8}\sqrt{\frac{\ln[(|u-v|)^2+1]}{|u-v|}}\right)^2 + \\ & \quad 2\left(\frac{\gamma'}{8}\sqrt{\frac{\ln[(|u'-v'|)^2+1]}{|u'-v'|}}\right)^2. \\ S_{qp_b}(\mathcal{A}u,\mathcal{A}u,\mathcal{S}v) &\leq \vartheta S_{qp_b}(u,u,v). \end{aligned}$$

Therefore $u^* \in X$ is a common fixed of $\mathcal{A}$ and $\mathcal{S}$, also a solution to integral equation (29). Hence the differential equation (28) has a solution.

4.2. Existence of a common solution of weakly compatible mappings for nonlinear fractional differential equation in quasi-partial S_b-metric Space

This subsection aims to provide an application of Theorem 3.1 to get a common solution of $\mathcal{A}$,$\mathcal{S}$-weakly compatible mappings for a nonlinear fractional differential equation, where we can apply a generalised mapping in quasi partial S_b-metric spaces.

We investigate the existence of a unique common fixed point for $\mathcal{A},\mathcal{S}$-weakly compatible mappings of the Caputo derivative with the fractional order of the nonlinear fractional differential equation.

This form of fractional derivative for a continuous function $f:[0,\infty)\to\mathbb{R}$ is given by Abdeljawad *et al.* [2] and Zahed *et al.* [64] as: Caputo fractional derivative of $f(t)$ order $\alpha>0$ is denoted by ${}^C\mathcal{D}^\alpha_f(t)$ and defined as

$${}^C\mathcal{D}^\alpha f(t) = \frac{1}{\Gamma(i-\alpha)}\int_0^t (t-\tau)^{i-\alpha-1}\eta^i(\tau)d\tau,$$

with $i=[\alpha]+1\in\mathbb{N}$, where $\alpha\in[i-1,i]$ and $[\alpha]$ denotes the greatest integers of α (i.e., the greatest part of α) and $\alpha:[0,\infty)\to\mathbb{R}$ is a continuous function.

We denote $X=C([0,1],\mathbb{R})$ the set of all continuous functions from $[0,1]$ into $\mathbb{R}$.

The Caputo fractional differential equation has several applications in mathematics, i.e., in image processing, Digital data processing, electrical signal, acoustics, physics, electrochemistry, radiotherapy and probability theory (one can see in [65]). The following nonlinear fractional differential equation is inspired by Baleanu *et al.* [9], Budhia *et al.* [14], Jarad *et al.* [32], Karapinar *et al.*[38] and Kanwal *et al.* [36].

Consider the following nonlinear fractional differential equation.

$$\begin{cases} {}^{C}\mathcal{D}^{\alpha}u(t) = f(t, u(t)), \; t \in (0,1), 1 < \alpha \leq 2, \\ \\ u(0) = 0, u(1) = \int_0^{\sigma} u(\tau)d\tau \; (0 < \sigma < 1) \end{cases} \tag{32}$$

where ${}^{C}\mathcal{D}^{\alpha}_{\tau}$ denotes the Caputo fractional derivative of order α and $f : [0,1] \times X \to X$ is a continuous function.

The nonlinear fractional differential Equation 32 can be written as

$$\begin{aligned} \mathrm{u(t)} \;\; = \;\; & \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, u(\tau))d\tau - \\ & \frac{2t}{(2-\sigma^2)\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} f(\tau, u(\tau))d\tau + \\ & \frac{2t}{(2-\sigma^2)\Gamma(\alpha)} \int_0^{\sigma} \left[\int_0^{\tau} (\tau - z)^{\alpha-1} f(z, u(z))dz \right] d\tau. \end{aligned} \tag{33}$$

A function $x \in C(I, X)$ is a solution of the fractional differential integral equation (33) if and only if x is a solution of the nonlinear fractional differential equation (32).

We define a quasi-partial S_b metric on X as

$$S_{qp_b}(u,u,v) = \left(\sup_{t \in [0,1]} |u(t) - v(t)| \right)^2 + \left(\sup_{t \in [0,1]} |u(t) - v(t)| \right)^2.$$

Then, (X, S_{qp_b}) is a complete quasi-partial S_b metric space.

Now, we prove the following theorem.

Theorem 4.2. *Suppose the following hypothesis hold:*

(i): *there exists $f \in C(I \times X, X)$ a continuous in the first and fifth variables;*

(iii): *there exists a continuous function $f : [0,1] \times \mathbb{R} \to \mathbb{R}_+$, such that*

$$|f(t, u(\tau) - f(t, v(\tau)| \leq 2\vartheta |u(\tau) - v(\tau)|^2,$$

for all $t \in [0,1]$ and for all $u, v \in X$ and a constant $\vartheta \in \left[0, \frac{1}{s}\right)$ such that

$$\vartheta = \left[\frac{t^{\alpha}(2-\sigma^2)(\alpha+1) + 2t(\alpha + \sigma^{(\alpha+1)} + 1)}{(2-\sigma^2)\Gamma(\alpha)(\alpha(\alpha+1))} \right]^2.$$

Then, the fractional differential Equation 32 has a common solution as a fixed point $u^* \in C(I, X)$.

Proof: Let us define $\mathcal{A}, \mathcal{S} : C([0,1]) \to C([0,1])$, with $\zeta \in \eta$ by

$$\begin{aligned} \mathcal{A}\mathrm{u(t)} &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, u(\tau)) d\tau - \\ & \frac{2t}{(2-\sigma^2)\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} f(\tau, u(\tau)) d\tau + \\ & \frac{2t}{(2-\sigma^2)\Gamma(\alpha)} \int_0^\sigma \left[\int_0^\tau (\tau - z)^{\alpha-1} f(z, u(z)) dz \right] d\tau, \end{aligned} \tag{34}$$

for $t \in [0,1]$, then $\mathcal{A}$ is continuous at the first and fifth variables. Suppose that

$$\mathcal{S}u(t) = \int_0^\tau (\tau - z)^{\alpha-1} f(z, u(z)) dz,$$

this implies that $\mathcal{S} \in \mathcal{A}$ and $\mathcal{A}$ posses a fixed point $u^* \in \mathcal{S}$. To prove the existence of a fixed point of $\mathcal{A}$, we prove that $\mathcal{A}$ is continuous in the first and fifth variables of the implicit function F_S and is a contraction. To show this, let $\mathcal{A}u \neq \mathcal{S}v$, for all $u, v \in [0,1]$. By the hypothesis of Theorem 4.2, we have

$$\begin{aligned} |\mathcal{A}\mathrm{u} - \mathcal{A}\mathrm{v}| &= 2 \Bigg| \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, u(\tau)) d\tau - \\ & \frac{2t}{(2-\nu^2)\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} f(\tau, u(\tau)) d\tau + \\ & \frac{2t}{(2-\sigma^2)\Gamma(\alpha)} \int_0^\sigma \left[\int_0^\tau (\tau - z)^{\alpha-1} f(z, u(z)) dz \right] d\tau \\ & - \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, v(s)) ds + \\ & \frac{2t}{(2-\sigma^2)\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} f(\tau, v(\tau)) d\tau - \\ & \frac{2t}{(2-\sigma^2)\Gamma(\alpha)} \int_0^\sigma \left[\int_0^\tau (\tau - z)^{\alpha-1} f(z, v(z)) dz \right] d\tau \Bigg|^2, \end{aligned}$$

$$
\begin{aligned}
&\leq 2\Bigg(\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}|f(\tau,u(\tau))-f(\tau,v(\tau))|d\tau + \\
&\quad \frac{2t}{(2-\sigma^2)\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}|f(\tau,u(\tau))-f(\tau,v(\tau))|d\tau, + \\
&\quad \frac{2t}{(2-\sigma^2)\Gamma(\alpha)}\int_0^\sigma \left[\int_0^\tau (\tau-z)^{\alpha-1}|f(z,u(z))-f(z,v(z))|dz\right]d\tau\Bigg)^2, \\
&\leq 2\Bigg(\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}|u(\tau)-v(\tau)|d\tau + \\
&\quad \frac{2t}{(2-\sigma^2)\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}|u(\tau)-v(\tau)|d\tau + \\
&\quad \frac{2t}{(2-\sigma^2)\Gamma(\alpha)}\int_0^\sigma \left[\int_0^\tau (\tau-z)^{\alpha-1}|u(z)-v(z)|dz\right]d\tau\Bigg)^2, \\
&= 2\Bigg(\frac{1}{\Gamma(\alpha)}\|u-v\|_\infty\int_0^t (t-\tau)^{\alpha-1}d\tau + \\
&\quad \frac{2t}{(2-\sigma^2)\Gamma(\alpha)}\|u-v\|_\infty\int_0^1 (1-\tau)^{\alpha-1}d\tau + \\
&\quad \frac{2t}{(2-\sigma^2)\Gamma(\alpha)}\|u-v\|_\infty\int_0^\sigma \left[\int_0^\tau (\tau-z)^{\alpha-1}dz\right]d\tau\Bigg)^2, \\
&\leq \left[\frac{t^\alpha}{\alpha\Gamma(\alpha)} + \frac{2t}{(2-\sigma^2)\alpha\Gamma(\alpha)} + \frac{2t\sigma^{\alpha+1}}{(2-\sigma^2)\alpha(\alpha+1)\Gamma(\alpha)}\right]^2 2\|u-v\|_\infty^2, \\
&\leq 2\vartheta\|u-v\|_\infty^2. \qquad (35)
\end{aligned}
$$

This implies that

$$\|\mathcal{A}u-\mathcal{A}v\|_\infty \leq 2\vartheta\|u-v\|_\infty^2.$$

Thus for each $u, v \in X$, we have

$$S_{qp_b}(\mathcal{A}u,\mathcal{A}u,\mathcal{A}v) \leq \vartheta S_{qp_b}(u,u,v). \qquad (36)$$

For $\vartheta \in [0, \frac{1}{s})$ and the condition $((FS1)-(FS2))$ shows that $\mathcal{A}$-$\mathcal{S}$ is a contraction mapping on X. Since all the hypotheses of Theorem (4.2) are satisfied. Therefore, there exists $u^* \in C(I)$ a common fixed point of $\mathcal{A}$ and $\mathcal{S}$, that is, u^* is a solution to fractional nonlinear differential equation (32).

Declarations

Availability of data and material

This clause does not apply to this paper.

Competing interests

The authors declare that they have no competing interests.

Funding

The authors declare that there is no funding available for this article.

Author's contributions

Both authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

Acknowledgement

The authors are thankful to the learned reviewer for his valuable comments.

References

[1] M. Abbas and G. Jungck, *Common fixed point results for non-commuting mappings without continuity in cone metric spaces*, J. Math. Anal. Appl. **341** (2008), no. 1, 416–420.

[2] T. Abdeljawad, R. P. Agarwal, E. Karapinar, and P. S. Kumari, *Solutions of the nonlinear integral equation and fractional differential equation using the technique of a fixed point with a numerical experiment in extended b-metric Space*, Symmetry **11** (2019), no. 5, 686.

[3] M. Ahmadullah, J. Ali, and M. Imdad, *Unified relation-theoretic metrical fixed point theorems under an implicit contractive condition with an application*, Fixed Point Theory Appl. (2016), no. 1, 1–15.

[4] J. Ali and M. Imdad, *An implicit function implies several contraction conditions*, Sarajevo J. Math. **4** (2008), no. 17, 269–285.

[5] A. H. Ansari, V. Gupta, and N. Mani, *C-class functions on some couple fixed point theorem in partially ordered S-metric spaces*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **68** (2019), no. 2, 1694–1708.

[6] H. Aydi, A. Felhi, and S. Sahmim, *Common fixed points via implicit contractions on b-metric-like spaces*, J. Nonlinear Sci. Appl. **10** (2017), no. 4, 1524–1537.

[7] H. Aydi, M. F. Bota, E. Karapinar, and S. Mitrović, *A fixed point theorem for set-valued quasi-contractions in b-metric spaces*, Fixed Point Theory Appl. **(1)** (2012), 1–8.

[8] H. Aydi, M. F. Bota, E. Karapinar, and S. Moradi, *A common fixed point for weak ϕ-contractions on b-metric spaces*, Fixed Point Theory **13** (2012), no. 2, 337–346.

[9] D. Baleanu, S. Rezapour, and H. Mohammadi, *Some existence results on nonlinear fractional differential equations*, Philos. Trans. R. Soc. Lond. Ser. A, Math. Phys. Eng. Sci. **371** (2013), no.1990, 1–7.

[10] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. **3** (1922), no. 1, 133–181.

[11] V. Berinde, *Approximating fixed points of implicit almost contractions*, Hacet J Math Stat. **40** (2012), no. 1, 93–102.

[12] V. Berinde and F. Vetro, *Common fixed points of mappings satisfying implicit contractive conditions*, Fixed Point Theory Appl. (2012), no. 1, 105.

[13] P. Borisut, P. Kumam, V. Gupta, and N. Mani, *Generalized (ψ, α, β)-Weak Contractions for Initial Value Problems*, Mathematics **7** (2019), no. 3, 266.

[14] L. Budhia, H. Aydi, A. H. Ansari, and D. Gopal, *Some new fixed point results in rectangular metric spaces with an application to fractional-order functional differential equations*, Nonlinear Anal.: Model. Control **25** (2020), no .4, 580–597.
[15] S. Chaipornjareansri, *Fixed point theorems for generalised weakly contractive mappings in S-metric spaces*, Thai J. Math. (2018), 50–62.
[16] C. Chifu, and G. Petruşel, *Fixed point results for multi-valued Hardy-Rogers contractions in b-metric spaces*, Filomat **31** (2017), no. 8, 2499–2507.
[17] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Univ. Ostrav. **1** (1993), 5–11.
[18] R. E. Edwards, *Functional Analysis Theory and Application*, Courier Corporation, 2012.
[19] K. S. Eke, B. Davvaz, and J. G. Oghonyon, *Relation-theoretic common fixed point theorems for a pair of implicit contractive maps in metric spaces*, Commun Math Appl. **10** (2019), no. 1, 159–168.
[20] M. Fréchet, *Sur quelques points du calcul fonctionnel*, Rendiconti del Circolo Matematico di Palermo **22** (1906), no. 1, 1–72.
[21] P. Gautam and S. Verma, *Fixed point via implicit contraction mapping in Quasi-partial b-metric spaces*, The Journal of Analysis (2021), 1–13.
[22] P. Gautam, L. M. Sanchez Ruiz, and S. Verma, *Fixed point of interpolative Rus-Reich-Ćirić contraction mapping on rectangular quasi-partial b-metric space*, Symmetry **13** (2020), no. 1, 1–32.
[23] P. Gautam, L. M. Sanchez Ruiz, and S. Verma, *Common fixed point results on generalised weak compatible mapping in quasi-partial b-metric Space*, J. Math **2021** (2021), Article ID 5526801.
[24] P. Gautam, V. N. Mishra, R. Ali, and S. Verma, *Interpolative Chatterjea and cyclic Chatterjea contraction on quasi-partial b-metric space*, AIMS Mathematics **6** (2021), no. 2, 1727–1742.
[25] D. Gopal, P. Kumam, and M. eds. Abbas, *Background and recent developments of metric fixed point theory*, CRC Press, 2017.
[26] A. Gupta, and P. Gautam, *Quasi-partial b-metric spaces and some related fixed point theorems*, Fixed Point Theory Appl. **1** (2015), 1–12.
[27] A. Gupta and P. Gautam, *Topological structure of quasi-partial b-metric spaces*, Int. J. Pure Math. Sci. **17** (2016), 8–18.
[28] A. Gupta and P. Gautam, *Some coupled fixed point theorems on quasi-partial b-metric spaces*, Int. J. Math. Anal. **9** (2015), no. 6, 293–306.
[29] V. Gupta, W. Shatanawi, and N. Mani, *Fixed point theorems for (ψ, α, β)-Geraghty contraction type maps in ordered metric spaces and some applications to integral and ordinary differential equations*, J. Fixed Point Theory Appl. **19** (2017), no. 2, 1251–1267.
[30] M. Imdad, S. Kumar, and M. S. Khan, *Remarks on some fixed point theorems satisfying implicit relations*, Radovi Mathematicki **11** (2002), 135–143.
[31] M. Imdad, R. Gubran, and M. Ahmadullah *Using an implicit function to prove common fixed point theorems*, prepreint (2016); arXiv:1605.05743.
[32] F. Jarad, T. Abdeljawad, and Z. Hammouch, *On a class of ordinary differential equations in the frame of Atangana-Baleanu fractional derivative*, Chaos Solitons Fractals **117** (2018), 16–20.
[33] G. Jungck, *Commuting mappings and fixed points*, Amer. Math. Monthly **83** (1976), no. 4, 261–263.
[34] G. Jungck, *Compatible mappings and common fixed points*, Int. J. Math. Math. Sci. **9** (1986), no. 4, 771–779.
[35] H. Kaneko and S. Sessa, *Fixed point theorem for compatible multi-valued and single-valued mappings*, Internat. J. Math. and Math. Sci. **12** (1989), 257–262.

[36] T. Kanwal, A. Hussain, H. Baghani, and M. de la Sen, *New fixed point theorems in orthogonal F-metric spaces with application to fractional differential equation*, Symmetry **12** (2020), no. 5, 832.

[37] E. Karapinar, I. Erhan, and A. Ozurk, *Fixed point theorems on quasi-partial metric spaces*, Math Comput. Model. **57** (2013), 2442–2448.

[38] E. Karapinar, T. Abdeljawad, and F. Jarad, *Applying new fixed point theorems on fractional and ordinary differential equations*, Adv. Difference Equ. **1** (2019), 1–25.

[39] E. Karapinar, A. Fulga, and A. Petrusel, *On Istratescu type contractions in b-metric spaces*, Mathematics **8** (2020), no.3, 388.

[40] J. K. Kim, S. Sedghi, A. Gholidahneh, and M. Rezaee, *Fixed point theorems in S-metric spaces*, East Asian, Math. J. **32** (2016), no. 5, 677–684.

[41] W. Kirk and N. Shahzad, *Fixed point theory in distance spaces*, Springer, 2014.

[42] S. G. Matthews, *Partial-metric topology*, Ann. N. Y. Acad. Sci. **728** (1994), 183–197.

[43] N. Mlaiki, A. Mukheimer, Y. Rohen, N. Souayah, and T. Abdeljawad, *Fixed point theorem for α-ϕ contractive mappings in S_b-metric Space*, J. Math. Anal. Appl. **8** (2017), no. 5, 40–46.

[44] S. Nizar, *A fixed point in partial S_b-metric spaces*, Analele Universitatii Ovidius Constanta-Seria Matematica **24** (2016), no. 3, 351–362.

[45] S. Nizar and M. Nabil, *A fixed point theorem in S_b-metric spaces*, J. Math. Computer Sci. **16** (2016), 131–139.

[46] N. Y. Ozgur and N. Tas, *New contractive conditions of integral type on complete S-metric spaces*, Math. Sci. **11** (2017), no. 3, 231–240..

[47] H. K. Pathak, *An introduction to nonlinear analysis and fixed point theory*, Springer, 2018.

[48] H. K. Pathak, *Fixed point theorems for weak compatible multi-valued and single-valued mappings*, Acta Math. Hungarica. **67** (1995), no. 1-2, 69–78.

[49] H. K. Pathak, M. S. Khan, and R. Tiwari, *A common fixed point theorem and its application to nonlinear integral equations*, Comput. Math. Appl. **53** (2007), no. 6, 961–971.

[50] V. Popa, *Some fixed point theorems for compatible mappings satisfying an implicit relation*, Demonstr. Math. **32** (1999), no. 1, 157–164.

[51] V. Popa, *Fixed points for non-surjective expansion mappings satisfying an implicit relation*, Buletinul s'tiint'ific Al Universitatii Baia Mare, Seria B, Fascicola Matematica-Informatica **18** (2002), 105–108.

[52] V. Popa, *A general fixed point theorem for weakly compatible mappings in compact metric spaces*, Turkish J. Math. **25** (2001), 465–474.

[53] V. Popa, *Fixed point theorems for implicit contractive mappings*, Studii si Cercetari Stiintifice Series: Mathematics, Universitatea din Bacau **7** (1997), 127–133.

[54] V. Popa, *Common fixed points of mappings satisfying implicit relations in partial-metric spaces*, J. Nonlinear Sci. Appl. **6** (2013), no. 3, 152–161.

[55] V. Popa and A. Patriciu, *Fixed point for compatible mappings in S-metric spaces*, Scientific Studies and Research Series Mathematics and Informatics **28** (2018), no. 2, 63–78.

[56] J. R. Roshan, V. Parvaneh, and Z. Kadelburg, *Common fixed point theorems for weakly isotone increasing mappings in ordered b-metric spaces*, J. Nonlinear Sci. Appl. **7** (2014), 229–245.

[57] S. Sedghi, N. Shobe, and A. Aliouche, *A generalisation of fixed point theorem in S-metric spaces*, Matematicki Vesnik **64** (2012), 258–266.

[58] S. Sessa, *On a weak commutativity condition of mappings in fixed point considerations*, Publ. Inst. Math. **32** (1982), no. 46, 149–153.

[59] K. A. Singh and M. Singh, *Common fixed point of four maps in S-metric spaces*, Math. Sci. **12** (2018), no. 2, 137–143.

[60] K. A. Singh and M. Singh, *Fixed point theorem for generalised β-ϕ Geraghty contraction type maps in S-metric Space*, Electron J Math Anal Appl. **8** (2020), no. 1, 273–283.
[61] W. Sintunavarat and P. Kumam, *Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces*, J. Appl Math. **2011**, 14 pages, DOI:10.1155/2011/637958.
[62] C. Vetro and F. Vetro, *Common fixed points of mappings satisfying implicit relations in partial metric spaces*, J. Nonlinear Sci. Appl. **6** (2013), no. 3, 152–161.
[63] F. Yan, Y. Su, and Q. Feng, *A new contraction mapping principle in partially ordered metric spaces and applications to ordinary differential equations*, Fixed Point Theory Appl. **2012** (2012), Article ID 152.
[64] H. Zahed, H. A. Fouad, S. Hristova, and J. Ahmad, *Generalized Fixed Point Results with Application to Nonlinear Fractional Differential Equations,* Mathematics **8** (2020), no. 7, 1168.
[65] U. Zölzer, *DAFX: Digital Audio Effects,* Ed. **21** (2020), no. 2, 48–49.

Lucas Wangwe
Department of Mathematics, College of Natural and Applied Sciences,
University of Dar es Salaam, Tanzania.
E-mail: wangwelucas@gmail.com

Santosh Kumar
Department of Mathematics, College of Natural and Applied Sciences,
University of Dar es Salaam, Tanzania.
E-mail: drsengar2002@gmail.com